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New approach to approximate the solution for the system of fractional order Volterra integro-differential equations

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ABSTRACT

The main aim of this article is the extension of Optimal Homotopy Asymptotic Method to the system of fractional order integro-differential equations. The systems of fractional order Volterra integro-differential equations (SFIDEs) are taken as test examples. The fractional order derivatives are defined in the Caputo fractional form and the optimal values of auxiliary constants are calculated using the well-known method of least squares. The results obtained by proposed scheme are very encouraging and show close resemblance with exact values. Hence it will be more appealing for the researchers to apply the proposed scheme to different fractional order systems arising in different fields of sciences especially in fluid dynamics and bio-engineering.

Introduction

Fractional calculus has been concerned with integration and differentiation of fractional (non-integer) order of the function. In recent years, fractional calculus has been revolutionized by its tremendous innovations, observed in different fields of science and technology, such as fractional dynamics, nonlinear oscillation, hereditary in mechanics of solids, visco-elastically damped structures, bio-engineering and continuum mechanics [1–6]. Therefore, researchers have paid enormous interest in this field. Dynamical behavior of mixed type lump solution [7], exact optical solution of perturbed nonlinear Schrödinger equation [8], nonlinear complex fractional emerging telecommunication model [9], explicit solution of nonlinear Zoomeron equation[10], optical soliton in nematic liquid crystals [11], two-hybrid technique coupled with integral transformation for caputo time fractional Navier-Stokes equation [12], Analysis of Fractionally-Damped generalized Bagley-Torvik equation [13], Brusselator reaction–diffusion system [14], Fractional order of biological system [15], time fractional lvancevic option pricing model [16], analysis of fractionally damped beams [17], model of vibration equation of large membranes [18], fractional Jeffrey fluid over inclined plane [19], thermal stratification of rotational second-grade fluid [20], long memory processes [21], heat-transfer properties of noble gases [22] and modelling the dynamic mechanical analysis [23]. A history of fractional differential operators can be found in [24]. Owing to its applications, researchers compel to extract its solutions, but exact solutions of all problems are difficult to find due to its nonlinearity. Therefore, researchers used analytical and numerical techniques for its approximate solutions. Numerical methods [25–28], perturbation methods [29–31], homotopy based method [32,33] and iterative techniques [34,35] are the main tools for obtaining the approximation of nonlinear problems. In the literature, researchers have used different techniques for the solution of fractional order integro-differential equations and their systems. Khan et al. implemented the Chebyshev wavelet method [36], Rahim et al. used the fractional alternative Legendre functions [37], Hamoud et al. applied the modified adomian decomposition method [38], Zedan et al. used the Chebyshev spectral method [39], and Zada et al. studied the impulsive coupled system [40].

The OHAM was introduced by Marinca et al. [41–43] for the solution of differential equations, and in a short period, different researchers have successfully implemented it for the solution of different problems

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in literature. Iqbal et al. applied this technique to Klein–Gordon equations and singular Lane–Emden type equation [44,45]. Sheikholeslami et al. used the proposed method for investigation of the laminar viscous flow and magnetohydrodynamic flow in a permeable channel [46,47]. Hashmi et al. obtained the solution of nonlinear Fredholm integral equations [48]. Nawaz et al. obtained an approximate solution of fractional Order Zakharov–Kuznetsov equations [49] and three-dimensional integral equations [50].

The System of fractional order Volterra integro-differential equations of the following form:

$$D^{\alpha}U(x) - T(x) - \int_0^x K(x,t)\,U(t)\,dt = 0, \qquad a\leqslant x\leqslant b,\ 0<\alpha\leqslant 1, \tag{1}$$

with initial condition, $U(0) = b_i,\ i = 1, 2, \ldots n$, where

$$\begin{cases} D^{\alpha}U(x) = \left[u_1^{\alpha}(x),\, u_2^{\alpha}(x),\, \ldots, u_n^{\alpha}(x)\right], \\ U(t) = [u_1(t),\, u_2(t),\, \ldots, u_n(t)], \\ T(x) = [t_1(x),\, t_2(x),\, \ldots, t_n(x)], \\ K(x,t) = [k_{ij}(x,t)], \quad i,j = 1,2,\ldots,n. \end{cases} \tag{2}$$

In Eq. (1), $T(x)$ are known functions, $K(x,t)$ are known kernels of integration, D^{α} is fractional differential operator and $U(x)$ are unknown functions to be determined.

In the literature, fractional derivatives and integrations of order $\alpha > 0$ are defined in a different ways. The two most frequently used definitions are the Riemann-Liouville and Caputo. The Riemann-Liouville fractional integration of order α is defined as:

$$J^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u(t)dt, \qquad x,\alpha>0 \tag{3}$$

where J^{α} denotes the fractional integral operator. The Caputo definition of fractional differential operator of order α is defined as:

$$D^{\alpha}u(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}u^{(m)}(t)dt, \qquad m-1<\alpha\leqslant m, \tag{4}$$

where D^{α} denotes the fractional order differential operator, first compute an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative in Caputo sense.

The main aim of this work is to extend the OHAM for SFIDE. The extended work is demonstrated by several numerical examples to approximate the solution of SFIDE which covers the exact solution in just few steps. OHAM reduced the involution of computational work as compare to the other existing methods in the literature. Error analysis and convergence of the proposed method also provided in this paper.

This paper is organized in six sections. Section "Formulation of OHAM" contains the basic formulation of OHAM. In Section "Convergence of OHAM" convergence of OHAM has been discussed. The applications of OHAM to SFIDE are discussed in Section "Numerical problem". Results and discussion are in Section "Results and discussions" while the conclusions are in Section "Conclusions".

Formulation of OHAM

Consider a general equation from Eq. (1) as follows

$$D^{\alpha}u(x) = t(x) + A(u), \quad 0\leqslant x\leqslant 1,\ 0<\alpha\leqslant 1, \tag{5}$$

where D^{α} denotes the Caputo or Riemann-Liouville fractional derivative operator, $t(x)$ is known function, A is an integral operator and $u(x)$ is unknown function to be determined.

The homotopy of OHAM [41–43], constructed as follow:

$$(1-p)\left(\frac{\partial^{\alpha}\psi(x;p)}{\partial x^{\alpha}} - t(x)\right) - \mathrm{H}(p)\left\{\frac{\partial^{\alpha}\psi(x;p)}{\partial x^{\alpha}} - A(\psi) - t(x)\right\} = 0, \quad x\in\Omega, \tag{6}$$

where $p\in[0,1]$ and $H(p) = \sum_{i\geqslant 1}p^i c_i$ is non zero auxiliary function for $p\neq 0, c_{i's}$ are auxiliary constants, the auxiliary constants increases the precision and competence of the method, when $p=0$ then$H(0)=0$. Expand $\psi(x;p)$ in Taylor's series about p, one can get

$$\psi(x;p) = u_0(x) + \sum_{i\geqslant 1}u_i(x)p^i \tag{7}$$

At p = 1, the series in Eq. (7) is observed to be convergent, therefore the approximate solution having auxiliary constants is

$$\tilde{u}(x,c_i) = u_0(x) + \sum_{i\geqslant 1}u_i(x) \tag{8}$$

Substitute Eq. (7) into the homotopy Eq. (6) and compare the coefficient of like power of p, one can obtain different order problems as follow

$$p^0:\ D^{\alpha}u_0(x) - t(x) = 0, \tag{9}$$

$$p^1:\ D^{\alpha}u_1(x) + c_1A(u_0) + (1+c_1)t(x) - (1+c_1)D^{\alpha}u_0(x) = 0, \tag{10}$$

$$p^2:\ D^{\alpha}u_2(x) + c_1A(u_1) + c_2A(u_0) + c_2(t(x) - D^{\alpha}u_0(x)) - (1+c_1)D^{\alpha}u_1(x) = 0, \tag{11}$$

$$p^3:D^{\alpha}u_3(x) + c_3A(u_0) + c_2A(u_1) + c_1A(u_2)\,c_3(t(x) - D^{\alpha}u_0(x)) - (1+c_1)D^{\alpha}u_2(x) - c_2D^{\alpha}u_1(x) = 0, \tag{12}$$

$$\begin{aligned} p^n:\quad & D^{\alpha}u_n(x) + \sum_{i=1}^{n}c_iA(u_{n-i}) + c_n\,(t(x) - D^{\alpha}u_0(x)) \\ & -(1+c_1)D^{\alpha}u_{n-1}(x) - \sum_{i=2}^{n-1}c_iD^{\alpha}u_{n-i}(x) = 0, \quad n = 4,5 \end{aligned} \tag{13}$$

Implement the fractional operator J^{α} to the series of problems from Eq. (9)–Eq. (13) to obtain a series of solutions. Using these solutions in Eq. (8), we will get OHAM solution $\tilde{u}(x,c_i)$. The approximate solution contains auxiliary constants; the optimal values of these constants are obtained through different methods. In the present work, we have used the least square method [51,52]. The method of least squares is a powerful technique for obtaining the values of auxiliary constants. By putting the optimal values of these constants in Eq. (8), we obtain the OHAM solution.

Convergence of OHAM

Theorem 1:. *As the series Eq. (8) converge to $u(x)$, where $u_m(x)\in \mathrm{L}(\mathrm{R}^+)$formed by zero order problem and mth order formation so that $u(x)$become the exact solution of Eq. (5) [49].*

Proof:. As the series

$$\sum_{m=1}^{\infty}u_{i,m}(x, C_1, C_2, C_3, \ldots, C_m), \tag{14}$$

converges, it can be written as

$$G_i(x) = \sum_{m=1}^{\infty}u_{i,m}(x, C_1, C_2, C_3, \ldots, C_m), \tag{15}$$

indicates that

$$\lim_{m\to\infty}\sum_{m=1}^{\infty}u_{i,m}(x, C_1, C_2, C_3, \ldots, C_m) = 0, \tag{16}$$

satisfies

$$u_{i,1}(x,C_1)+\sum_{m=2}^{n}u_{i,m}(x,\overrightarrow{C}_m)-\sum_{m=2}^{n}u_{i,m-1}(x,\overrightarrow{C}_{m-1})=u_{i,2}(x,\overrightarrow{C}_2)+u_{i,1}(x,\overrightarrow{C}_1)+\ldots+u_{i,n}(x,\overrightarrow{C}_n)-u_{i,n-1}(x,\overrightarrow{C}_{n-1})=u_{i,n}(x,\overrightarrow{C}_n), \tag{17}$$

by mean of Eq. (15) we have

$$u_{i,1}(x,C_1)+\sum_{m=2}^{n}u_{i,m}(x,\overrightarrow{C}_m)-\sum_{m=2}^{n}u_{i,m-1}(x,\overrightarrow{C}_{m-1})=\lim_{n\to\infty}u_{i,n}(x,\overrightarrow{C}_n) \tag{18}$$

By using linear operator L_i

$$L_i(u_{i,1}(x,C_1))+\sum_{m=2}^{\infty}L_i\left(u_{i,m}(x,\overrightarrow{C}_m)\right)-\sum_{m=2}^{n}L_i\left(u_{i,m-1}(x,\overrightarrow{C}_{m-1})\right)=L_i\left(u_{i,1}(x,\overrightarrow{C}_1)\right)+L_i\sum_{m=2}^{\infty}u_{i,m}(x,\overrightarrow{C}_m)-L_i\sum_{m=2}^{n}u_{i,m-1}(x,\overrightarrow{C}_{m-1})=0 \tag{19}$$

which satisfies

$$L_i(u_{i,1}(x,C_1))+L_i\sum_{m=2}^{\infty}\left(u_{i,m}(x,\overrightarrow{C}_m)\right)-L_i\sum_{m=2}^{\infty}\left(u_{i,m-1}(x,\overrightarrow{C}_{m-1})\right)=\sum_{m=2}^{\infty}C_j\left[L_i\left(u_{i,m-j}(x,C_{m-j})\right)+N_{i,m-j}\left(u_{i,m-1}(x,C_{m-1})\right)\right]+s_i(x)=0. \tag{20}$$

Now if $C_j, j=1,2,3,\ldots,$ is appropriately chosen then equation tends to

$L_i(u_i(x)+\beta))=0,$

which is the required exact solution.

Numerical problem

The consistency and effectiveness of the proposed method is checked by solving System of fractional order integro-differential equations of Volterra type.

Problem:. *Consider the System of Volterra type fractional order integro-differential equations*

$$D^\alpha u(x)-\frac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+2\alpha)}-\int_0^x(x-t)u(t)dt-\int_0^x(x-t)v(t)dt=0,$$
$$D^\alpha v(x)+\frac{2x^{2+3\alpha}}{2+9\alpha+9\alpha^2}+\frac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+2\alpha)}-\int_0^x(x-t)u(t)dt+\int_0^x(x-t)v(t)dt=0,$$
$$u(0)=v(0)=0,\quad 0\leqslant x\leqslant 1,\ 0<\alpha\leqslant 1, \tag{21}$$

with exact solution $u(x)x^{3\alpha}$ and $v(x)=-x^{3\alpha}$.

Using the proposed algorithm, one can obtain the following different order problems and its solutions.

Zero order problem

$$p^0:\begin{cases} D^\alpha u_0(x)-\dfrac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+2\alpha)}=0,\\ D^\alpha v_0(x)+\dfrac{2x^{2+3\alpha}}{2+9\alpha+9\alpha^2}+\dfrac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+2\alpha)}=0,\end{cases} \tag{22}$$

Its solution is

$$\begin{cases} u_0(x)=\dfrac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+3\alpha)},\\ v_0(x)=-x^{3\alpha}(1+\dfrac{2x^{2+\alpha}\Gamma(1+3\alpha)}{\Gamma(3+4\alpha)}),\end{cases} \tag{23}$$

First order problem:

$$p^1:\begin{cases} D^\alpha u_1(x)+\dfrac{c_1x^{2+3\alpha}}{2+9\alpha(1+\alpha)}-c_1x^{2+3\alpha}\left(\dfrac{1}{2+9\alpha+9\alpha^2}+\dfrac{2x^{2+\alpha}\Gamma(1+3\alpha)}{\Gamma(5+4\alpha)}\right)=0,\\ D^\alpha v_1(x)+\dfrac{2c_1x^{2+3\alpha}(x^{2+\alpha}\Gamma(3+3\alpha)+\Gamma(5+4\alpha))}{(2+9\alpha(1+\alpha))\Gamma(5+4\alpha)}=0,\end{cases} \tag{24}$$

Its solution is

$$\begin{cases} u_1(x)=\dfrac{2c_1x^{4+5\alpha}\Gamma(1+3\alpha)}{\Gamma(5+5\alpha)},\\ v_1(x)=-\dfrac{2c_1x^{2+4\alpha}\Gamma(3+3\alpha)(5x^{2+\alpha}\Gamma(5+4\alpha)+4(3+4\alpha)\Gamma(6+5\alpha))}{5(2+9\alpha+9\alpha^2)\Gamma(5+4\alpha)\Gamma(5+5\alpha)},\end{cases} \tag{25}$$

Second order problem:

$$p^2:\begin{cases} D^{\alpha}u_2(x)-\dfrac{2(c_1+2c_1^2+c_2)x^{4+4\alpha}\Gamma(1+3\alpha)}{\Gamma(5+4\alpha)}=0,\\ D^{\alpha}v_2(x)+2x^{2+3\alpha}\left(\dfrac{c_1+c_1^2+c_2}{2+9\alpha(1+\alpha)}-x^{2+\alpha}\Gamma(1+3\alpha)(\dfrac{c_1+2c_1^2+c_2}{\Gamma(5+4\alpha)}+\dfrac{2c_1^2x^{2+\alpha}}{\Gamma(7+5\alpha)})\right)=0,\end{cases} \tag{26}$$

Its solution is

$$\begin{cases} u_2(x)=\dfrac{2(c_1+2c_1^2+c_2)x^{4+5\alpha}\Gamma(1+3\alpha)}{\Gamma(5+5\alpha)},\\ v_2(x)=-2x^{2+4\alpha}\left(\dfrac{(c_1+c_1^2+c_2)\Gamma(3+3\alpha)}{(2+9\alpha+9\alpha^2)\Gamma(3+4\alpha)}+\dfrac{(c_1+2c_1^2+c_2)x^{2+\alpha}\Gamma(1+3\alpha)}{\Gamma(5+5\alpha)}+\dfrac{2c_1^2x^{4+2\alpha}\Gamma(1+3\alpha)}{\Gamma(7+6\alpha)}\right)\end{cases} \tag{27}$$

By adding zero order, first order and second order solution, we get

$$\begin{cases} u(x)=\dfrac{3x^{2\alpha}\alpha\Gamma(3\alpha)}{\Gamma(1+3\alpha)}+\dfrac{2c_1x^{4+5\alpha}\Gamma(1+3\alpha)}{\Gamma(5+5\alpha)}+\dfrac{2(c_1+2c_1^2+c_2)x^{4+5\alpha}\Gamma(1+3\alpha)}{\Gamma(5+5\alpha)},\\ v(x)=-x^{3\alpha}(1+\dfrac{2x^{2+\alpha}\Gamma(1+3\alpha)}{\Gamma(3+4\alpha)})-\dfrac{2c_1x^{2+4\alpha}\Gamma(3+3\alpha)(5x^{2+\alpha}\Gamma(5+4\alpha)+4(3+4\alpha)\Gamma(6+5\alpha))}{5(2+9\alpha+9\alpha^2)\Gamma(5+4\alpha)\Gamma(5+5\alpha)}\\ \quad -2x^{2+4\alpha}\left(\dfrac{(c_1+c_1^2+c_2)\Gamma(3+3\alpha)}{(2+9\alpha+9\alpha^2)\Gamma(3+4\alpha)}+\dfrac{(c_1+2c_1^2+c_2)x^{2+\alpha}\Gamma(1+3\alpha)}{\Gamma(5+5\alpha)}+\dfrac{2c_1^2x^{4+2\alpha}\Gamma(1+3\alpha)}{\Gamma(7+6\alpha)}\right)\end{cases} \tag{28}$$

The approximate second order solution in Eq. (28), contain auxiliary constants. Using method of least square [51,52], to get the values of these constant.

Values of constants at different value of α, for u(x).

α	c_1	c_1
0.25	−0.004522783988986472	−0.49098511692171015
0.5	−0.005687847665037152	−0.4886728325079634
0.75	0.06349273791357	−0.6330324761698597
1	0	−0.5000000000000002

Values of constants at different value of α, for v(x).

α	c_1	c_2
0.25	−0.9817053094352104	0.000053982003528413334
0.5	−0.9919400621497756	0.0000142435424524696
0.75	−0.9965795635954773	$3.356546710735383\times10^{-6}$
1	−0.9985892246701309	$7.203686513695456\times10^{-7}$

with these constants and related value of α, the approximate solution given in Eq. (28) becomes

Table 1
OHAM and Exact Solution at $\alpha = 1$.

x	Exact $u(x)$	Exact $v(x)$	OHAM $u(x)$	OHAM $v(x)$	Errors of $u(x)$	Errors of $v(x)$
0	0	0	0	0	0	0
0.1	0.001	−0.001	0.001	−0.001	1.65343×10^{-14}	4.50846×10^{-14}
0.2	0.008	−0.008	0.008	−0.008	8.46561×10^{-12}	2.84388×10^{-12}
0.3	0.027	−0.027	0.027	−0.027	3.25446×10^{-10}	3.11276×10^{-11}
0.4	0.064	−0.064	0.064	−0.064	4.33439×10^{-9}	1.61467×10^{-10}
0.5	0.125	−0.125	0.125	−0.125	3.22937×10^{-8}	5.36165×10^{-10}
0.6	0.216	−0.216	0.216	−0.216	1.66629×10^{-7}	1.27783×10^{-9}
0.7	0.343	−0.343	0.343	−0.343	6.67222×10^{-7}	2.24772×10^{-9}
0.8	0.512	−0.512	0.512	−0.512	2.21921×10^{-6}	2.77411×10^{-9}
0.9	0.729	−0.729	0.729	−0.729	6.40576×10^{-6}	2.03210×10^{-9}
1	1	−1	0.999	−1	1.65344×10^{-5}	1.99075×10^{-9}

	c_1	c_2	c_3
$u(x)$	−1.052563282048944	0.001306475904828404	−0.00030297888452953563
$v(x)$	−0.9155355959777859	−0.033242696481974185	0.005169957367184157

$$\alpha = 0.25\ :\quad \begin{cases} u(x) = x^{0.75} - 0.004971540333066124x^{5.25}, \\ v(x) = -x^{0.75} - 0.0001190730383214914x^{3.} + 0.0003566259018268417x^{5.25} \\ -0.0002524487669562700 4x^{7.5}, \end{cases} \tag{29}$$

$$\alpha = 0.5\ :\quad \begin{cases} u(x) = x^{1.5} - 0.0007103777283920224x^{6.5}, \\ v(x) = -0.9999999999999999x^{1.5} - 0.0000087743268078 20858x^{4} \\ +0.0000226983071494 8665x^{6.5} - 0.0000144179667522 43654x^{9}, \end{cases} \tag{30}$$

$$\alpha = 0.75\ :\quad \begin{cases} u(x) = x^{2.25} - 0.0001071384358656 8148x^{7.75}, \\ v(x) = -x^{2.25} - 6.396906551285021 \times 10^{-7}x^{5} \\ +0.0000014660177523 7348x^{7.75} - 8.510824872520814 \times 10^{-7}x^{10.5}, \end{cases} \tag{31}$$

$$\alpha = 1\ :\quad \begin{cases} u(x) = x^{3} - 0.0000165343915343 91542x^{9}, \\ v(x) = x^{3}(-0.9999999999999999 - 4.517759471130337\times 10^{-8}x^{3} \\ +9.314979222963254 \times 10^{-8}x^{6} - 4.996294490677074 \times 10^{-8}x^{9}). \end{cases} \tag{32}$$

Problem 2. *Consider the following System of fractional order integro-differential equations of Volterra type.*

Table 2
OHAM and Exact solution at $\alpha = 0.75$.

x	Exact $u(x)$	Exact $v(x)$	OHAM$u(x)$	OHAM $v(x)$	Errors of $u(x)$	Errors of $v(x)$
0	0	0	0	0	0	0
0.1	0.005623	−0.005623	0.005623	−0.005623	1.90522×10^{-12}	6.37086×10^{-12}
0.2	0.026749	−0.026749	0.026749	−0.026749	4.10136×10^{-10}	1.99128×10^{-10}
0.3	0.066607	−0.066607	0.066607	−0.066607	9.49806×10^{-9}	1.42724×10^{-9}
0.4	0.127243	−0.127243	0.127243	−0.127243	8.82897×10^{-8}	5.39877×10^{-9}
0.5	0.210224	−0.210224	0.210224	−0.210224	4.97694×10^{-7}	1.37679×10^{-8}
0.6	0.316840	−0.316840	0.316838	−0.316840	2.04464×10^{-6}	2.57509×10^{-8}
0.7	0.448199	−0.448199	0.448192	−0.448199	6.75235×10^{-6}	3.52318×10^{-8}
0.8	0.605275	−0.605275	0.605256	−0.605275	1.90061×10^{-5}	3.12826×10^{-8}
0.9	0.788943	−0.788943	0.788896	−0.788943	4.73505×10^{-5}	1.13406×10^{-8}
1	1	−1	0.999893	−1	1.07138×10^{-4}	2.47554×10^{-8}

$$\begin{gathered} D^{\frac{1}{2}}u(x) + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{2(\sqrt{x} + \frac{4x^{3/2}}{3})}{\sqrt{\pi}} - \int_0^x u(t)dt - \int_0^x v(t)dt = 0, \\ D^{\frac{1}{2}}v(x) + \frac{x^2}{2} - \frac{x^4}{4} - \frac{8x^{3/2}(5+6x)}{15\sqrt{\pi}} - \int_0^x u(t)dt + \int_0^x v(t)dt = 0, \\ u(0) = v(0) = 0,\ \ 0\leqslant x\leqslant 1. \end{gathered} \tag{33}$$

The exact solution of system in Eq. (33) is $u(x) = x + x^2$ and $v(x) = x^2 + x^3$.

Using the proposed algorithm of OHAM, one can get the following 3rd order approximate solution. Auxiliary constants are obtained though method of least square.
3rd order approximate solution:

$$\begin{aligned} u(x) = x + x^2(1 + \sqrt{x}(0.000176193014854796 92 + x(0.000201363445548305 34 \\ -0.000992916521469032 8\sqrt{x} + 0.0000671211485160951 7x \\ -0.000397166608587616 1x^{3/2} + 0.001232805077099195 3x^{2} \\ +0.000758649278214889 5x^{3} - 0.000925495054782162 7x^{7/2} \\ +0.000151729855642977 77x^{4} - 0.000231373763695540 67x^{9/2}))), \end{aligned}$$

$$\begin{aligned} v(x) = x^2(0.9999999999999998 - 0.0000472105502288803\sqrt{x} \\ +0.9999999999999998x + 0.0000179849715157548 84x^{5/2} \\ +0.001360970412191149 5x^{3} - 0.001898431983681742x^{7/2} \\ +0.000680485206095574 7x^{4} + 0.000233653167222368 15x^{11/2} \\ -0.000152263274924647 67x^{6} - 0.0000507544249748825 5x^{7}). \end{aligned}$$

Results and discussions

In this work, OHAM is extended to solution of SFIDE. The values of auxiliary constants for $u(x)$ and $v(x)$ have been calculated by a well-known method of least square. The results of the tested examples are presented in the form of tables and graphs. Tables 1–4 show the 2nd order OHAM and exact solution, and their absolute errors for problem 1. Table 5 show 3rd order OHAM and exact solution for problem 2. Figs. 1 and 2 are the plots of OHAM and exact solutions, assigned different values to α for problem 1. Fig. 3 show the solution plot of exact and OHAM solution for problem 2. These tables and figures clearly show the consistency and effectiveness of OHAM for solution of System of fractional order integro-differential equations of Volterra type.

Conclusions

In the present attempt, solutions of System of fractional order integro-differential equations are obtained through a consistent and reliable algorithm of OHAM. The procedure and methodology of the proposed method are simple, straight forward and fast convergence to

Table 3
OHAM and Exact solution at $\alpha = 0.5$.

x	Exact $u(x)$	Exact $v(x)$	OHAM $u(x)$	OHAM$v(x)$	Errors of $u(x)$	Errors of $v(x)$
0	0	0	0	0	0	0
0.1	0.0316228	−0.0316228	0.0316228	−0.0316228	2.24641×10^{-10}	8.70269×10^{-10}
0.2	0.089442	−0.089442	0.089442	−0.089442	2.03322×10^{-8}	1.33966×10^{-8}
0.3	0.164317	−0.164317	0.164316	−0.164317	2.83647×10^{-7}	6.22926×10^{-8}
0.4	0.252982	−0.252982	0.252980	−0.252982	1.84026×10^{-6}	1.69602×10^{-7}
0.5	0.353553	−0.353553	0.353546	−0.353554	7.84864×10^{-6}	3.25772×10^{-7}
0.6	0.464758	−0.464758	0.464732	−0.464758	2.56728×10^{-5}	4.62145×10^{-7}
0.7	0.585662	−0.585662	0.585592	−0.585662	6.99241×10^{-5}	4.54288×10^{-7}
0.8	0.715542	−0.715542	0.715375	−0.715542	1.66561×10^{-4}	2.07068×10^{-7}
0.9	0.853815	−0.853815	0.853457	−0.853815	3.58151×10^{-4}	1.01136×10^{-7}
1	1	−1	0.99929	−1	7.10378×10^{-4}	4.93986×10^{-7}

Table 4
OHAM and Exact solution at $\alpha = 0.25$.

x	Exact $u(x)$	Exact $v(x)$	OHAM $u(x)$	OHAM $v(x)$	Errors of $u(x)$	Errors of $v(x)$
0	0	0	0	0	0	0
0.1	0.177828	−0.177828	0.177828	−0.177828	2.7957×10^{-8}	1.17076×10^{-7}
0.2	0.299070	−0.299070	0.299069	−0.299071	1.06389×10^{-6}	8.77713×10^{-7}
0.3	0.405360	−0.405360	0.405351	−0.405363	8.94082×10^{-6}	2.60386×10^{-6}
0.4	0.502973	−0.502973	0.502933	−0.502978	4.04861×10^{-5}	4.97806×10^{-6}
0.5	0.594604	−0.594604	0.594473	−0.594610	1.30642×10^{-4}	6.90731×10^{-6}
0.6	0.681732	−0.681732	0.681391	−0.681738	3.40240×10^{-4}	6.78723×10^{-6}
0.7	0.765286	−0.765286	0.764521	−0.765289	7.64286×10^{-4}	3.41155×10^{-6}
0.8	0.845897	−0.845897	0.844356	−0.845895	1.54068×10^{-3}	2.20012×10^{-6}
0.9	0.924021	−0.924021	0.921162	−0.924017	2.85933×10^{-3}	3.75618×10^{-6}
1	1	−1	0.995028	−1.000010	4.97154×10^{-3}	1.48959×10^{-5}

Table 5
OHAM and Exact solution of problem 2.

x	Exact $u(x)$	Exact $v(x)$	OHAM $u(x)$	OHAM $v(x)$	$u(x)$ absolute errors	$v(x)$ absolute errors
0	0	0	0	0	0	0
0.1	0.11	0.011	0.11000	0.01099	5.23755×10^{-7}	1.40431×10^{-7}
0.2	0.24	0.048	0.24000	0.04799	2.39109×10^{-6}	6.23357×10^{-7}
0.3	0.39	0.117	0.39000	0.11699	4.69768×10^{-6}	9.53971×10^{-7}
0.4	0.56	0.224	0.56000	0.22400	6.01984×10^{-6}	7.15477×10^{-8}
0.5	0.75	0.375	0.75000	0.37500	5.76996×10^{-6}	4.25905×10^{-6}
0.6	0.96	0.576	0.96000	0.57601	5.16376×10^{-6}	1.38683×10^{-5}
0.7	1.19	0.833	1.19001	0.83303	7.27850×10^{-6}	3.13758×10^{-5}
0.8	1.44	1.152	1.44002	1.15206	1.55702×10^{-5}	5.89781×10^{-5}
0.9	1.71	1.539	1.71003	1.53910	3.01342×10^{-5}	9.75276×10^{-5}
1	2	2	2.00004	2.00014	4.09099×10^{-5}	1.44434×10^{-4}

the exact solution. The presented technique does not require discretization like other numerical methods and has no need for the small or large parameter on the problem unlike other perturbation methods. Another merit of the proposed method is that the convergent region is optimally controlled. The accuracy of the presented technique is a point of interest, and a valid reason for researchers to implement this technique for the different systems of nonlinear problems of fractional order arising in different field of science and technology.

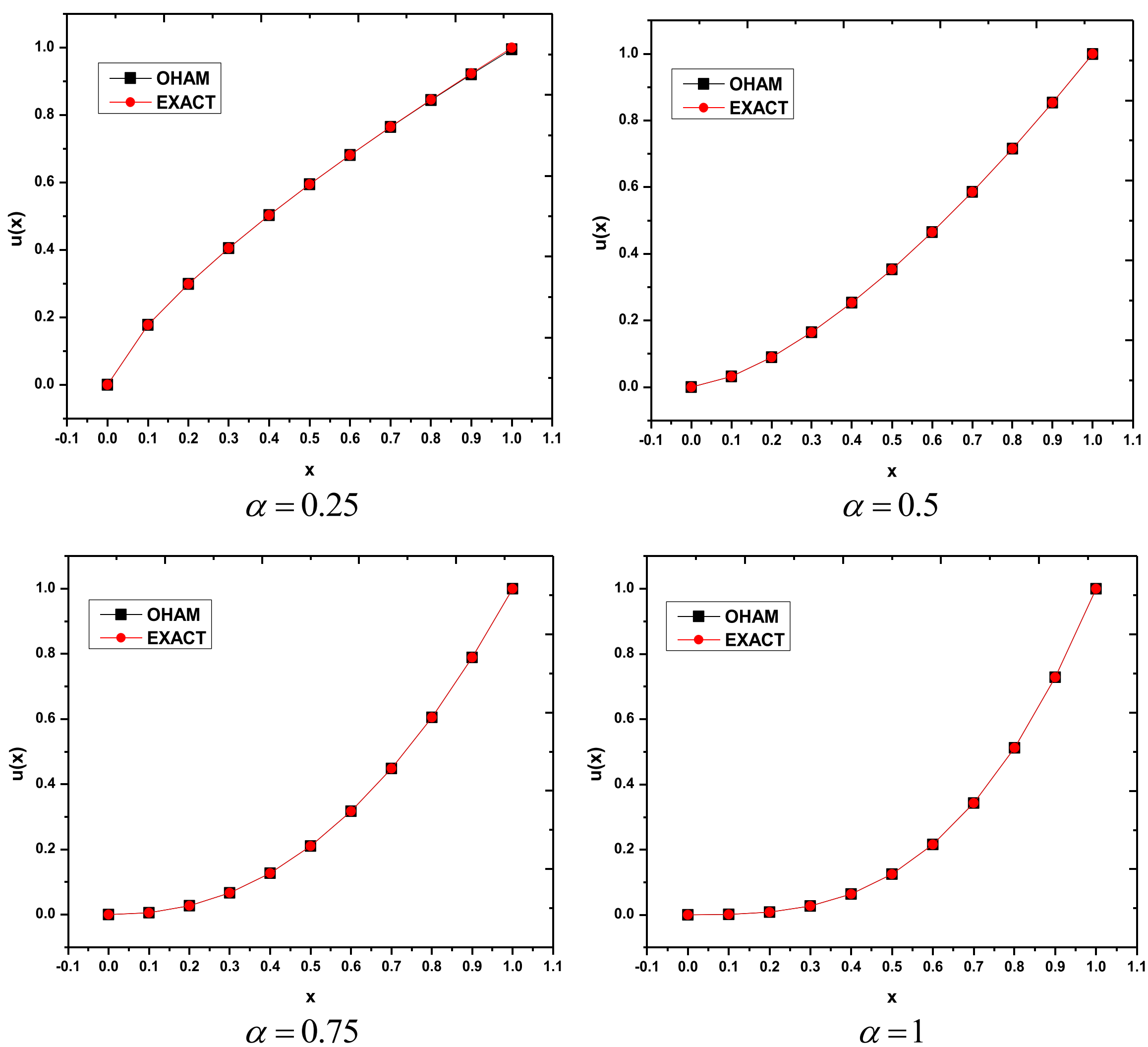

Fig. 1. Solution plots of exact and OHAM, for $u(x)$.

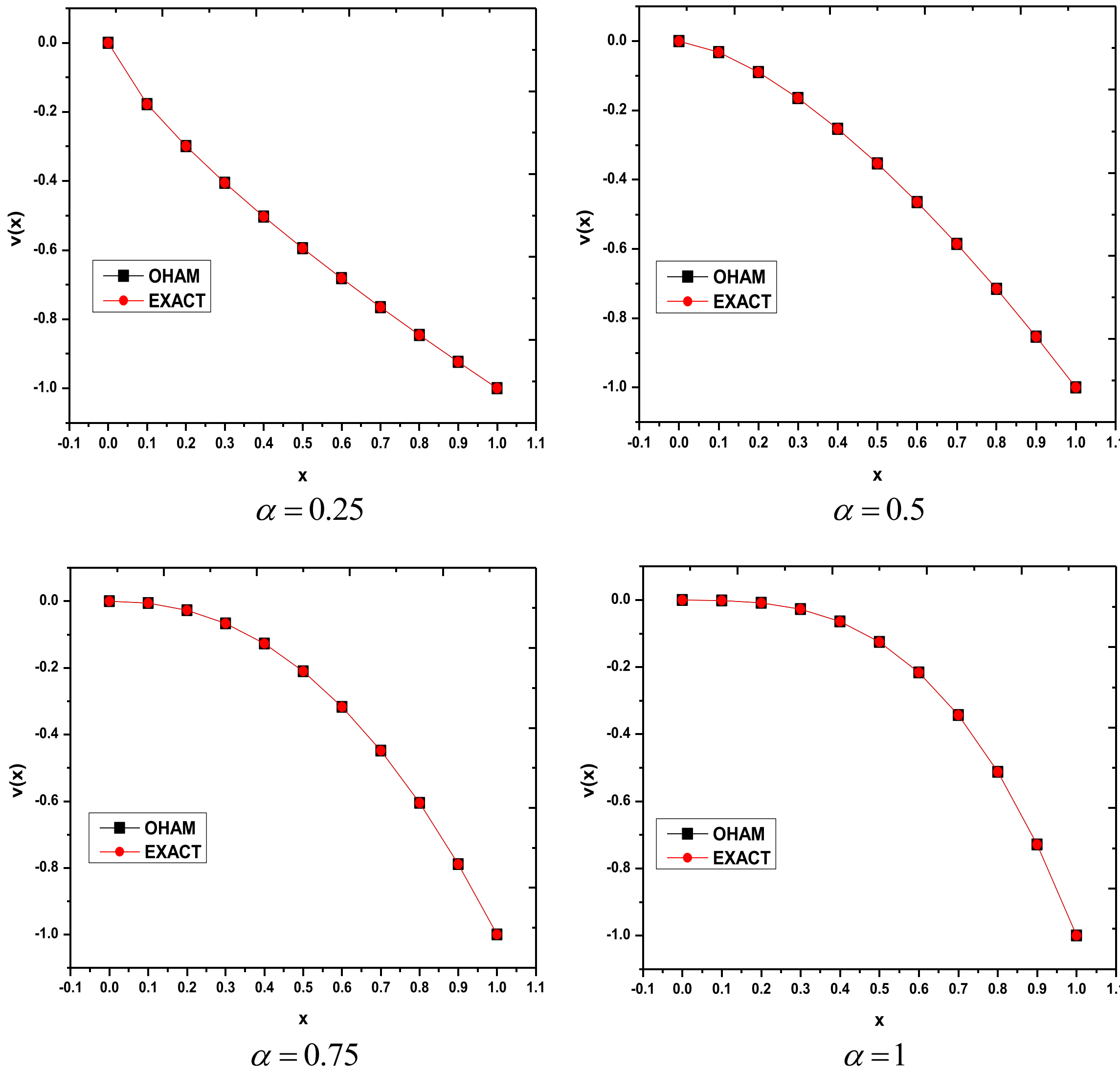

Fig. 2. Solution plots of exact and OHAM, for $v(x)$.

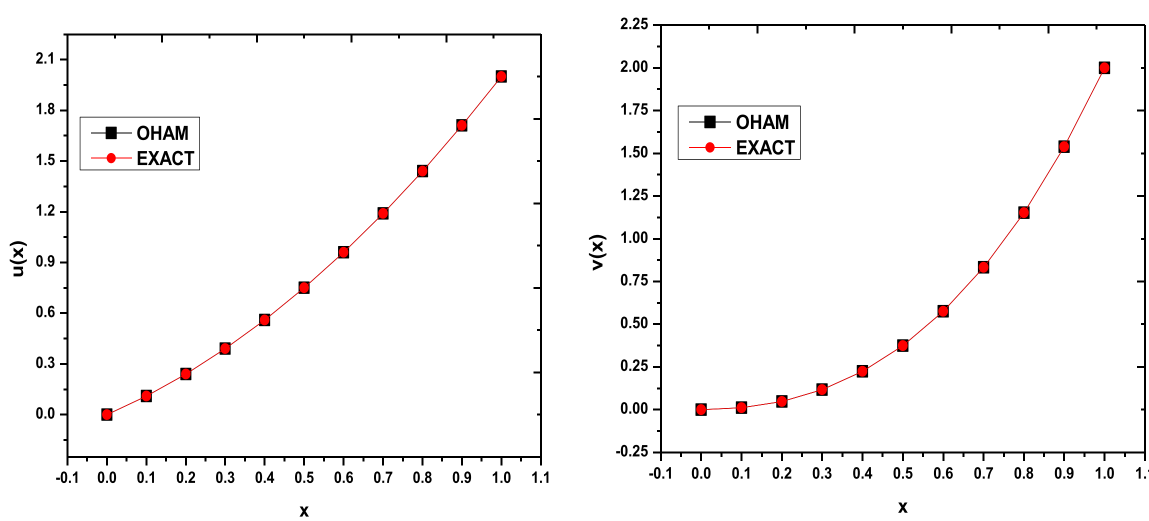

Fig. 3. Solution plot of exact and OHAM for problem 2.

CRediT authorship contribution statement

Muhammad Akbar: Conceptualization, Formal analysis, Investigation, Software, Supervision, Validation, Visualization, Writing - original draft, Writing - review & editing. **Rashid Nawaz:** Conceptualization, Formal analysis, Investigation, Software, Supervision, Visualization, Writing - original draft, Writing - review & editing. **Sumbal Ahsan:** Conceptualization, Formal analysis, Investigation, Software, Supervision, Visualization, Writing - original draft, Writing - review & editing. **Kottakkaran Sooppy Nisar:** Writing - original draft, Methodology, Project administration, Validation, Visualization, Writing - review & editing. **Abdel-Haleem Abdel-Aty:** Data curation, Formal analysis, Funding acquisition, Methodology, Project administration, Resources, Visualization, Writing - review & editing. **Hichem Eleuch:** Formal analysis, Investigation, Resources, Writing - review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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