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Review on nonlocal continuum mechanics: Physics, material applicability, and mathematics

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ABSTRACT

The classical continuum mechanics assumes that a material is a composition of an infinite number of particles each of which is a point that can only move and interact with its nearest neighbors. This classical mechanics has limited applications where it fails to describe the discrete structure of the material or to reveal many of the microscopic phenomena, e.g., micro-deformation and micro-dislocation. This observation motivated the need for a general point of view that instills the fact that the material particle is a volume element that would deform and rotate, and the material is generally a multiscale material. In addition, the particle's equilibrium should not be considered in isolation from its nonlocal interactions with other particles of the material. Material models with these features are the nonlocal microcontinuum theories.

Whereas review articles and books on microcontinuum theories and nonlocal mechanics would be found in the literature, no review that extensively deals with the fundamentals of nonlocal mechanics from the physics, material, and mathematical points of view has been presented so far. There is a current scientific debate on the benefits of applying nonlocal theories to various fields of mechanics. This is due to a lack of understanding of the physics behind these theories. In addition, questions on the applicability of nonlocal mechanics for various materials are not answered yet. Furthermore, mathematicians revealed paradoxes and complications of finding solutions of nonlocal field problems. In this review, we shed light on these folders. We give extensive interpretations on the physics of nonlocal mechanics of particles and elastic continua, and the applicability of nonlocal mechanics to multiscale materials and single-scale materials is interpreted. In addition, the existing complications of solving nonlocal field problems, and the various methods and approaches to overcome these complications are collected and discussed from the physical and material points of view. Furthermore, we define the open forums that would be considered in future studies on nonlocal mechanics.

1. Introduction

Continuum mechanics is the most practical approach to describe the mechanics of various types of materials. In the classical approach of continuum mechanics, a material is composed of a set of idealized-infinitesimal material particles, each of which is a mass point that only interacts with its neighbor particles. The stress field in the classical continuum mechanics is expressible based on Noll's local constitutive axioms of deformations of simple materials (Truesdell, 1977). Thus, the stress at a point depends on the motions of the points of the continuum (or the deformation field that generates due to these motions), which are located within a finite distance from that point. This finite distance depends on the local material properties of the continuum. Nevertheless, the existence of long-range interactions between non-neighbor particles was reported in various theoretical and experimental studies, and the effects of these interactions on the dispersion of elastic waves and mechanical properties of materials have been extensively documented (e.g., Bouchet et al., 2010; Di Paola et al., 2010; Zingales, 2011). In addition, both natural and man-made materials have complicated microstructures with exceptional micro-and-nano-scale phenomena. The classical continuum mechanics fails to describe physical phenomena in which the long-range interactions play a major role. In addition, it cannot observe many of the micro-/nano-scale phenomena, especially the ones that require breaking down the material into multiple scales. In other words, the properties and behaviors of materials captured by the classical continuum mechanics are invariant with respect to time and length scales, and more notably size effects cannot be captured by this classical mechanics (Hütter, 2017). It has shown a

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clear failure to capture microscopic phenomena such as micro-rotation, micro-deformation, micro-dislocation, or micro-twinning (Mindlin, 1964; Eringen, 2002; Neff et al., 2014; Shaat and Abdelkefi, 2016a). Because of the aforementioned limitations of the classical continuum mechanics, various types of generalized models of continuum mechanics – such as micromorphic, micropolar, nonlocal, and high-order strain/rotation gradient mechanics – have been developed. These generalized models reflected effects of long-range interactions on the material mechanics, captured many of the micro-/nano-scale phenomena and the material dispersion properties (Kunin, 1984; Shaat, 2019), and modeled the realistic microstructures of materials (Kunin, 1968, 1982; 1983; Maugin, 1993; Capriz, 1989; Gurtin and Podio-Guidugli, 1992).

Advanced theories of continuum mechanics – dated back to Voigt (1887) – in which the material is modeled as a colony of independently moving and/or deforming particles have been developed. These theories are now called *microcontinuum field theories* (Eringen, 1999, 2002). Voigt (1887) early studied the elasticity and piezoelectricity of crystals in consideration of the crystallographic-rotation of polar molecules, which were considered rigid objects in a crystal structure. Subsequently, the Cosserat theory (Cosserat and Cosserat, 1909) was developed assuming that the material is a composition of rigid particles, and the particles exhibit independent rotations. In the Cosserat theory, the continuum is subjected to surface and body couples, which are independent of the surface and body forces. This theory was then further extended by Gunther (1958), Grioli (1960), Aero and Kuvshinskii (1961), and Schaefer (1962). Eringen (1966, 1968a, 1969) developed the micropolar theory and the microstretch theory for linear and nonlinear elasticities. The micropolar theory is a generalized version of the Cosserat theory, which accounts for both static and dynamic effects of the micro-rotation fields. In 1964, Eringen (1964a,b) developed the micromorphic theory. In the context of the micromorphic theory, the material is a composition of a large number of material particles, and each one of these particles can move, deform, and rotate. The micromorphic theory is a generalized model of continua, which can be simplified to the micropolar/Cosserat theory and microstretch theory (Eringen, 1968b). Recently, various simplified versions of the general micromorphic theory have been developed (Neff et al., 2014; Romano et al., 2016; Barbagallo et al., 2017; Shaat, 2018a). A fundamental feature of the microcontinuum theories is the formulation of the kinematical variables considering independent microstructures. This resulted in special kinematical variables such as the gradients of the micro-deformation fields and the coupling variables that measure the difference between the micro- and macro-scale fields. Because of these special kinematical variables, the microcontinuum theories are capable of describing the physical microstructures of materials (Capriz, 1989; Gurtin and Podio-Guidugli, 1992), exploring the material dispersion characteristics (Kunin, 1968, 1982; 1983, 1984; Trovalusci and Pau, 2014; Shaat, 2019), and the multiscale modeling of materials (Trovalusci, 2014; Shaat, 2018a).

Nonlocal mechanics may be viewed as a general material model. The microcontinuum theories, e.g., micromorphic, micropolar and microstretch theories, reveal nonlocality (Kunin, 1984; Maugin, 1993). Sources of the nonlocality in these theories are the gradients of the micro-deformations and the kinematical variables of the coupling between the different degrees of freedom. This nonlocality can be understood by realizing the ability of these theories to describe the spatial dispersion of the elastic waves (Kunin, 1984; Shaat, 2019). Thus, the nonlocal fields are implicitly modeled by these theories (Kunin, 1984; Maugin, 2017; Tuna and Trovalusci, 2020). However, other theories represented the nonlocal fields using convolutions of the kinetical and kinematical variables. These convolutions model the long-range effects of the material's kinematical degrees of freedom. Despite all microcontinuum theories reveal nonlocal fields implicitly, the continuum theories that utilize convolution-type constitutive equations of the nonlocal fields are commonly named in the literature 'nonlocal theories'. This has also led to the classification of the nonlocal theories into 'explicit' and 'implicit' nonlocal theories (Kunin, 1968, 1982; 1983, 1984; Tuna and Trovalusci, 2020; Polizzotto, 2001). Nonetheless, a classification of the nonlocal theories is not abstractive, as many factors should be considered. For example, the nonlocal field due the gradient and the coupling kinematical variables may be different than the nonlocal field that is modeled by the convolutions of the kinetical and kinematical variables.

The first nonlocal continuum model was developed by Kröner (1967) in which the long-range effect of the cohesive forces was modeled. Then, Eringen and co-workers (Eringen and Edelen, 1972; Eringen, 1972) developed the current version of nonlocal mechanics. They derived attenuation functions that describe the decay of the long-range interactions between two particles with the distance between them. These functions gave the constitutive equations and the balance equations of the continuum theory depend on integral functionals of the kinematical variables. In the early versions of the nonlocal theory, it resembled the classical mechanics theory in modeling the particles of the material as mass points that exhibit only translational motions. Subsequently, the *nonlocal microcontinuum mechanics* was developed as a more general version of nonlocal mechanics (Eringen, 1999, 2002). A unified approach of nonlocal field theories for elastic solids, viscous fluids, electromagnetic solids and fluids, memory-dependent elastic solids, and media with microstructure was developed and presented in a book by Eringen (2002).

Various models of nonlocal mechanics were developed. Polizzotto (2001) reformulated Eringen's nonlocal elasticity for elastic continua with discontinuities by deriving attenuation functions that depend on the geodetical path of the nonlocal field. A nonlocal elasticity model that depended on the strain-difference of two distinct points in the continuum was represented by Polizzotto et al. (2004). Eringen (2006) formulated a nonlocal model in which the balance equations are formed without nonlocal residuals, and singularities and discontinuities are smoothed out. These forms of nonlocal mechanics were effectively used to model elastic continua with non-smooth deformation phenomena (Lazopoulos, 2006, 2016) and peridynamic continua (Silling et al., 2003; Silling and Lehoucq, 2010). In 2000, the peridynamics nonlocal theory was introduced by Silling (2000) as a computational tool of nonlocal mechanics. In addition, applications of the fractional calculus into the nonlocal continuum mechanics were early introduced by Gubenko (1957), Rostovtsev (1959) and recently by Lazopoulos (2006), Di Paola et al. (2009a,b), Cottone et al. (2009a,b), Drapaca and Sivaloganathan (2012), Challamel et al. (2013), Atanackovic et al. (2014a,b), Tarasov (2014), and Sumelka and Blaszczyk (2014). Fractional calculus is an effective mathematical approach that can describe physical phenomena in continuous matter using fractional derivatives with nonlocal influence (Oldham and Spanier, 1974; Poldubny, 1999).

Although several review articles (Arash and Wang, 2012; Rafii-Tabar et al., 2016; Askari et al., 2017; Behera and Chakraverty, 2017; Srinivasa and Reddy, 2017) and books (Gopalakrishnan and Narendar, 2013; Karličić et al., 2015; Ghavanloo et al., 2019) on the theoretical foundation and applications of nonlocal mechanics would be found in the literature, reviews that extensively deal with the fundamentals of nonlocal mechanics from the physics, material, and mathematics points of view have not been presented so far. Whereas theories of nonlocal mechanics have great potential to model advanced materials, the current implementations of these theories indicate a scientific debate on their benefits to various fields of mechanics. Many of the current implementations of nonlocal mechanics lack the crucial understanding of the physics behind these theories. In addition, the phenomena that would be captured by these theories and their applicability to various materials are still open. One may observe in the literature that the mechanics of the same material has been studied using different theories. Values were, as well, assigned to various material parameters defined by the theories of nonlocal mechanics with neither clarifications nor connections to the investigated materials. In addition, an extensive debate on the mathematics of nonlocal mechanics is currently found in the literature. The mathematical considerations of nonlocal mechanics have explained the paradoxes and complications of finding solutions of nonlocal field problems. Nonetheless, in the recent years, the mathematics of nonlocal mechanics in isolation from its physics have led to many questionable interpretations of nonlocal mechanics. In this review, we discuss new insights into the aforementioned folders by giving extensive interpretations on the physics behind the various theories of nonlocal mechanics, the material applicability of each one of these theories, and the mathematics of nonlocal mechanics. Nonlocal mechanics of particles and nonlocal mechanics of elastic continua are discussed. In addition, the applicability of various theories of nonlocal mechanics to single-scale and multiscale materials is examined. Moreover, the existing complications of solving nonlocal field problems are reviewed. The various efforts to overcome these complications are also collected and discussed from the physical and material points of view. Aspects of the thermodynamics and the variational principles of theories of nonlocal mechanics are, however, beyond the scope of this review. These aspects have been fully established in early studies of nonlocal mechanics and have been well interpreted by Edelen and Laws (1971), Eringen (1999, 2002), and Polizzotto (2003a,b).

The physics of nonlocal mechanics is discussed in Sections 2 and 3. Then, the nonlocal mechanics of multiscale materials and single-scale materials is discussed in Sections 4 and 5. The different micro-continuum theories are reviewed in Section 4. The essential equations of these theories are presented in the context of nonlocal mechanics. In addition, their applicability to multiscale materials is discussed. The nonlocal theory and its reduction to the strain gradient and rotation gradient theories are presented in Section 5. In addition, the applicability of nonlocal theories to single-scale materials is interpreted. The mathematical aspects of nonlocal mechanics are reviewed and discussed in Section 6. The solutions of nonlocal field equations for unbounded and bounded domains are reviewed and compared explaining the complications and limitations of the nonlocal models. Finally, the open forums and the future prospects of research on nonlocal mechanics are defined and presented in Sections 7 and 8.

2. Nonlocal mechanics of particles

One way to understand the mechanics of a continuous matter is to study first the mechanics of the discrete system of particles that form this matter. Then, by averaging procedures that postulate the representation of macroscopic fields by the volume-average of appropriate microscopic quantities of a material particle, the passage from the discrete system of particles to a continuum model can be established (Murdoch, 1985). This is needed specially to understand the nonlocal mechanics of continuous matter. Nonlocal mechanics stems from a generalized lattice model with neighbor and non-neighbor interactions. Early models of nonlocal mechanics depended on representative lattices with short- and long-range interactions. For example, Gazis and Wallis (1965) solved a model of a one-dimensional lattice of semi-infinite length with direct and indirect interactions. Ishii (1968) investigated the effects of the long-range interactions on the vibrational properties of monatomic lattices. Eringen and Kim (1977) modeled an atomic lattice to formulate a strain-based nonlocal continuum model. Rosenau (1987) studied a generalized lattice with an interaction that includes two closest neighbors. Recently, lattice models with long-range effects were used to explain the nonlocal and strain gradient theories. For instance, Tarasov (2015) reformulated the strain gradient theory as a discrete lattice with nearest-neighbor and next-neighbor interactions. Porubov et al. (2018) studied a square-two-dimensional lattice with different orders of nonlocality. Wang et al. (2018, 2019) studied the wave propagation in one- and two-dimensional lattices with long-range effect. Challamel et al. (2018, 2019) proposed generalized lattice models with neighbor and non-neighbor interactions. The majority of the lattice models of nonlocal mechanics in the literature depended on monoatomic lattices with atoms/particles represented as mass points under neighbor and non-neighbor interactions. In this section, we study the nonlocal mechanics of a generalized discrete system of particles. In the context of this model, each particle is a volume element that undergoes interactions with its direct neighbors and non-neighbors.

Consider a discrete system of particles, as shown in Fig. 1(a). According to the classical theory of mechanics, each one of these particles is a point that undergoes interactions with its direct-neighbor particles. Each one of these direct interactions forms a local force field. From the lattice mechanics, the local force, $\mathbf{F}$, between two neighbor particles (n and m) can be defined as follows:

$$\mathbf{F}_{nm} = \mathbf{K}\cdot(\mathbf{u}_n - \mathbf{u}_m) \quad (1)$$

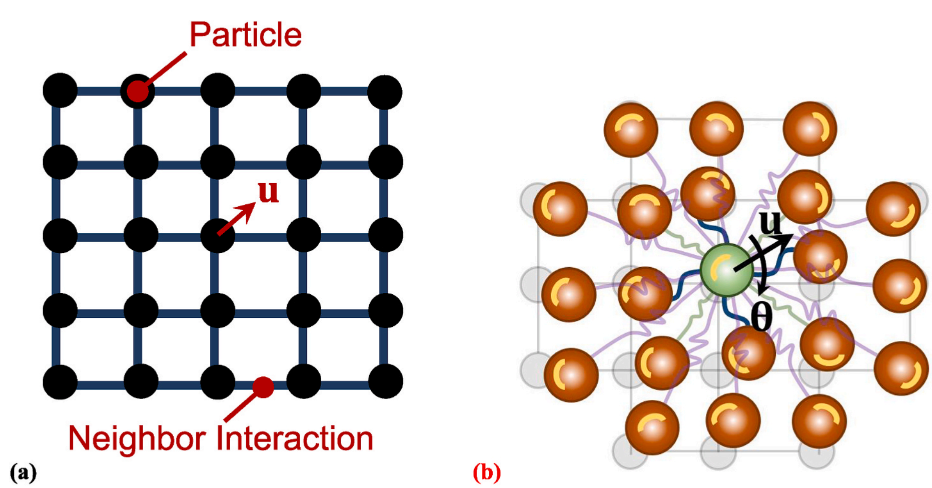

Fig. 1. Classical mechanics versus nonlocal mechanics. (a) Classical mechanics: A discrete system of particles each of which exhibits a displacement **u**. Each particle interacts only with its neighbor particles. No non-neighbor interactions are generated. (b) Nonlocal mechanics: A discrete system of hard-sphere particles each of which exhibits a displacement **u** and a rotation θ. The interactions of the central particle with all other particles of the system are represented. In addition to direct neighbor interactions, non-neighbor interactions are generated (Colors are used to distinct the different levels of interactions of the central particle with other particles). (For interpretation of the references to colour in this figure legend, the reader is referred to the Web version of this article.)

where $\mathbf{K}$ is an equivalent stiffness of an interaction force between two particles, and $\mathbf{u}$ is the displacement field expressing the translational motion of the particle.

Under the influence of neighbor interactions, the condition of the dynamic equilibrium of a particle (n) secretes the following balance equations (Shaat, 2015a, 2017):

$$\sum_{m=1}^{N_m} \mathbf{F}_{nm} = \bar{m}\ddot{u}$$
$$\sum_{m=1}^{N_m} (\mathbf{X} \times \mathbf{F}_{nm}) = \mathbf{X} \times \bar{m}\ddot{\mathbf{u}} \tag{2}$$

where $\mathbf{X} \times \mathbf{F}$ denotes a moment field that is generated due to the couple of force acting on the particle. N_m is the number of direct-neighbors to particle (n), and $\bar{m}$ is the particle's mass. The dot denotes a time derivative.

A more generalized model of the considered discrete system of particles can be developed by replacing the particles with their hard-sphere counterparts, as shown in Fig. 1(b). Each particle is a volume element that can rotate and move, and it undergoes interactions with all the particles of the system. In addition to neighbor interactions, the particle would be put in an interaction with a non-neighbor particle, and a non-neighbor interaction field would be generated. The non-neighbor interaction is usually weak, but – in some occasions – it would strongly contribute to the particle's mechanics, and therefore it should be considered.

The interactions of the hard-sphere model in Fig. 1(b) are non-traditional. These interactions generate force and couple fields. The force field due to a direct-neighbor interaction between two particles n and m is as defined in equation (1). In addition to this local force field, a local couple field is generated, and it can be written in the form:

$$\mathbf{M}_{nm} = \tilde{\mathbf{K}}\cdot(\theta_n - \theta_m) \tag{3}$$

where $\tilde{\mathbf{K}}$ is an equivalent moment-stiffness of the interaction couple between two particles n and m. θ is the rotation field expressing the rotations of the particle about its center point.

In addition to the local force ($\mathbf{F}$) and the local couple ($\mathbf{M}$), non-neighbor force ($\mathcal{F}$) and non-neighbor couple ($\mathcal{M}$) that depend on the distance between two non-neighbor particles are generated. The non-neighbor force between two non-neighbor particles (n and j) is a functional of the distance between them $|\mathbf{x}_n - \mathbf{x}_j|$, as follows:

$$\mathcal{F}_{nj} = \mathbf{k}_r(|\mathbf{x}_n - \mathbf{x}_j|)\cdot(\mathbf{u}_n - \mathbf{u}_j) \tag{4}$$

where $\mathbf{k}_r(|\mathbf{x}_n - \mathbf{x}_j|)$ is a stiffness that is equivalent to the interaction force between two non-neighbor particles. The non-neighbor couple between two non-neighbor particles (n and j) is also a functional of the distance between them $|\mathbf{x}_n - \mathbf{x}_j|$, as follows:

$$\mathcal{M}_{nj} = \tilde{\mathbf{k}}_r(|\mathbf{x}_n - \mathbf{x}_j|)\cdot(\theta_n - \theta_j) \tag{5}$$

where $\tilde{\mathbf{k}}_r(|\mathbf{x}_n - \mathbf{x}_j|)$ is a moment-stiffness that is equivalent to the interaction moment between two non-neighbor particles.

The concept of nonlocal mechanics has been developed to model the particle equilibrium under the influence of neighbor and non-neighbor interactions. Therefore, the equilibrium of the system of hard-sphere particles in Fig. 1(b) can be expressed, as follows:

$$\sum_{m=1}^{N_m} \mathbf{F}_{nm} + \sum_{j=1}^{N_j} \mathcal{F}_{nj} = \bar{m}\ddot{\mathbf{u}}$$
$$\sum_{m=1}^{N_m} (\mathbf{X} \times \mathbf{F}_{nm}) + \sum_{j=1}^{N_j} (\mathbf{X} \times \mathcal{F}_{nj}) + \sum_{m=1}^{N_m} \mathbf{M}_{nm} + \sum_{j=1}^{N_j} \mathcal{M}_{nj} = (\mathbf{X} \times \bar{m}\ddot{\mathbf{u}}) + \bar{J}\ddot{\theta} \tag{6}$$

where $\bar{J}$ is the rotatry-inertia of the particle.

3. Nonlocal mechanics of elastic continua

The nonlocal model of the discrete system of particles presented in Section 2 is extended here to a generalized model of nonlocal mechanics of elastic continua. Therefore, consider a continuum occupies a volume V and is bounded by a surface S that consists of a large number of hard-sphere particles. The dynamic equilibrium of the system of particles that form the continuum can be written in the following form based on equation (6), as follows:

$$\int_V \mathbf{F}(\mathbf{x})dV(\mathbf{x}) + \int_V \int_V \mathcal{F}(\mathbf{x}, \mathbf{x}')dV(\mathbf{x}')dV(\mathbf{x}) + \int_S \mathbf{T}dS = \int_V \bar{m}\ddot{\mathbf{u}}(\mathbf{x})dV(\mathbf{x}) \tag{7}$$

$$\int_V ([\mathbf{X} \times \mathbf{F}(\mathbf{x})] + \mathbf{M}(\mathbf{x}))dV(\mathbf{x}) + \int_V \int_V ([\mathbf{X} \times \mathcal{F}(\mathbf{x}, \mathbf{x}')] + \mathcal{M}(\mathbf{x}, \mathbf{x}'))dV(\mathbf{x}')dV(\mathbf{x}) + \int_S ([\mathbf{X} \times \mathbf{T}] + \mathbf{C})dS = \int_V ([\mathbf{X} \times \bar{m}\ddot{\mathbf{u}}(\mathbf{x})] + \bar{J}\ddot{\theta}(\mathbf{x}))dV(\mathbf{x}) \tag{8}$$

where $\mathbf{F}$ is a vector of the local body forces, which locally act on the material particle at point $\mathbf{x}$. These local forces would include the local interactions between the material particle and its neighbor particles and any other local forces (including gravitational and external forces) that act on the particle. In addition to the local body forces, $\int_{\mathcal{V}} \mathcal{F}(\mathbf{x}, \mathbf{x}')dV(\mathbf{x}')$ is a vector of the nonlocal residual body forces. The nonlocal body force is mainly due to the interaction that is generated between the material particle at point $\mathbf{x}$ and a non-neighbor particle located at point $\mathbf{x}'$. Similarly, $\mathbf{M}$ is a vector of the local body couple, and $\int_V ([\mathbf{X} \times \mathcal{F}(\mathbf{x}, \mathbf{x}')] + \mathcal{M}(\mathbf{x}, \mathbf{x}'))dV(\mathbf{x}')$ is a vector of the nonlocal residual body couple. $\mathbf{T}dS$ is an external surface traction force, and $\mathbf{C}dS$ is an external surface traction couple. The external surface tractions ($\mathbf{T}$ and $\mathbf{C}$) are balanced with the body forces and couples and the inertia of the particles of the elastic continuum according to equations (7) and (8). According to the considered nonlocal model, each of the surface tractions $\mathbf{T}$ and $\mathbf{C}$ can be decomposed into two tractions, as follows:

$$\mathbf{T} = \mathbf{n}\cdot\boldsymbol{\sigma} + \mathbf{n}\cdot\boldsymbol{\sigma}_r \tag{9}$$

$$\mathbf{C} = \mathbf{n}\cdot\boldsymbol{\mu} + \mathbf{n}\cdot\boldsymbol{\mu}_r \tag{10}$$

where $\mathbf{n}$ is the unit normal vector. $\boldsymbol{\sigma}$ is a local force-stress tensor, and $\boldsymbol{\sigma}_r$ is a nonlocal residual force-stress tensor. In addition, $\boldsymbol{\mu}$ is a local couple-stress tensor, and $\boldsymbol{\mu}_r$ is a residual couple-stress tensor. The local stresses ($\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$) are measures related to the direct-neighbor interactions while the nonlocal residual stresses ($\boldsymbol{\sigma}_r$ and $\boldsymbol{\mu}_r$) are measures conjugate to the non-neighbor interactions.

By the substitution of equations (9) and (10) into equations (7) and (8) and by applying the divergence theorem, equations (7) and (8) can be rewritten as follows:

$$\nabla\cdot\boldsymbol{\sigma}(\mathbf{x}) + \nabla\cdot\boldsymbol{\sigma}_r(\mathbf{x}) + \mathbf{F}(\mathbf{x}) + \int_V \mathcal{F}(\mathbf{x}, \mathbf{x}')dV(\mathbf{x}') = \rho\ddot{\mathbf{u}}(\mathbf{x}) \tag{11}$$

$$\nabla\cdot\boldsymbol{\mu}(\mathbf{x}) + \nabla\cdot\boldsymbol{\mu}_r(\mathbf{x}) - \hat{\boldsymbol{\sigma}}(\mathbf{x}) - \hat{\boldsymbol{\sigma}}_r(\mathbf{x}) + \mathbf{M}(\mathbf{x}) + \int_V ([\mathbf{X} \times \mathcal{F}(\mathbf{x}, \mathbf{x}')] + \mathcal{M}(\mathbf{x}, \mathbf{x}'))dV(\mathbf{x}') = \rho_m J\ddot{\theta}(\mathbf{x}) \tag{12}$$

where ρ is a macroscopic mass density while ρ_m denotes the mass density of the material particle, and J is a micro-inertia. $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}_r$ are local stress-vector and nonlocal stress-vector, respectively. The components of these vectors are defined as $\hat{\sigma}_i = \varepsilon_{ijk}\sigma_{jk}$ and $\hat{\sigma}_{ri} = \varepsilon_{ijk}\sigma_{r\,jk}$ where σ_{jk} and $\sigma_{r\,jk}$ are the components of the local stress tensor $\boldsymbol{\sigma}$ and the nonlocal residual stress tensor $\boldsymbol{\sigma}_r$, respectively.

Equations (11) and (12) represent a model of elastic continua in the context of nonlocal mechanics. This model assumes the continuum a composition of rigid particles each of which exhibits 3-displacements and 3-rotations as kinematical degrees of freedom. Eliminating all nonlocal residuals from equations (11) and (12), the model reduces to the micropolar model (Eringen, 1999, 2002). In addition, the model (equation (11) and (12)) gives the nonlocal micropolar model (Eringen, 1999, 2002) by merging the local and the nonlocal residual fields into total nonlocal fields, i.e., $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\sigma}_r$ and $\mathbf{m} = \boldsymbol{\mu} + \boldsymbol{\mu}_r$ where $\boldsymbol{\tau}$ and $\mathbf{m}$ are nonlocal force-stress and couple-stress tensors, respectively. A more general model of nonlocal continuum mechanics is the one that considers the particles' deformation (i.e., strain) along with their rotational and translational motions. This model is the *nonlocal micromorphic theory* (Eringen, 1999, 2002).

The models presented in Sections 2 and 3 explained the physics of the *nonlocal microcontinuum theories*. In Sections 4 and 5, these theories are reviewed, and their applicability to multiscale and single-scale materials are discussed.

4. Nonlocal mechanics of multiscale materials

4.1. Multiscale materials

Multiscale materials can be defined as materials that require a multiscale modeling approach to capture one or more of their physical phenomena, which occur at different length and/or time scales (Nieminen, 2002; Liu et al., 2010; Elliott, 2011; Shaat, 2018a). Materials – in general – exhibit and secrete different phenomena at the different length and time scales. Scale-dependent phenomena are nontraditional phenomena, which cannot be observed using the classical mechanics of materials. The classical mechanics cannot observe the discrete structure dependence of the material properties (Chen and Lipton, 2013; Shaat, 2018a). In addition, the material mechanics in relation to the material's microstructural features (e.g., micro-slipping and twinning), geometries, and/or topologies (e.g., bandgap) are unpredictable using the classical mechanics. This is because it lacks measures and length-scales, which are needed to capture the scale-dependent phenomena of materials. To observe two or more of these phenomena, a multiscale modeling approach is required. In this approach, the material is modeled as a multiscale material whose mechanics and properties depend on various length and time scales (Capecchi et al., 2011). Hierarchical multiscale modeling (Nieminen, 2002; Berendsen, 2005; Bouvard et al., 2009) and hybrid multiscale modeling (Ogata et al., 2001; Miller and Tadmor, 2007, 2009; Bernstein et al., 2009) are the two main approaches of multiscale modeling. In a hierarchical multiscale model, the material is broken down into multiple scales, and models for the different scales are run independently. To couple the independently working models, coupling parameters are usually introduced. However, a hybrid multiscale model works as a single package where models of the different scales are run concurrently (Elliott, 2011).

Models used for multiscale modeling range from quantum mechanics to continuum mechanics (Bouvard et al., 2009; Liu et al., 2010). Beyond the quantum mechanics level, atomistic and molecular dynamics models are used to capture the nanoscale phenomena of multiscale materials. These models depend on empirical interatomic potentials (O'connor and Biersack, 1986) or quantum-molecular mechanics hybrid potentials (Bernstein et al., 2009). At microscales (0.1–10μm), molecular dynamics models are computationally expensive, and, therefore, a mesoscale model can be used. In mesoscale modeling, redundant degrees of freedom are eliminated by grouping the nanostructures of the material into a microstructure, e.g., atoms into particles (coarse graining) (Elliott, 2011). At the microscale, micromechanics models can be used as well. In these models, representative volume elements (RVEs) are defined for the different scales to represent the hierarchical structure of the multiscale material. At the macroscale, continuum mechanics is usually used. Approaches of molecular-continuum mechanics for multiscale materials have been early proposed (Voigt, 1887; Poincaré, 1892). These approaches can be dated back to Voigt and Poincaré whose work on molecular theory of elasticity put the basis of the 'multi-constant theory' and the 'multi-body potential theory' (Capecchi et al., 2011; Podio-Guidugli, 2019; Trovalusci, 2014). Models of continuum mechanics depend on constitutive relations, which should be properly identified for multiple scale materials. These constitutive relations can be determined using atomistic and experimental models combined by multiscale procedures (Trovalusci and Masiani, 1999; Trovalusci and Pau, 2014) or optimization approaches (Tuna and Kirca, 2019).

Whereas various reviews on multiscale modeling approaches can be found in the literature, (e.g., Bouvard et al., 2009; Liu et al., 2010; Elliott, 2011), reviews that focused on continuum mechanics of multiscale materials are scarce. Models of the continuum mechanics of multiscale materials depend on constitutive equations that can capture the different scale-dependent phenomena of the multiscale material. The constitutive equations of multiscale materials are different than those of the classical theory of continuum mechanics. These constitutive equations incorporate measures and length scales that can relate the material's phenomena and properties to their corresponding scales. In this section, we focus on the continuum mechanics of multiscale materials. The different models of continuum mechanics are reviewed, and their applicability to multiscale materials are discussed.

4.2. Nonlocal fields of multiscale materials

Multiscale materials exhibit various nonlocal fields. Polycrystalline materials, for example, exhibit various micro-deformation patterns, which arise due to the fluctuation of the mechanical properties, the heterogeneity nature of the microstructure, and/or the lattice misorientations (Shaat, 2018a,b; Sharma et al., 2018). Many of the macroscopic behaviors, e.g., fatigue and damage, of a polycrystalline material depend on the local and nonlocal interactions between the heterogeneities and misorientations of the microstructural lattices. Therefore, nonlocal models of the plasticity, fatigue, and damage of polycrystalline materials were developed (Fish and Oskay, 2005; Benedetti and Aliabadi, 2015; Lebensohn and Needleman, 2016; Sharma et al., 2018). In polymers, the nonlocal dielectric response of dipolar polymers was revealed (Warner and Cates, 1993). In addition, the nonlocal effect of the molecular dynamics of a long-chain polylactic acid was discussed (Li et al., 2017a). It was demonstrated that the elastic modulus of polymers should be determined accounting for the nonlocal molecular interactions, and molecular dynamics simulations should be adjusted according to a nonlocal-based length scale (Li et al., 2017a). Single crystals – as well – exhibit various nonlocal fields. Experimental investigations on single crystals have shown that their dissociation energies and interatomic potentials have nonlocal long-range contributions (Dalgarno and Davison, 1967; Leroy and Bernstein, 1970; Mostepanenko and Sokolov, 1993). The nonlocality of an interatomic force would cover the whole crystal, or it is effective only within the interatomic distance range and has a short-range effect (Demiray, 1977). Electromagnetic forces in the crystal usually produce acoustic and optical phonons, both, with long-range effects (Demiray, 1977; Jian and Zhi-da, 1992).

The nonlocal mechanics of multiscale materials can be captured using *nonlocal continuum mechanics*. In the context of a nonlocal continuum mechanics model, a multiscale material consists of interconnected multiple sub-domains over, both, the short-range and long-range. The constitutive equations of these sub-domains constitute material coefficients, which are functions of the distance between two non-neighbor particles of the sub-domain. Next, the various models of nonlocal mechanics of multiscale materials are reviewed and discussed.

4.3. Nonlocal micromorphic theory

The nonlocal micromorphic theory (Eringen, 1964b, 1968a,b, 1999, 2002) is an effective model of the continuum mechanics of multiscale materials. In the framework of the micromorphic theory, the material is represented consisting of a large number of deformable material particles (Eringen, 1964b, 1968a,b; 1999; Shaat, 2018a). The boundary value problem of a solid material that occupies a volume, V, and is bounded by a surface, S, based on the nonlocal micromorphic theory constitutes the following essential equations:

1: Particle's DOFs:		
$\mathbf{u} = \boldsymbol{\delta}_i u_i$	Displacement Vector (3-Components)	
$\boldsymbol{\psi} = \boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij}$	Micro-deformation Tensor (9-Components)	
2: Kinematical Variables:		
$\mathbf{s} = \frac{1}{2}(\boldsymbol{\psi} + \boldsymbol{\psi}^{\mathrm{T}})$	Microstrain Tensor (6-Components)	
$\boldsymbol{\gamma} = \nabla\mathbf{u} - \boldsymbol{\psi}$	Relative Deformation Tensor (9-Components)	
$\boldsymbol{\chi} = \nabla\boldsymbol{\psi}$	Micro-deformation Gradient Tensor (27-Components)	
3: Equations of Motion:		
$\nabla\cdot\boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho\ddot{\mathbf{u}}(\mathbf{x})$	$\forall\mathbf{x}\in V$ (3-Equations)	
$\nabla\cdot\mathbf{m}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}) - \mathbf{t}(\mathbf{x}) + \mathbf{h}(\mathbf{x}) = \rho_m J\ddot{\boldsymbol{\psi}}(\mathbf{x})$	$\forall\mathbf{x}\in V$ (9-Equations)	
4: Boundary Conditions (BCs):		
$\mathbf{n}\cdot\boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x})$ or $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})$	$\forall\mathbf{x}\in S$ (3-BCs)	
$\mathbf{n}\cdot\mathbf{m}(\mathbf{x}) = \mathbf{M}(\mathbf{x})$ or $\boldsymbol{\psi}(\mathbf{x}) = \boldsymbol{\Psi}(\mathbf{x})$	$\forall\mathbf{x}\in S$ (9-BCs)	
5: Strain Energy Density Function:		
$W(\mathbf{x}) = W(\mathbf{s}(\mathbf{x}), \boldsymbol{\gamma}(\mathbf{x}), \boldsymbol{\chi}(\mathbf{x}))$		
6: Constitutive Equations:	$\forall\mathbf{x}\in V$	
$\mathbf{t}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\mathbf{s}(\mathbf{x})$	Microstress Tensor	
$\boldsymbol{\tau}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\boldsymbol{\gamma}(\mathbf{x})$	Cauchy-type Stress Tensor	
$\mathbf{m}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\boldsymbol{\chi}(\mathbf{x})$	Double-stress (Hyperstress) Tensor	(13)

where ρ is a macroscopic mass density while ρ_m denotes the mass density of the material particle. J is a micro-inertia. $\mathbf{f}$ is a body force vector, and $\mathbf{h}$ is a body moment tensor. $\mathbf{T}$ is a surface traction force vector, and $\mathbf{M}$ is a surface traction moment tensor. $\mathbf{n}$ is the unit normal vector. $\mathbf{U}(\mathbf{x})$ and $\boldsymbol{\Psi}(\mathbf{x})$ are prescribed displacement vector and micro-deformation tensor at a boundary surface, respectively.

According to equation (13), the micromorphic theory treats each particle of the continuum as an embedded elastic domain that exhibits 12 independent degrees of freedom (3-displacements, 3-microrotations, and 6-microstrains). $\mathbf{u}$ is a displacement vector of 3 components u_i ($\boldsymbol{\delta}_i$ is the unit vector in $i-$direction), and $\boldsymbol{\psi}$ is a general-dyadic has 9-independent components, ψ_{ij}, that represent the different patterns of the micro-deformation of the particle. This general tensor can be decomposed into a symmetric tensor, $\mathbf{s}$, and a skew-symmetric tensor, $\boldsymbol{\theta}$, as follows:

$$\boldsymbol{\psi} = \mathbf{s} + \boldsymbol{\theta}$$
$$\textit{with}$$
$$\mathbf{s} = \tfrac{1}{2}(\boldsymbol{\psi} + \boldsymbol{\psi}^{\mathrm{T}})\ \textit{and}\ \boldsymbol{\theta} = \tfrac{1}{2}(\boldsymbol{\psi} - \boldsymbol{\psi}^{\mathrm{T}}) \tag{14}$$

where the superscript, T, indicates the tensor transpose. The symmetric tensor, $\mathbf{s}$, represents the microstrains of the material particle, and it has 6-independent components, s_{ij}. $\boldsymbol{\theta}$ is the micro-rotation tensor that consists of 3-independent components that represent the micro-rotations of the particle.

The strain energy of the micromorphic theory (equation $(13)_5$) depends on 42 kinematical variables, which are defined in equation $(13)_2$. In addition to the microstrain tensor $\mathbf{s}$, $\boldsymbol{\chi} = \nabla\boldsymbol{\psi}$ is a triadic that represents the gradient of the micro-deformation tensor (∇ is the gradient operator), and $\boldsymbol{\gamma}$ is a dyadic that represents the relative deformation (i.e., difference between macroscopic and microscopic deformations). The relative deformation tensor, $\boldsymbol{\gamma}$, is a coupling tensor that relates the material deformations at two different scales. Thanks to this tensor, the micromorphic theory has the potential of multiscale materials' modeling.

The dynamic equilibrium of a micromorphic continuum is governed by 12 equations of motion and 12 boundary conditions, as given in equations $(13)_3$ and $(13)_4$. In the context of the nonlocal micromorphic theory, these equations are formulated based on the global balance requirement where the stress fields ($\mathbf{t}$, $\boldsymbol{\tau}$, and $\mathbf{m}$) at a point ($\mathbf{x}$) are formed depending on not only the strain fields at this point ($\mathbf{x}$) but also the strain fields at other points ($\mathbf{x}'$) of the micromorphic continuum. In equation (13), $\mathbf{t}$ is the microstress tensor, and $\boldsymbol{\tau}$ is a Cauchy-type stress tensor. $\boldsymbol{\tau}$ is a dyadic that represents the residual stresses due to the difference between the macro-deformation field and the micro-deformation field. $\mathbf{m}$ is a double-stress tensor, which is a triadic that is conjugate to the micro-deformation gradient tensor $\boldsymbol{\chi}$. The components of $\mathbf{t}$ and $\boldsymbol{\tau}$ tensors have the dimension of a force per unit area while the components of $\mathbf{m}$ have the dimension of a moment per unit area.

According to Eringen's formulation of the nonlocal micromorphic mechanics, the strain energy density function was defined for the nonlocal micromorphic theory, as follows (Eringen, 1999, 2002):

$$\begin{aligned} W(\mathbf{x},\mathbf{x}') = {} & a_{ijkl}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{kl}(\mathbf{x}) + b_{ijkl}(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}')s_{kl}(\mathbf{x}) \\ & + c_{ijklmn}(\mathbf{x},\mathbf{x}')\chi_{ijk}(\mathbf{x}')\chi_{lmn}(\mathbf{x}) + d_{ijkl}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')s_{kl}(\mathbf{x}) + d_{ijkl}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x})s_{kl}(\mathbf{x}') \\ & + e_{ijklm}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')\chi_{klm}(\mathbf{x}) + e_{ijklm}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x})\chi_{klm}(\mathbf{x}') \\ & + f_{ijklm}(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}')\chi_{klm}(\mathbf{x}) + f_{ijklm}(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x})\chi_{klm}(\mathbf{x}') \end{aligned} \tag{15}$$

where the strain energy depends on the kinematical variables at two different points of the micromorphic continuum, $\mathbf{x}$ and $\mathbf{x}'$. Based on the nonlocal concept discussed in Sections 2 and 3, the constitutive equations of the nonlocal micromorphic theory (equation $(13)_6$) are obtained in the form:

$$t_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial s_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V (b_{klij}(\mathbf{x}, \mathbf{x}') s_{kl}(\mathbf{x}') + d_{klij}(\mathbf{x}, \mathbf{x}') \gamma_{kl}(\mathbf{x}') + f_{ijklm}(\mathbf{x}, \mathbf{x}') \chi_{klm}(\mathbf{x}')) dV(\mathbf{x}') \tag{16}$$

$$\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \gamma_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V (a_{klij}(\mathbf{x}, \mathbf{x}') \gamma_{kl}(\mathbf{x}') + d_{ijkl}(\mathbf{x}, \mathbf{x}') s_{kl}(\mathbf{x}') + e_{ijklm}(\mathbf{x}, \mathbf{x}') \chi_{klm}(\mathbf{x}')) dV(\mathbf{x}') \tag{17}$$

$$m_{ijk}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \chi_{ijk}(\mathbf{x})} dV(\mathbf{x}') = \int_V (c_{lmnijk}(\mathbf{x}, \mathbf{x}') \chi_{lmn}(\mathbf{x}') + e_{lmijk}(\mathbf{x}, \mathbf{x}') \gamma_{lm}(\mathbf{x}') + f_{lmijk}(\mathbf{x}, \mathbf{x}') s_{lm}(\mathbf{x}')) dV(\mathbf{x}') \tag{18}$$

where a_{ijkl}, b_{ijkl}, d_{ijkl}, c_{ijklmn}, e_{ijklm}, and f_{ijklm} are material coefficients, which are functions of the distance between two non-neighbor particles of the micromorphic continuum ($|\mathbf{x} - \mathbf{x}'|$). For homogeneous-isotropic linear elastic materials, these material coefficients have the form (Eringen, 1999, 2002; Chen and Lee, 2003):

$$\begin{aligned}
e_{ijklm} &= f_{ijklm} = 0 \\
a_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \eta \delta_{il} \delta_{jk} \\
b_{ijkl} &= \lambda_m \delta_{ij} \delta_{kl} + \mu_m \delta_{ik} \delta_{jl} + \mu_m \delta_{il} \delta_{jk} \\
d_{ijkl} &= \lambda_c \delta_{ij} \delta_{kl} + \mu_c \delta_{ik} \delta_{jl} + \mu_c \delta_{il} \delta_{jk} \\
c_{ijklmn} &= c_1(\delta_{ij} \delta_{kl} \delta_{mn} + \delta_{in} \delta_{jk} \delta_{lm}) \\
&\quad + c_2(\delta_{ij} \delta_{km} \delta_{ln} + \delta_{ik} \delta_{jn} \delta_{lm}) + c_3 \delta_{ij} \delta_{kn} \delta_{lm} \\
&\quad + c_4 \delta_{il} \delta_{jk} \delta_{mn} + c_5(\delta_{ik} \delta_{jl} \delta_{mn} + \delta_{im} \delta_{jk} \delta_{ln}) \\
&\quad + c_6 \delta_{ik} \delta_{jm} \delta_{ln} + c_7 \delta_{il} \delta_{jm} \delta_{kn} + c_8(\delta_{im} \delta_{jn} \delta_{kl} + \delta_{in} \delta_{jl} \delta_{km}) \\
&\quad + c_9 \delta_{il} \delta_{jn} \delta_{km} + c_{10} \delta_{im} \delta_{jl} \delta_{kn} + c_{11} \delta_{in} \delta_{jm} \delta_{kl}
\end{aligned} \tag{19}$$

where δ_{ij} is the Kronecker delta. λ_m and μ_m are the Lamé constants of the material's microstructure. λ, μ, and η were defined as the material coefficients of a material confined between two material particles while λ_c and μ_c were introduced as coupling parameters that depend on the difference between the microscopic and macroscopic fields (Shaat, 2018a). The substitution of equation (19) into equation (15) gives the strain energy density function in the form:

$$\begin{aligned}
W(\mathbf{x}, \mathbf{x}') &= \lambda(\mathbf{x}, \mathbf{x}') \gamma_{ii}(\mathbf{x}') \gamma_{jj}(\mathbf{x}) + \mu(\mathbf{x}, \mathbf{x}') \gamma_{ij}(\mathbf{x}') \gamma_{ij}(\mathbf{x}) \\
&\quad + \mu(\mathbf{x}, \mathbf{x}') \eta(\mathbf{x}, \mathbf{x}') \gamma_{ij}(\mathbf{x}') \gamma_{ji}(\mathbf{x}) + \lambda_m(\mathbf{x}, \mathbf{x}') s_{ii}(\mathbf{x}') s_{jj}(\mathbf{x}) \\
&\quad + 2\mu_m(\mathbf{x}, \mathbf{x}') s_{ij}(\mathbf{x}') s_{ij}(\mathbf{x}) \\
&\quad + \lambda_c(\mathbf{x}, \mathbf{x}')(\gamma_{ii}(\mathbf{x}') s_{jj}(\mathbf{x}) + \gamma_{ii}(\mathbf{x}) s_{jj}(\mathbf{x}')) \\
&\quad + 2\mu_c(\mathbf{x}, \mathbf{x}')(\gamma_{ij}(\mathbf{x}') s_{ij}(\mathbf{x}) + \gamma_{ij}(\mathbf{x}) s_{ij}(\mathbf{x}')) \\
&\quad + c_1(\mathbf{x}, \mathbf{x}')(\chi_{iik}(\mathbf{x}') \chi_{kjj}(\mathbf{x}) + \chi_{ijj}(\mathbf{x}') \chi_{kki}(\mathbf{x})) \\
&\quad + c_2(\mathbf{x}, \mathbf{x}')(\chi_{iik}(\mathbf{x}') \chi_{jkj}(\mathbf{x}) + \chi_{iji}(\mathbf{x}') \chi_{kkj}(\mathbf{x})) \\
&\quad + c_3(\mathbf{x}, \mathbf{x}') \chi_{iik}(\mathbf{x}') \chi_{jjk}(\mathbf{x}) + c_4(\mathbf{x}, \mathbf{x}') \chi_{ijj}(\mathbf{x}') \chi_{ikk}(\mathbf{x}) \\
&\quad + c_5(\mathbf{x}, \mathbf{x}')(\chi_{iji}(\mathbf{x}') \chi_{jkk}(\mathbf{x}) + \chi_{ijj}(\mathbf{x}') \chi_{kik}(\mathbf{x})) \\
&\quad + c_6(\mathbf{x}, \mathbf{x}') \chi_{iji}(\mathbf{x}') \chi_{kjk}(\mathbf{x}) + c_7(\mathbf{x}, \mathbf{x}') \chi_{ijk}(\mathbf{x}') \chi_{ijk}(\mathbf{x}) \\
&\quad + c_8(\mathbf{x}, \mathbf{x}')(\chi_{ijk}(\mathbf{x}') \chi_{kij}(\mathbf{x}) + \chi_{ijk}(\mathbf{x}') \chi_{jki}(\mathbf{x})) \\
&\quad + c_9(\mathbf{x}, \mathbf{x}') \chi_{ijk}(\mathbf{x}') \chi_{ikj}(\mathbf{x}) + c_{10}(\mathbf{x}, \mathbf{x}') \chi_{ijk}(\mathbf{x}') \chi_{jik}(\mathbf{x}) \\
&\quad + c_{11}(\mathbf{x}, \mathbf{x}') \chi_{ijk}(\mathbf{x}') \chi_{kji}(\mathbf{x})
\end{aligned} \tag{20}$$

and the substitution of equation (19) into equations (16)-(18) yields the constitutive equations in the form:

$$t_{ij}(\mathbf{x}) = \int_V (\lambda_m(\mathbf{x}, \mathbf{x}') s_{ll}(\mathbf{x}') \delta_{ij} + 2\mu_m(\mathbf{x}, \mathbf{x}') s_{ij}(\mathbf{x}') + \lambda_c(\mathbf{x}, \mathbf{x}') \gamma_{ll}(\mathbf{x}') \delta_{ij} + \mu_c(\mathbf{x}, \mathbf{x}') \gamma_{ij}(\mathbf{x}') + \mu_c(\mathbf{x}, \mathbf{x}') \gamma_{ji}(\mathbf{x}')) dV(\mathbf{x}') \tag{21}$$

$$\tau_{ij}(\mathbf{x}) = \int_V (\lambda(\mathbf{x}, \mathbf{x}') \gamma_{ll}(\mathbf{x}') \delta_{ij} + \mu(\mathbf{x}, \mathbf{x}') \gamma_{ij}(\mathbf{x}') + \mu(\mathbf{x}, \mathbf{x}') \eta(\mathbf{x}, \mathbf{x}') \gamma_{ji}(\mathbf{x}') + \lambda_c(\mathbf{x}, \mathbf{x}') s_{ll}(\mathbf{x}') \delta_{ij} + 2\mu_c(\mathbf{x}, \mathbf{x}') s_{ij}(\mathbf{x}')) dV(\mathbf{x}') \tag{22}$$

$$\begin{aligned}
m_{ijk}(\mathbf{x}) &= \int_V (c_1(\mathbf{x}, \mathbf{x}')(\chi_{kll}(\mathbf{x}') \delta_{ij} + \chi_{lli}(\mathbf{x}') \delta_{jk}) \\
&\quad + c_2(\mathbf{x}, \mathbf{x}')(\chi_{lkl}(\mathbf{x}') \delta_{ij} + \chi_{llj}(\mathbf{x}') \delta_{ik}) + c_3(\mathbf{x}, \mathbf{x}') \chi_{llk}(\mathbf{x}') \delta_{ij} \\
&\quad + c_4(\mathbf{x}, \mathbf{x}') \chi_{ill}(\mathbf{x}') \delta_{jk} + c_5(\mathbf{x}, \mathbf{x}')(\chi_{jll}(\mathbf{x}') \delta_{ik} + \chi_{lil}(\mathbf{x}') \delta_{jk}) \\
&\quad + c_6(\mathbf{x}, \mathbf{x}') \chi_{ljl}(\mathbf{x}') \delta_{ik} + c_7(\mathbf{x}, \mathbf{x}') \chi_{ijk}(\mathbf{x}') \\
&\quad + c_8(\mathbf{x}, \mathbf{x}')(\chi_{kij}(\mathbf{x}') + \chi_{jki}(\mathbf{x}')) + c_9(\mathbf{x}, \mathbf{x}') \chi_{ikj}(\mathbf{x}') \\
&\quad + c_{10}(\mathbf{x}, \mathbf{x}') \chi_{jik}(\mathbf{x}') + c_{11}(\mathbf{x}, \mathbf{x}') \chi_{kji}(\mathbf{x}')) dV(\mathbf{x}')
\end{aligned} \tag{23}$$

It should be mentioned that Mindlin developed the microstructure theory (Mindlin, 1964), which is similar to the micromorphic theory for linear elastic materials. The microstructure theory can be obtained from the micromorphic theory by observing the following changes over the kinematical variables: $s_{ij} \to \varepsilon_{ij}$, $\gamma_{ij} \to 2\varepsilon_{ij} - \gamma_{ji}$, and $\chi_{ijk} \to -\chi_{kji}$ (Eringen, 1968b).

Utilizing the gradient operator for the Cartesian coordinates, $\nabla = \boldsymbol{\delta}_x \blacksquare_{,x} + \boldsymbol{\delta}_y \blacksquare_{,y} + \boldsymbol{\delta}_z \blacksquare_{,z}$ (where $\blacksquare_{,\zeta}$ denotes a derivative with respect to ζ), and the cylindrical coordinates, $\nabla = \boldsymbol{\delta}_r \blacksquare_{,r} + \boldsymbol{\delta}_\theta \frac{1}{r} \blacksquare_{,\theta} + \boldsymbol{\delta}_z \blacksquare_{,z}$, the kinematical variables and equations of motion of the micromorphic theory can be written in the Cartesian and cylindrical coordinates, as follows:

Cartesian Coordinates:

$$\begin{aligned}
\mathbf{s} &= \tfrac{1}{2}(\boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij} + \boldsymbol{\delta}_j \boldsymbol{\delta}_i \psi_{ji}) \\
\boldsymbol{\gamma} &= \boldsymbol{\delta}_i \boldsymbol{\delta}_j u_{j,i} - \boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij} \\
\boldsymbol{\chi} &= \boldsymbol{\delta}_i \boldsymbol{\delta}_j \boldsymbol{\delta}_k \psi_{jk,i}
\end{aligned}$$

Cylindrical Coordinates:

$$\begin{aligned}
\mathbf{s} &= \tfrac{1}{2}(\boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij} + \boldsymbol{\delta}_j \boldsymbol{\delta}_i \psi_{ji}) \\
\boldsymbol{\gamma} &= \boldsymbol{\delta}_r \boldsymbol{\delta}_i u_{i,r} + \boldsymbol{\delta}_\theta \boldsymbol{\delta}_i \frac{1}{r} u_{i,\theta} + \boldsymbol{\delta}_\theta \boldsymbol{\delta}_\theta \frac{u_r}{r} - \boldsymbol{\delta}_\theta \boldsymbol{\delta}_r \frac{u_\theta}{r} + \boldsymbol{\delta}_z \boldsymbol{\delta}_i u_{i,z} - \boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij} \\
\boldsymbol{\chi} &= \boldsymbol{\delta}_r \boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij,r} + \boldsymbol{\delta}_\theta \boldsymbol{\delta}_\theta \boldsymbol{\delta}_j \frac{1}{r} \psi_{rj} - \boldsymbol{\delta}_\theta \boldsymbol{\delta}_r \boldsymbol{\delta}_j \frac{1}{r} \psi_{\theta j} + \boldsymbol{\delta}_\theta \boldsymbol{\delta}_i \boldsymbol{\delta}_\theta \frac{1}{r} \psi_{ir} - \boldsymbol{\delta}_\theta \boldsymbol{\delta}_i \boldsymbol{\delta}_r \frac{1}{r} \psi_{i\theta} + \boldsymbol{\delta}_\theta \boldsymbol{\delta}_i \boldsymbol{\delta}_j \frac{1}{r} \psi_{ij,\theta} + \boldsymbol{\delta}_z \boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij,z}
\end{aligned} \tag{24}$$

Cartesian Coordinates:

$$\begin{aligned}
&\boldsymbol{\delta}_i \tau_{ji,j} + \boldsymbol{\delta}_i f_i = \boldsymbol{\delta}_i \rho \ddot{u}_i \\
&\boldsymbol{\delta}_j \boldsymbol{\delta}_k m_{ljk,l} + \boldsymbol{\delta}_j \boldsymbol{\delta}_k \tau_{jk} - \boldsymbol{\delta}_j \boldsymbol{\delta}_k t_{jk} + \boldsymbol{\delta}_j \boldsymbol{\delta}_k h_{jk} = \boldsymbol{\delta}_j \boldsymbol{\delta}_k \rho_m J \ddot{\psi}_{jk}
\end{aligned}$$

Cylindrical Coordinates:

$$\begin{aligned}
&\boldsymbol{\delta}_i \tau_{ri,r} + \frac{1}{r}(\boldsymbol{\delta}_\theta \tau_{\theta r} + \boldsymbol{\delta}_i \tau_{ri} - \boldsymbol{\delta}_r \tau_{\theta\theta} + \boldsymbol{\delta}_i \tau_{\theta i,\theta}) + \boldsymbol{\delta}_i \tau_{zi,z} + \boldsymbol{\delta}_i f_i = \boldsymbol{\delta}_i \rho \ddot{u}_i \\
&\boldsymbol{\delta}_j \boldsymbol{\delta}_k m_{rjk,r} + \frac{1}{r}(\boldsymbol{\delta}_j \boldsymbol{\delta}_k m_{rjk} + \boldsymbol{\delta}_\theta \boldsymbol{\delta}_k m_{\theta rk} - \boldsymbol{\delta}_r \boldsymbol{\delta}_k m_{\theta\theta k} + \boldsymbol{\delta}_j \boldsymbol{\delta}_\theta m_{\theta jr} - \boldsymbol{\delta}_j \boldsymbol{\delta}_r m_{\theta j\theta} + \boldsymbol{\delta}_j \boldsymbol{\delta}_k m_{\theta jk,\theta}) + \boldsymbol{\delta}_j \boldsymbol{\delta}_k m_{zjk,z} + \boldsymbol{\delta}_j \boldsymbol{\delta}_k \tau_{jk} - \boldsymbol{\delta}_j \boldsymbol{\delta}_k t_{jk} + \boldsymbol{\delta}_j \boldsymbol{\delta}_k h_{jk} = \boldsymbol{\delta}_j \boldsymbol{\delta}_k \rho_m J \ddot{\psi}_{jk}
\end{aligned} \tag{25}$$

4.4. Nonlocal microstretch theory

The microstretch theory was first introduced by Eringen (1969). The microstretch theory can be obtained from the micromorphic theory by eliminating the shear micro-strains of the material particle, i.e., $\mathbf{s} - \mathrm{tr}(\mathbf{s}) = 0$ (Eringen, 1999, 2002; Lazar and Maugin, 2007). Thus, in the microstretch theory, the material particle is a volume element has 1-dilatational stretch, 3-rotational, and 3-translational degrees of freedom. Accordingly, the boundary value problem of an elastic continuum can be written in the framework of the microstretch theory, as follows:

$$
\begin{array}{|lll|}
\hline
\text{1: Particle's DOFs:} & & \\
\mathbf{u} = \boldsymbol{\delta}_i u_i & \text{Displacement Vector (3-Components)} & \\
\varphi = s_{rr}/3 & \text{Dilatational Microstretch (Scalar Field)} & \\
\boldsymbol{\theta} = \boldsymbol{\delta}_i \theta_i & \text{Micro-rotation Vector (3-Components)} & \\
\text{2: Kinematical Variables:} & & \\
\varphi = s_{rr}/3 & \text{Dilatational Microstretch (Scalar Field)} & \\
\mathbf{R} = \nabla\boldsymbol{\theta} & \text{Rotation Gradient Tensor (9-Components)} & \\
\boldsymbol{\gamma} = \nabla\mathbf{u} - \boldsymbol{\theta}\times\mathbf{I} & \text{Relative Deformation Tensor (9-Components)} & \\
\mathbf{e} = \nabla\varphi & \text{Dilatational Microstretch Gradient Vector (3-Components)} & \\
\text{3: Equations of Motion:} & & \\
\nabla\cdot\boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho\ddot{\mathbf{u}}(\mathbf{x}) & \forall\mathbf{x}\in V & \text{(3-Equations)} \\
\nabla\cdot\mathbf{m}(\mathbf{x}) + \hat{\boldsymbol{\tau}}(\mathbf{x}) + \hat{\mathbf{h}}(\mathbf{x}) = \rho_m J\ddot{\boldsymbol{\theta}}(\mathbf{x}) & \forall\mathbf{x}\in V & \text{(3-Equations)} \\
\nabla\cdot\boldsymbol{\mu}(\mathbf{x}) + \mathrm{tr}\big(\boldsymbol{\tau}(\mathbf{x}) - \mathbf{t}(\mathbf{x})\big) + l(\mathbf{x}) = \rho_m J\ddot{\varphi}(\mathbf{x}) & \forall\mathbf{x}\in V & \text{(1-Equation)} \\
\text{4: Boundary Conditions:} & & \\
\mathbf{n}\cdot\boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) \quad \text{or} \quad \mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) & \forall\mathbf{x}\in S & \text{(3-BCs)} \\
\mathbf{n}\cdot\mathbf{m}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) \quad \text{or} \quad \boldsymbol{\theta}(\mathbf{x}) = \boldsymbol{\Theta}(\mathbf{x}) & \forall\mathbf{x}\in S & \text{(3-BCs)} \\
\mathbf{n}\cdot\boldsymbol{\mu}(\mathbf{x}) = L(\mathbf{x}) \quad \text{or} \quad \varphi(\mathbf{x}) = \Phi(\mathbf{x}) & \forall\mathbf{x}\in S & \text{(1-BC)} \\
\text{5: Strain Energy Density Function:} & & \\
W(\mathbf{x}) = W\big(\varphi(\mathbf{x}), \boldsymbol{\gamma}(\mathbf{x}), \mathbf{R}(\mathbf{x}), \mathbf{e}(\mathbf{x})\big) & & \\
\text{6: Constitutive Equations:} & \forall\mathbf{x}\in V & \\
\mathbf{t}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\varphi(\mathbf{x}) & \text{Dilatational Microstress Tensor} & \\
\boldsymbol{\tau}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\boldsymbol{\gamma}(\mathbf{x}) & \text{Cauchy-type Stress Tensor} & \\
\mathbf{m}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\mathbf{R}(\mathbf{x}) & \text{Couple Stress Tensor} & \\
\boldsymbol{\mu}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\mathbf{e} & \text{Dilatational Wryness-Stress Vector} & \\
\hline
\end{array}
\tag{26}
$$

where $\phi = s_{rr}/3$ is the dilatational microstretch (micro-dilatation), and θ is the micro-rotation vector with θ_i are its components. The components of the micro-rotation vector in terms of the components of the micro-rotation tensor are $\theta_i = \frac{1}{2}\varepsilon_{ijk}\vartheta_{kj}$ where ε_{ijk} is the permutation symbol, and $\vartheta_{ij} = \frac{1}{2}(\psi_{ij} - \psi_{ji})$ are the components of the micro-rotation tensor, which is the skew-symmetric part of the micro-deformation tensor $\boldsymbol{\psi}$. $\mathbf{R}$ is introduced as a micro-rotation gradient tensor. $\hat{\boldsymbol{\tau}}$ is a stress-vector with components $\hat{\tau}_i = \varepsilon_{ijk}\tau_{jk}$ where τ_{jk} are the components of the Cauchy-type stress tensor $\boldsymbol{\tau}$. $\hat{\mathbf{h}}$ is a body couple vector, and l is a dilatational body couple scalar field. In addition to the surface traction force vector $\mathbf{T}$, a surface traction couple vector $\mathbf{C}$ and a dilatational moment scalar field L are introduced at the boundary of a microstretch continuum. $\mathbf{U}$, $\boldsymbol{\Theta}$, and Φ are prescribed displacement vector, rotation vector, and dilatational microstretch scalar field at a boundary surface of the continuum.

The strain energy of the microstretch theory depends on 22 kinematical variables (equation (26)$_2$), which can be formed in the context of nonlocal mechanics, as follows (Eringen, 1999, 2002):

$$
\begin{aligned}
W(\mathbf{x}, \mathbf{x}') = {} & a_{ijkl}(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{kl}(\mathbf{x}) + b(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}')\phi(\mathbf{x}) + c_{ij}(\mathbf{x}, \mathbf{x}')e_i(\mathbf{x}')e_j(\mathbf{x}) \\
& + c_{ijk}(\mathbf{x}, \mathbf{x}')(e_i(\mathbf{x}')R_{jk}(\mathbf{x}) + e_i(\mathbf{x})R_{jk}(\mathbf{x}')) \\
& + c_{ijkl}(\mathbf{x}, \mathbf{x}')R_{ij}(\mathbf{x}')R_{kl}(\mathbf{x}) + d_{ij}(\mathbf{x}, \mathbf{x}')(\gamma_{ij}(\mathbf{x}')\phi(\mathbf{x}) + \gamma_{ij}(\mathbf{x})\phi(\mathbf{x}')) \\
& + e_{ijk}(\mathbf{x}, \mathbf{x}')(\gamma_{ij}(\mathbf{x}')e_k(\mathbf{x}) + \gamma_{ij}(\mathbf{x})e_k(\mathbf{x}')) + e_{ijkl}(\mathbf{x}, \mathbf{x}') \\
& (\gamma_{ij}(\mathbf{x}')R_{kl}(\mathbf{x}) + \gamma_{ij}(\mathbf{x})R_{kl}(\mathbf{x}')) \\
& + f_i(\mathbf{x}, \mathbf{x}')(\phi(\mathbf{x}')e_i(\mathbf{x}) + \phi(\mathbf{x})e_i(\mathbf{x}')) \\
& + f_{ij}(\mathbf{x}, \mathbf{x}')(\phi(\mathbf{x}')R_{ij}(\mathbf{x}) + \phi(\mathbf{x})R_{ij}(\mathbf{x}'))
\end{aligned}
\tag{27}
$$

where a_{ijkl}, b, c_{ij}, c_{ijk}, c_{ijkl}, d_{ij}, e_{ijk}, e_{ijkl}, f_i, f_{ij} are the material coefficients of the microstretch theory. These coefficients can be related to the ones of the micromorphic theory, as follows (Lazar and Maugin, 2007):

$$
\begin{aligned}
& b = b_{llkk} \\
& c_{ij} = c_{illjkk} \\
& c_{ijk} = -\varepsilon_{kmn}c_{illjmn} \\
& c_{ijkl} = \varepsilon_{jmn}\varepsilon_{lpq}c_{imnkpq} \\
& d_{ij} = d_{ijkk} \\
& e_{ijk} = e_{ijkll} \\
& e_{ijkl} = -\varepsilon_{lmn}e_{ijkmn} \\
& f_i = f_{kkill} \\
& f_{ij} = -\varepsilon_{jmn}f_{kkimn}
\end{aligned}
\tag{28}
$$

Accordingly, the constitutive equations in the context of the nonlocal microstretch theory will have the form:

$$
\begin{aligned}
t_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial\phi(\mathbf{x})}dV(\mathbf{x}') = \int_V & (b(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}') + d_{ij}(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}') \\
& + f_i(\mathbf{x}, \mathbf{x}')e_i(\mathbf{x}') + f_{ij}(\mathbf{x}, \mathbf{x}')R_{ij}(\mathbf{x}'))dV(\mathbf{x}')
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial\gamma_{ij}(\mathbf{x})}dV(\mathbf{x}') = \int_V & (a_{klij}(\mathbf{x}, \mathbf{x}')\gamma_{kl}(\mathbf{x}') + d_{ij}(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}') \\
& + e_{ijk}(\mathbf{x}, \mathbf{x}')e_k(\mathbf{x}') + e_{ijkl}(\mathbf{x}, \mathbf{x}')R_{kl}(\mathbf{x}'))dV(\mathbf{x}')
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
m_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial R_{ij}(\mathbf{x})}dV(\mathbf{x}') = \int_V & (c_{kij}(\mathbf{x}, \mathbf{x}')e_k(\mathbf{x}') + c_{ijkl}(\mathbf{x}, \mathbf{x}')R_{kl}(\mathbf{x}') \\
& + e_{klij}(\mathbf{x}, \mathbf{x}')\gamma_{kl}(\mathbf{x}') + f_{ij}(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}'))dV(\mathbf{x}')
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\mu_i(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial e_i(\mathbf{x})}dV(\mathbf{x}') = \int_V & (c_{ji}(\mathbf{x}, \mathbf{x}')e_j(\mathbf{x}') + c_{ijk}(\mathbf{x}, \mathbf{x}')R_{jk}(\mathbf{x}') \\
& + e_{jki}(\mathbf{x}, \mathbf{x}')\gamma_{jk}(\mathbf{x}') + f_i(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}'))dV(\mathbf{x}')
\end{aligned}
\tag{32}
$$

For homogeneous-isotropic linear elastic materials, the constitutive equations are expressed as follows:

$$t_{ij}(\mathbf{x}) = \int_V ((3\lambda_m(\mathbf{x}, \mathbf{x}') + 2\mu_m(\mathbf{x}, \mathbf{x}'))\phi(\mathbf{x}')\delta_{ij} + \lambda_c(\mathbf{x}, \mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + \mu_c(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}') + \mu_c(\mathbf{x}, \mathbf{x}')\gamma_{ji}(\mathbf{x}'))dV(\mathbf{x}') \tag{33}$$

$$\tau_{ij}(\mathbf{x}) = \int_V (\lambda(\mathbf{x}, \mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + \mu(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}') + \mu(\mathbf{x}, \mathbf{x}')\eta(\mathbf{x}, \mathbf{x}')\gamma_{ji}(\mathbf{x}') + (3\lambda_c(\mathbf{x}, \mathbf{x}') + 2\mu_c(\mathbf{x}, \mathbf{x}'))\phi(\mathbf{x}')\delta_{ij})dV(\mathbf{x}') \tag{34}$$

$$m_{ij}(\mathbf{x}) = \int_V (\alpha_0(\mathbf{x}, \mathbf{x}')\varepsilon_{rji}e_r(\mathbf{x}') + \beta_1(\mathbf{x}, \mathbf{x}')R_{rr}(\mathbf{x}')\delta_{ij} + \beta_2(\mathbf{x}, \mathbf{x}')R_{ji}(\mathbf{x}') + \beta_3(\mathbf{x}, \mathbf{x}')R_{ij}(\mathbf{x}'))dV(\mathbf{x}') \tag{35}$$

$$\mu_i(\mathbf{x}) = \int_V (\alpha_1(\mathbf{x}, \mathbf{x}')e_i(\mathbf{x}') + \alpha_0(\mathbf{x}, \mathbf{x}')\varepsilon_{ijk}R_{jk}(\mathbf{x}'))dV(\mathbf{x}') \tag{36}$$

where the material coefficients λ, λ_m, λ_c, μ, μ_m, μ_c, η are the same as those of the micromorphic theory. The material coefficients α_0, α_1, β_1, β_2, β_3 are expressed in terms of those of the micromorphic theory, as follows (Eringen, 1999):

$$\begin{aligned}
\alpha_0 &= 3c_1 - 3c_2 + c_4 - c_6 - c_9 + c_{11} \\
\alpha_1 &= 6c_1 + 6c_2 + 9c_3 + c_4 + 2c_5 + c_6 + 3c_7 + 2c_8 + c_9 + 3c_{10} + c_{11} \\
\beta_1 &= 2c_8 - c_9 - c_{11} \\
\beta_2 &= -c_4 + 2c_5 - c_6 \\
\beta_3 &= c_4 - 2c_5 + c_6 + 2c_7 - 2c_8 + c_9 - 2c_{10} + c_{11}
\end{aligned} \tag{37}$$

The kinematical variables and equations of motion of the microstretch theory can be written in the Cartesian and cylindrical coordinates, as follows:

Cartesian Coordinates:

$$\phi = s_{ll}/3$$
$$\mathbf{R} = \delta_i\delta_j\theta_{j,i}$$
$$\boldsymbol{\gamma} = \delta_i\delta_j u_{j,i} + \delta_i\delta_j\varepsilon_{ijm}\theta_m$$
$$\mathbf{e} = \delta_i\phi_{,i}$$

Cylindrical Coordinates:

$$\phi = s_{ll}/3$$
$$\mathbf{R} = \delta_r\delta_i\theta_{i,r} + \delta_\theta\delta_i\frac{1}{r}\theta_{i,\theta} + \delta_\theta\delta_\theta\frac{\theta_r}{r} - \delta_\theta\delta_r\frac{\theta_\theta}{r} + \delta_z\delta_i\theta_{i,z}$$
$$\boldsymbol{\gamma} = \delta_r\delta_i u_{i,r} + \delta_\theta\delta_i\frac{1}{r}u_{i,\theta} + \delta_\theta\delta_\theta\frac{u_r}{r} - \delta_\theta\delta_r\frac{u_\theta}{r} + \delta_z\delta_i u_{i,z} + \delta_i\delta_j\varepsilon_{ijm}\theta_m$$
$$\mathbf{e} = \delta_r\phi_{,r} + \delta_\theta\frac{1}{r}\phi_{,\theta} + \delta_z\phi_{,z} \tag{38}$$

Cartesian Coordinates:

$$\delta_i\tau_{ji,j} + \delta_i f_i = \delta_i\rho\ddot{u}_i$$
$$\delta_i m_{ji,j} + \delta_i\hat{\tau}_i + \delta_i\hat{h}_i = \delta_i\rho_m J\ddot{\theta}_i$$
$$\mu_{i,i} + \tau_{ll}/3 - t_{ll}/3 + l = \rho_m J\ddot{\phi}$$

Cylindrical Coordinates:

$$\delta_i\tau_{ri,r} + \frac{1}{r}(\delta_\theta\tau_{\theta r} + \delta_i\tau_{ri} - \delta_r\tau_{\theta\theta} + \delta_i\tau_{\theta i,\theta}) + \delta_i\tau_{zi,z} + \delta_i f_i = \delta_i\rho\ddot{u}_i$$
$$\delta_i m_{ri,r} + \frac{1}{r}(\delta_\theta m_{\theta r} + \delta_i m_{ri} - \delta_r m_{\theta\theta} + \delta_i m_{\theta i,\theta}) + \delta_i m_{zi,z} + \delta_i\hat{\tau}_i + \delta_i\hat{h}_i = \delta_i\rho_m J\ddot{\theta}_i$$
$$\frac{1}{r}(r\mu_{r,r} + \mu_r) + \frac{1}{r}\mu_{\theta,\theta} + \mu_{z,z} + \tau_{ll}/3 - t_{ll}/3 + l = \rho_m J\ddot{\phi} \tag{39}$$

4.5. Nonlocal micropolar theory

In the context of the Micropolar/Cosserat theories (Cosserat and Cosserat, 1909; Eringen, 1966), the elastic domain is represented containing a large number of rigid material particles, each of which exhibits displacements and rotations (Shaat, 2015a; Shaat and Abdelkefi, 2016a). The micropolar theory can be obtained from the micromorphic theory by eliminating the microstrain field, i.e., $\mathbf{s} = 0$. It can also be obtained from the microstretch theory by eliminating the dilatational microstretch, i.e., $\phi = 0$. Thus, the material particle in the micropolar theory has only 3-translational and 3-rotational motions. The essential equations of the boundary value problems according to the micropolar theory can be written as follows:

1: Particle's DOFs:

$\mathbf{u} = \boldsymbol{\delta}_i u_i$ Displacement Vector (3-Components)

$\boldsymbol{\theta} = \boldsymbol{\delta}_i \theta_i$ Micro-rotation Vector (3-Components)

2: Kinematical Variables:

$\mathbf{R} = \nabla\boldsymbol{\theta}$ Rotation Gradient Tensor (9-Components)

$\boldsymbol{\gamma} = \nabla\mathbf{u} - \boldsymbol{\theta}\times\mathbf{I}$ Relative Deformation Tensor (9-Components)

3: Equations of Motion:

$\nabla\cdot\boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho\ddot{\mathbf{u}}(\mathbf{x})$ $\forall\mathbf{x}\in V$ (3-Equations)

$\nabla\cdot\mathbf{m}(\mathbf{x}) + \hat{\boldsymbol{\tau}}(\mathbf{x}) + \hat{\mathbf{h}}(\mathbf{x}) = \rho_m J\ddot{\boldsymbol{\theta}}(\mathbf{x})$ $\forall\mathbf{x}\in V$ (3-Equations)

4: Boundary Conditions:

$\mathbf{n}\cdot\boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x})$ or $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})$ $\forall\mathbf{x}\in S$ (3-BCs)

$\mathbf{n}\cdot\mathbf{m}(\mathbf{x}) = \mathbf{C}(\mathbf{x})$ or $\boldsymbol{\theta}(\mathbf{x}) = \boldsymbol{\Theta}(\mathbf{x})$ $\forall\mathbf{x}\in S$ (3-BCs)

5: Strain Energy Density Function:

$W(\mathbf{x}) = W(\boldsymbol{\gamma}(\mathbf{x}), \mathbf{R}(\mathbf{x}))$

6: Constitutive Equations: $\forall\mathbf{x}\in V$

$\boldsymbol{\tau}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\boldsymbol{\gamma}(\mathbf{x})$ Cauchy-type Stress Tensor

$\mathbf{m}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\mathbf{R}(\mathbf{x})$ Couple Stress Tensor

(40)

The strain energy of the micropolar theory can be, consequently, formed in the context of nonlocal mechanics, as follows (Eringen, 1999, 2002):

$$W(\mathbf{x}, \mathbf{x}') = a_{ijkl}(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{kl}(\mathbf{x}) + c_{ijkl}(\mathbf{x}, \mathbf{x}')R_{ij}(\mathbf{x}')R_{kl}(\mathbf{x}) + e_{ijkl}(\mathbf{x}, \mathbf{x}')(\gamma_{ij}(\mathbf{x}')R_{kl}(\mathbf{x}) + \gamma_{ij}(\mathbf{x})R_{kl}(\mathbf{x}')) \tag{41}$$

where only a_{ijkl}, , and e_{ijkl} are the material coefficients. The constitutive equations in the context of the nonlocal micropolar theory can be written in the form:

$$\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \gamma_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V (a_{klij}(\mathbf{x}, \mathbf{x}')\gamma_{kl}(\mathbf{x}') + e_{ijkl}(\mathbf{x}, \mathbf{x}')R_{kl}(\mathbf{x}'))dV(\mathbf{x}') \tag{42}$$

$$m_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial R_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V (c_{ijkl}(\mathbf{x}, \mathbf{x}')R_{kl}(\mathbf{x}') + e_{klij}(\mathbf{x}, \mathbf{x}')\gamma_{kl}(\mathbf{x}'))dV(\mathbf{x}') \tag{43}$$

For homogeneous isotropic-linear elastic materials, the constitutive equations can be expressed as follows:

$$\tau_{ij}(\mathbf{x}) = \int_V (\lambda(\mathbf{x}, \mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + \mu(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}') + \mu(\mathbf{x}, \mathbf{x}')\eta(\mathbf{x}, \mathbf{x}')\gamma_{ji}(\mathbf{x}'))dV(\mathbf{x}') \tag{44}$$

$$m_{ij}(\mathbf{x}) = \int_V (\beta_1(\mathbf{x}, \mathbf{x}')R_{rr}(\mathbf{x}')\delta_{ij} + \beta_2(\mathbf{x}, \mathbf{x}')R_{ji}(\mathbf{x}') + \beta_3(\mathbf{x}, \mathbf{x}')R_{ij}(\mathbf{x}'))dV(\mathbf{x}') \tag{45}$$

4.6. Nonlocal relaxed micromorphic theory

Neff et al. (2014) developed the relaxed micromorphic model. This model has been formulated with the aim of simplifying the original micromorphic theory. The original micromorphic theory depends on many kinematical variables and material coefficients. The relaxed micromorphic theory was developed depending on the symmetric part of the relative deformation tensor and the curl of the micro-deformation tensor. The curl of the micro-deformation tensor was introduced as a micro-dislocation tensor. In the relaxed micromorphic theory, the Cauchy-type stress and the microstress tensors are symmetric tensors while the double-stress tensor is a general tensor. The relaxed micromorphic model still can fully describe all the micro-deformations where the equations of motion are still 12 equations. In addition, the constitutive equations of the relaxed micromorphic model for isotropic linear elastic materials depend on 7 material coefficients only. The reduction in the number of the required material coefficients makes the relaxed micromorphic model implementable when compared to the original micromorphic theory. The boundary value problem in the context of the relaxed micromorphic theory can be written, as follows:

1: Particle's DOFs:
$\mathbf{u} = \boldsymbol{\delta}_i u_i$ — Displacement Vector (3-Components)
$\boldsymbol{\psi} = \boldsymbol{\delta}_i \boldsymbol{\delta}_j \psi_{ij}$ — Micro-deformation Tensor (9-Components)
2: Kinematical Variables:
$\mathbf{s} = \frac{1}{2}(\boldsymbol{\psi} + \boldsymbol{\psi}^{\mathrm{T}})$ — Microstrain Tensor (6-Components)
$\boldsymbol{\gamma} = \boldsymbol{\varepsilon} - \mathbf{s}$ — Relative Deformation Tensor (6-Components)
$\boldsymbol{\chi} = -\boldsymbol{\nabla} \times \boldsymbol{\psi}$ — Micro-dislocation Tensor (9-Components)
3: Equations of Motion:
$\boldsymbol{\nabla} \cdot \boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho \ddot{\mathbf{u}}(\mathbf{x})$ — $\forall \mathbf{x} \in V$ (3-Equations)
$-\boldsymbol{\nabla} \times \mathbf{m}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}) - \mathbf{t}(\mathbf{x}) + \mathbf{h}(\mathbf{x}) = \rho_m J \ddot{\boldsymbol{\psi}}(\mathbf{x})$ — $\forall \mathbf{x} \in V$ (9-Equations)
4: Boundary Conditions:
$\mathbf{n} \cdot \boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x})$ or $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})$ — $\forall \mathbf{x} \in S$ (3-BCs)
$\mathbf{n} \cdot \mathbf{m}(\mathbf{x}) = \mathbf{M}(\mathbf{x})$ or $\boldsymbol{\psi}(\mathbf{x}) = \boldsymbol{\Psi}(\mathbf{x})$ — $\forall \mathbf{x} \in S$ (9-BCs)
5: Strain Energy Density Function:
$W(\mathbf{x}) = W(\mathbf{s}(\mathbf{x}), \boldsymbol{\gamma}(\mathbf{x}), \boldsymbol{\chi}(\mathbf{x}))$
6: Constitutive Equations: $\forall \mathbf{x} \in V$
$\mathbf{t}(\mathbf{x}) = \partial W(\mathbf{x}) / \partial \mathbf{s}(\mathbf{x})$ — Microstress Tensor
$\boldsymbol{\tau}(\mathbf{x}) = \partial W(\mathbf{x}) / \partial \boldsymbol{\gamma}(\mathbf{x})$ — Cauchy-type Stress Tensor
$\mathbf{m}(\mathbf{x}) = \partial W(\mathbf{x}) / \partial \boldsymbol{\chi}(\mathbf{x})$ — Double-Stress Tensor

(48)

The micropolar theory depends on 6 material coefficients for isotropic-linear elastic materials, which can be related to those of the micromorphic theory as defined in equation (37).

The kinematical variables and equations of motion of the micropolar theory can be written in the Cartesian and cylindrical coordinates, as follows:

Cartesian Coordinates:
$\mathbf{R} = \delta_i \delta_j \theta_{j,i}$
$\boldsymbol{\gamma} = \delta_i \delta_j u_{j,i} + \delta_i \delta_j \varepsilon_{ijm} \theta_m$
Cylindrical Coordinates:

$$\mathbf{R} = \delta_r \delta_i \theta_{i,r} + \delta_\theta \delta_i \frac{1}{r}\theta_{i,\theta} + \delta_\theta \delta_\theta \frac{\theta_r}{r} - \delta_\theta \delta_r \frac{\theta_\theta}{r} + \delta_z \delta_i \theta_{i,z}$$
$$\boldsymbol{\gamma} = \delta_r \delta_i u_{i,r} + \delta_\theta \delta_i \frac{1}{r} u_{i,\theta} + \delta_\theta \delta_\theta \frac{u_r}{r} - \delta_\theta \delta_r \frac{u_\theta}{r} + \delta_z \delta_i u_{i,z} + \delta_i \delta_j \varepsilon_{ijm} \theta_m \tag{46}$$

Cartesian Coordinates:
$\delta_i \tau_{ji,j} + \delta_i f_i = \delta_i \rho \ddot{u}_i$
$\delta_i m_{ji,j} + \delta_i \hat{\tau}_i + \delta_i \hat{h}_i = \delta_i \rho_m J \ddot{\theta}_i$
Cylindrical Coordinates:

$$\delta_i \tau_{ri,r} + \frac{1}{r}(\delta_\theta \tau_{\theta r} + \delta_i \tau_{ri} - \delta_r \tau_{\theta\theta} + \delta_i \tau_{\theta i,\theta}) + \delta_i \tau_{zi,z} + \delta_i f_i = \delta_i \rho \ddot{u}_i$$
$$\delta_i m_{ri,r} + \frac{1}{r}(\delta_\theta m_{\theta r} + \delta_i m_{ri} - \delta_r m_{\theta\theta} + \delta_i m_{\theta i,\theta}) + \delta_i m_{zi,z} + \delta_i \hat{\tau}_i + \delta_i \hat{h}_i = \delta_i \rho_m J \ddot{\theta}_i \tag{47}$$

In addition, the strain energy density function and the constitutive equations of the relaxed micromorphic theory were defined, as follows (*here, it is written in the context of nonlocal mechanics*):

$$W(\mathbf{x}, \mathbf{x}') = a_{ijkl}(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{kl}(\mathbf{x}) + b_{ijkl}(\mathbf{x}, \mathbf{x}')s_{ij}(\mathbf{x}')s_{kl}(\mathbf{x}) + c_{ijkl}(\mathbf{x}, \mathbf{x}')\chi_{ij}(\mathbf{x}')\chi_{kl}(\mathbf{x}) \tag{49}$$

$$t_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial s_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V b_{klij}(\mathbf{x}, \mathbf{x}')s_{kl}(\mathbf{x}')dV(\mathbf{x}') \tag{50}$$

$$\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \gamma_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V a_{klij}(\mathbf{x}, \mathbf{x}')\gamma_{kl}(\mathbf{x}')dV(\mathbf{x}') \tag{51}$$

$$m_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \chi_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V c_{klij}(\mathbf{x}, \mathbf{x}')\chi_{kl}(\mathbf{x}')dV(\mathbf{x}') \tag{52}$$

where a_{ijkl}, b_{ijkl}, and c_{ijkl} are material coefficients. It should be noted that the coupling between the different kinematical variables is neglected, and the coefficients d_{ijkl}, f_{ijklm} and e_{ijklm} are zeros, in the context of the relaxed micromorphic theory.

For homogeneous-isotropic linear elastic materials, these material coefficients have the form (Neff et al., 2014):

$$\begin{aligned} a_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \\ b_{ijkl} &= \lambda_m\delta_{ij}\delta_{kl} + \mu_m\delta_{ik}\delta_{jl} + \mu_m\delta_{il}\delta_{jk} \\ c_{ijkl} &= \alpha_1\delta_{ij}\delta_{kl} + \alpha_2\delta_{ik}\delta_{jl} + \alpha_3\delta_{il}\delta_{jk} \end{aligned} \tag{53}$$

The substitution of equation (53) into equations (49)-(52) gives:

$$\begin{aligned} W(\mathbf{x},\mathbf{x}') &= \lambda(\mathbf{x},\mathbf{x}')\gamma_{ii}(\mathbf{x}')\gamma_{jj}(\mathbf{x}) + 2\mu(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{ij}(\mathbf{x}) + \lambda_m(\mathbf{x},\mathbf{x}')s_{ii}(\mathbf{x}')s_{jj}(\mathbf{x}) \\ &+ 2\mu_m(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}')s_{ij}(\mathbf{x}) + \alpha_1(\mathbf{x},\mathbf{x}')\chi_{ii}(\mathbf{x}')\chi_{jj}(\mathbf{x}) \\ &+ \alpha_2(\mathbf{x},\mathbf{x}')\chi_{ij}(\mathbf{x}')\chi_{ij}(\mathbf{x}) + \alpha_3(\mathbf{x},\mathbf{x}')\chi_{ij}(\mathbf{x}')\chi_{ji}(\mathbf{x}) \end{aligned} \tag{54}$$

$$t_{ij}(\mathbf{x}) = \int_V (\lambda_m(\mathbf{x},\mathbf{x}')s_{ll}(\mathbf{x}')\delta_{ij} + 2\mu_m(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}'))dV(\mathbf{x}') \tag{55}$$

$$\tau_{ij}(\mathbf{x}) = \int_V (\lambda(\mathbf{x},\mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + 2\mu(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}'))dV(\mathbf{x}') \tag{56}$$

$$m_{ij}(\mathbf{x}) = \int_V (\alpha_1(\mathbf{x},\mathbf{x}')\chi_{ll}(\mathbf{x}')\delta_{ij} + \alpha_2(\mathbf{x},\mathbf{x}')\chi_{ij}(\mathbf{x}') + \alpha_3(\mathbf{x},\mathbf{x}')\chi_{ji}(\mathbf{x}'))dV(\mathbf{x}') \tag{57}$$

Shaat (2018a) developed the reduced micromorphic theory. The reduced micromorphic theory is a generalized model that can capture all *microstructural deformation patterns* but with lower number of *microstructural degrees of freedom*. It was revealed that some of the degrees of freedom considered by the original micromorphic theory are redundant, where the same microstructural deformation can be captured using more than one of the particle's degrees of freedom. The reduced micromorphic theory was developed by modeling all microstructural deformation patterns, and each pattern is captured by only one degree of freedom of the particle. This has resulted in a great reduction in the equations of motion, number of kinematical variables, and number of the material coefficients. Thus, all the microstructural deformations of a multiscale material can be captured using only 9 equations of motion, and only 8 material coefficients are required for an isotropic material (Shaat, 2018a).

The reduced micromorphic theory captures all the microstructural deformation patterns employing 3 displacements and 6 microstrains of the material particle. It can be obtained from the original micromorphic theory be eliminating the micro-rotations of the particle and the gradient of the dilatational microstretch. Thus, the boundary value problem in the context of the reduced micromorphic theory has the form (Shaat, 2018a):

1: Particle's DOFs:		
$\mathbf{u} = \boldsymbol{\delta}_i u_i$	Displacement Vector (3-Components)	
$\mathbf{s} = \boldsymbol{\delta}_i\boldsymbol{\delta}_j s_{ij}$	Microstrain Tensor (6-Components)	
2: Kinematical Variables:		
$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \mathbf{u}\nabla)$	Macrostrain Tensor (6-Components)	
$\boldsymbol{\gamma} = \boldsymbol{\varepsilon} - \mathbf{s}$	Relative Deformation Tensor (6-Components)	
$\boldsymbol{\chi} = \nabla\mathbf{s} - \nabla\mathrm{tr}(\mathbf{s})$	Microstrain Gradient Tensor (9-Components)	
3: Equations of Motion:		
$\nabla\cdot\boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho\ddot{\mathbf{u}}(\mathbf{x})$		$\forall\mathbf{x}\in V$ (3-Equations)
$\nabla\cdot\mathbf{m}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}) - \mathbf{t}(\mathbf{x}) + \mathbf{h}(\mathbf{x}) = \rho_m J\ddot{\mathbf{s}}(\mathbf{x})$		$\forall\mathbf{x}\in V$ (6-Equations)
4: Boundary Conditions:		
$\mathbf{n}\cdot\boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x})$	or $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})$	$\forall\mathbf{x}\in S$ (3-BCs)
$\mathbf{n}\cdot\mathbf{m}(\mathbf{x}) = \mathbf{M}(\mathbf{x})$	or $\mathbf{s}(\mathbf{x}) = \mathbf{S}(\mathbf{x})$	$\forall\mathbf{x}\in S$ (6-BCs)
5: Strain Energy Density Function:		
$W(\mathbf{x}) = W(\mathbf{s}(\mathbf{x}),\boldsymbol{\gamma}(\mathbf{x}),\boldsymbol{\chi}(\mathbf{x}))$		
6: Constitutive Equations:		$\forall\mathbf{x}\in V$
$\mathbf{t}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\mathbf{s}(\mathbf{x})$	Microstress Tensor	
$\boldsymbol{\tau}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\boldsymbol{\gamma}(\mathbf{x})$	Cauchy-type Stress Tensor	
$\mathbf{m}(\mathbf{x}) = \partial W(\mathbf{x})/\partial\boldsymbol{\chi}(\mathbf{x})$	Double-Stress Tensor	

(58)

where $\mathbf{S}$ is a prescribed microstrain at the boundary.

It follows from equation (58) that the strain energy of the reduced micromorphic model depends on two symmetric dyadic tensors, $\boldsymbol{\gamma}$ and $\mathbf{s}$, and a triadic tensor, $\boldsymbol{\chi}$. The triadic tensor, $\boldsymbol{\chi}$, is the gradient of the shear microstrains. With only 9 equations of motion and 9 boundary conditions, the reduced micromorphic theory represents simple and yet efficient model of multiscale materials. It can capture all microstructural deformations of a multiscale material but with lower number of microstructural degrees of freedom. This makes the reduced micromorphic theory implementable when compared to the original micromorphic theory.

According to the reduced micromorphic theory, the strain energy density function can be written in the context of nonlocal mechanics, as

4.7. Nonlocal reduced micromorphic theory

The original micromorphic theory was proposed with the aim of developing a generalized theory that can capture all *microstructural degrees of freedom*. In the context of the micromorphic theory, 12 equations of motion are required to fully describe all degrees of freedom of the material particle. However, the constitutive equations of this original theory depend on many material coefficients; more than 1000 material coefficients are required for anisotropic materials, and 18 material coefficients are required for isotropic materials (Neff et al., 2014, 2017; Barbagallo et al., 2017). This, indeed, limits the applicability of the original micromorphic theory.

follows:

$$W(\mathbf{x},\mathbf{x}') = a_{ijkl}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{kl}(\mathbf{x}) + b_{ijkl}(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}')s_{kl}(\mathbf{x}) + c_{ijklmn}(\mathbf{x},\mathbf{x}')\chi_{ijk}(\mathbf{x}')\chi_{lmn}(\mathbf{x}) + d_{ijkl}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')s_{kl}(\mathbf{x}) + d_{ijkl}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x})s_{kl}(\mathbf{x}') + e_{ijklm}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')\chi_{klm}(\mathbf{x}) + e_{ijklm}(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x})\chi_{klm}(\mathbf{x}') + f_{ijklm}(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}')\chi_{klm}(\mathbf{x}) + f_{ijklm}(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x})\chi_{klm}(\mathbf{x}') \tag{59}$$

and the constitutive equations will have the form:

$$t_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x},\mathbf{x}')}{\partial s_{ij}(\mathbf{x})}dV(\mathbf{x}') = \int_V (b_{klij}(\mathbf{x},\mathbf{x}')s_{kl}(\mathbf{x}') + d_{klij}(\mathbf{x},\mathbf{x}')\gamma_{kl}(\mathbf{x}') + f_{ijklm}(\mathbf{x},\mathbf{x}')\chi_{klm}(\mathbf{x}'))dV(\mathbf{x}') \tag{60}$$

$$\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x},\mathbf{x}')}{\partial \gamma_{ij}(\mathbf{x})}dV(\mathbf{x}') = \int_V (a_{klij}(\mathbf{x},\mathbf{x}')\gamma_{kl}(\mathbf{x}') + d_{ijkl}(\mathbf{x},\mathbf{x}')s_{kl}(\mathbf{x}') + e_{ijklm}(\mathbf{x},\mathbf{x}')\chi_{klm}(\mathbf{x}'))dV(\mathbf{x}') \tag{61}$$

$$m_{ijk}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x},\mathbf{x}')}{\partial \chi_{ijk}(\mathbf{x})}dV(\mathbf{x}') = \int_V (c_{lmnijk}(\mathbf{x},\mathbf{x}')\chi_{lmn}(\mathbf{x}') + e_{lmijk}(\mathbf{x},\mathbf{x}')\gamma_{lm}(\mathbf{x}') + f_{lmijk}(\mathbf{x},\mathbf{x}')s_{lm}(\mathbf{x}'))dV(\mathbf{x}') \tag{62}$$

For homogeneous-isotropic linear elastic materials, the material coefficients, a_{ijkl}, b_{ijkl}, d_{ijkl}, c_{ijklmn}, e_{ijklm}, f_{ijklm}, have the form:

$$\begin{aligned} e_{ijklm} &= f_{ijklm} = 0 \\ a_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \\ b_{ijkl} &= \lambda_m\delta_{ij}\delta_{kl} + \mu_m\delta_{ik}\delta_{jl} + \mu_m\delta_{il}\delta_{jk} \\ d_{ijkl} &= \lambda_c\delta_{ij}\delta_{kl} + \mu_c\delta_{ik}\delta_{jl} + \mu_c\delta_{il}\delta_{jk} \\ c_{ijklmn} &= \xi_1\delta_{ij}\delta_{km}\delta_{ln} + \xi_2\delta_{il}\delta_{jm}\delta_{kn} + \xi_3\delta_{im}\delta_{jn}\delta_{kl} \end{aligned} \tag{63}$$

Consequently, the strain energy density function and the constitutive equations will have the form:

$$W(\mathbf{x}) = \lambda(\mathbf{x},\mathbf{x}')\gamma_{ii}(\mathbf{x}')\gamma_{jj}(\mathbf{x}) + 2\mu(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{ij}(\mathbf{x}) + \lambda_m(\mathbf{x},\mathbf{x}')s_{ii}(\mathbf{x}')s_{jj}(\mathbf{x}) + 2\mu_m(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}')s_{ij}(\mathbf{x}) + \lambda_c(\mathbf{x},\mathbf{x}')(\gamma_{ii}(\mathbf{x}')s_{jj}(\mathbf{x}) + \gamma_{ii}(\mathbf{x})s_{jj}(\mathbf{x}')) + 2\mu_c(\mathbf{x},\mathbf{x}')(\gamma_{ij}(\mathbf{x}')s_{ij}(\mathbf{x}) + \gamma_{ij}(\mathbf{x})s_{ij}(\mathbf{x}')) + \xi_1(\mathbf{x},\mathbf{x}')\chi_{iik}(\mathbf{x}')\chi_{jkj}(\mathbf{x}) + \xi_2(\mathbf{x},\mathbf{x}')\chi_{ijk}(\mathbf{x}')\chi_{ijk}(\mathbf{x}) + \xi_3(\mathbf{x},\mathbf{x}')\chi_{ijk}(\mathbf{x}')\chi_{kij}(\mathbf{x}) \tag{64}$$

$$t_{ij}(\mathbf{x}) = \int_V (\lambda_m(\mathbf{x},\mathbf{x}')s_{ll}(\mathbf{x}')\delta_{ij} + 2\mu_m(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}') + \lambda_c(\mathbf{x},\mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + 2\mu_c(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}'))dV(\mathbf{x}') \tag{65}$$

$$\tau_{ij}(\mathbf{x}) = \int_V (\lambda(\mathbf{x},\mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + 2\mu(\mathbf{x},\mathbf{x}')\gamma_{ij}(\mathbf{x}') + \lambda_c(\mathbf{x},\mathbf{x}')s_{ll}(\mathbf{x}')\delta_{ij} + 2\mu_c(\mathbf{x},\mathbf{x}')s_{ij}(\mathbf{x}'))dV(\mathbf{x}') \tag{66}$$

$$m_{ijk}(\mathbf{x}) = \int_V (\xi_1(\mathbf{x},\mathbf{x}')(\chi_{lkl}(\mathbf{x}')\delta_{ij} + \chi_{llj}(\mathbf{x}')\delta_{ik}) + \xi_2(\mathbf{x},\mathbf{x}')\chi_{ijk}(\mathbf{x}') + \xi_3(\mathbf{x},\mathbf{x}')(\chi_{kij}(\mathbf{x}') + \chi_{jki}(\mathbf{x}')))dV(\mathbf{x}') \tag{67}$$

where ξ_i material coefficients can be related to c_i material coefficients of the original micromorphic theory, as follows:

$$\begin{aligned} \xi_1 &= 2c_2 + c_3 + c_6 \\ \xi_2 &= c_7 + c_9 \\ \xi_3 &= 2c_8 + c_{10} + c_{11} \end{aligned} \tag{68}$$

where $c_3 = c_6$ and $c_{10} = c_{11}$.

4.8. Applicability of nonlocal microcontinuum theories

Materials that would require a multiscale modeling approach and a microcontinuum model are numerous. For example, materials such as polycrystalline materials, composites, superlattices, framework and molecular crystals, and metamaterials are multiscale materials, and many of their properties would depend on the material's microstructure. The microcontinuum theories including the micromorphic, microstretch, micropolar, relaxed micromorphic, and reduced micromorphic theories can effectively model the mechanics of these multiscale materials. These theories can emulate the discrete structure of multiscale materials by the definition of sub-continua for the different scales of the material. The microcontinuum theories as presented in the preceding sections are applicable for two-scale materials with uniform microstructures. For multiple-scale materials, the microcontinuum theory can be sequentially applied starting from the lower scale following the hierarchical structure of the material (Franck et al., 2007; Shaat, 2018a), e.g., Franck et al. (2007) proposed a multi-scale micromorphic model for non-uniform/heterogeneous microstructures.

The original microcontinuum theories were developed in the 1960s–1980s. In the 1990s, these theories have been further extended and applied to model the elasticity and plasticity of materials. During 2003–2006, Chen and co-workers (Chen and Lee, 2003, 2004; Zeng et al., 2006) connected the microcontinuum theories to the lattice mechanics and developed an approach of determining the material coefficients of the micromorphic theory for isotropic-linear elastic materials. In 2007, the micromorphic theory was implemented for multiscale modeling, and a multiscale micromorphic theory was developed (Franck et al., 2007). A micropolar peridynamics model was represented by Chowdhury et al. (2015). In the 2000s, the microcontinuum theories were further implemented to capture the size effects of structures such as plates and shells. For instance, the behavior of plates under thermal stresses were investigated based on the microstretch theory (Nappa, 1995; Ieşan and Quintanilla, 2005). Models of the static bending of linear elastic plates and shells were developed based on the micropolar theory (Sargsyan, 2012; Zozulya, 2018). In addition, micromorphic finite element models have been recently proposed (Isbuga and Regueiro, 2011; Ansari et al., 2016, 2017; Norouzzadeh et al., 2019; Bazdid-Vahdati et al., 2019). The reduced micromorphic theory was used in orthogonal curvilinear elastic problems (El Dhaba, 2020). Recently, a complete set of constitutive relations and field equations for the linear thermoelastic relaxed micromorphic continuum has been presented by Khurana et al. (2020). In addition, a novel micromorphic beam theory has been developed on the basis of the reduced micromorphic theory (Shaat et al., 2020).

Next, we discuss the applicability of the microcontinuum theories to various multiscale materials. Understanding the applicability of these theories would pave the way for their implementations and considerations in the future investigations on the mechanics of multiscale materials.

4.8.1. Applicability to single crystalline materials

Single crystalline materials are the perfect form of the material, which is preferred over their polycrystalline counter parts in many applications. The properties of single crystalline materials are generally superior to those of polycrystalline materials. Single crystalline materials have been used for making advanced micro/nano-scale systems, e.g., mechanical resonators (Li et al., 2003; Li and Chou, 2003; Tao et al., 2014). Mechanical resonators made of single crystalline materials have shown exceptional dynamical characteristics in comparison to the ones made of polycrystalline materials. For example, quality factors exceeding one million were obtained using single crystalline diamond nano-resonators (Tao et al., 2014).

The properties of single crystalline materials are directly related to their atomic structures, the way by which atoms are packed to form a unit cell or a molecule, and the way by which they move inside the crystal structure. Phonons are the mechanical description of the vibrational modes by which the unit cell oscillates inside a crystal structure. Acoustic phonons describe the translational vibrations of the unit cells inside the crystal structure while optical phonons represent the vibrations of the atoms inside a unit cell.

The mechanics of a crystal is robustly related to its phonons. Most of the mechanical properties of crystals can be figured out from their dispersion relations. It was observed that the acoustic phonons of a crystal are conjugate to the macro-displacement fields while the optical phonons are conjugate to the micro-deformation fields of the single crystalline material. Thus, to capture the optical properties of a single crystal using a continuum model, it would be modeled as a multiscale material, and a microcontinuum theory would be used. For instance, in the context of a micromorphic model of single crystalline materials, a single crystal is a continuum with deformable unit cells/molecules. The applicability of the different microcontinuum theories to different crystalline materials has been discussed by Chen et al. (2004) and Shaat and Abdelkefi (2016a).

The choice of the microcontinuum theory to model the mechanics of a certain single crystalline material mainly depends on the crystal structure and the type of bonding between its atoms. For example, the micropolar theory can be used to model the mechanics of molecular and framework crystals. The bonds between atoms inside the molecules of these crystals are relatively strong (covalent or ionic bonds), but molecules themselves are weakly bonded together (Callister and Rethwisch, 2007). Therefore, the molecules are very stiff, and their microstrains can be neglected, i.e., $s_{ij} = 0$. In addition, the difference between the bonding strength inside and outside the molecules is conjugate to a difference between the macro-rotation and micro-rotation fields. This indicates that the micropolar theory can effectively model the mechanics of molecular and framework crystals. As for covalent and ionic crystals, a micromorphic theory is recommended. Because of the difference between the potentials inside and outside their unit cells, the properties of covalent and ionic crystals usually depend on their both acoustic and optical phonons. Optical phonons are modeled by the micro-deformation fields, ψ_{ij} and χ_{ijk}, of the micromorphic theory (Chen et al., 2004; Shaat and Abdelkefi, 2016a).

The mechanics of single crystalline materials would depend on the long-range effects of the interatomic interactions. Experimental investigations on single crystalline materials have shown that the dissociation energies and the interatomic potentials have nonlocal long-range contributions (Dalgarno and Davison, 1967; Leroy and Bernstein, 1970; Mostepanenko and Sokolov, 1993). Furthermore, experimental measurements and theoretical investigations of the elastic and plastic properties of single crystals have demonstrated that these properties are size-dependents (Fleck et al., 1994; Li et al., 2003a; Shaat, 2019), and this has been attributed to the long-range effects of the interatomic interactions. Investigations on the dispersion characteristics of materials have revealed that the elastic properties of single crystalline materials are not only material-dependent but also size-dependent, and the size dependence of their elastic properties is due to non-neighbor interactions inside the crystal structure (Shaat, 2019). To model the long-range effects of single crystalline materials, a nonlocal microcontinuum theory is recommended.

4.8.2. Applicability to advanced materials

Advanced materials are being developed with the promises of enhancing a wide range of engineering and medical applications (Capolino, 2009; Surjadi et al., 2019). Many of these advanced material are artificial metamaterials. Artificial metamaterials possess non-conventional properties that do not exist in natural materials. The design of metamaterials depends on engineering and/or altering the material microstructure to promote certain microscopic fields. These micro-fields – when promoted – impart certain optical, mechanical, electrical, magnetic, and thermal properties, which are exceptional and have never been observed in natural materials. For instance, an early acoustic metamaterial was made of specially-shaped and parallel plates to give metal-lens antennas that can detect meter-wavelengths (Kock, 1946, 1948). In addition, a metamaterial with negative electric permittivity and negative magnetic permeability was developed (Veselago, 1968).

Recently, mechanical metamaterials have been developed with negative Poisson's ratios, negative effective moduli, and/or negative effective densities (Ding et al., 2007; Wu et al., 2007; Grima et al., 2007, 2008; Yang et al., 2008; Lee et al., 2009, 2010; Ren et al., 2015; Qu et al., 2017; Lakes, 2017; Shaat and El Dhaba, 2019). The design of these mechanical metamaterials depended of hierarchical (Gatt et al., 2015), chiral (Grima et al., 2008; Wu et al., 2018), kirigami (Cho et al., 2014; Neville et al., 2016), and lattice unit cells (Yuan et al., 2017; Li et al., 2017b; Coulais et al., 2017). These extraordinary mechanical properties of the metamaterials were attributed to the activation of certain microscopic deformations (Shaat and Wagih, 2020). For instance, a metamaterial with specially designed porous microstructure that activates different microrotation fields gave a giant Poisson's ratio of -12 (Poisson's ratio of natural materials are within the range [0,0.5]) (Caddock and Evans, 1989). In addition, the contrast between the interatomic forces along the different crystallographic directions was the reason behind the negative Poisson's ratios observed for many of the FCC and BCC crystals (Milstein and Huang, 1979; Baughman et al., 1998). Furthermore, lattice metamaterials with specially designed unit cells of curved/straight struts gave giant negative Poisson's ratios and static nonreciprocity (Li et al., 2017b; Coulais et al., 2017; Shaat and Wagih, 2020; Chen et al., 2017; Shaat et al., 2020). Recently, it was observed that promoting the coupling between micro-deformations and macro-deformation fields of composite metamaterials would give composites with giant and/or negative shear moduli (Shaat and El Dhaba, 2019). It had been a challenge to give a metamaterial whose Poisson's ratio is independent of the applied strain (Shaat and Wagih, 2020). Poisson's ratio of a metamaterial being tailorable as the material is strained was a must. Recently, it was revealed that the inhibition of the micro-rotations and the enhancement of the microstrains would give a metamaterial with strain-independent Poisson's ratio (Shaat and Wagih, 2020). Shaat and Wagih (2020) developed a new class of metamaterials that can give giant and strain-independent Poisson's ratio of -16 (the highest Poisson's ratio ever measured).

The quantum physics and mechanics of materials are different from their Newtonian counterparts. The realization of the discrepancies between quantum and Newtonian physics has led to the discovery of topological insulators (Kane and Mele, 2005; Moore, 2010), photonics (Shalaev, 2007; Saleh and Teich, 2007; Jensen and Sigmund, 2011), and phononics (Laude, 2015). Despite bridging the gap between quantum and Newtonian mechanics of materials using models of continuum mechanics would seem impossible, the micromorphic theory has recently revealed topological characteristics, and therefore it has been suggested to bridge this gap (Shaat, 2020a,b). The topological mechanics of micromorphic metamaterials have revealed analogies to topological insulators in achieving mechanical deformation immunity and non-reciprocity (Shaat, 2020a,b; Coulais et al., 2017; Shaat et al., 2020; Shaat et al., 2020).

It follows from the previous discussion that the theory of advanced materials depends on the material design to promote or activate certain microstructural degrees of freedom and/or microstructural deformations. Therefore, models of mechanics of advanced materials should incorporate special measures that can relate the microstructural phenomena to the material's exceptional properties. Unfortunately, these models have not been fully established yet, and the development of these models is an open forum. Existing models depended on the lattice mechanics modeling (Zhou et al., 2012; Li and Wang, 2016; Li et al., 2019; Huang and Zhou, 2019; Ghavanloo and Fazelzadeh, 2019; Aivaliotis et al., 2020) and the effective medium theory (Wu et al., 2007) to model the unit cells of metamaterials. Another approach was the finite element analysis of the metamaterial's unit cell (Wang et al., 2011b; Ai and Gao, 2017; Li et al., 2019; Dong et al., 2019; Fallah et al., 2019; Koutsianitis et al., 2019; Zhang et al., 2020). Whereas models of the unit cells or representative volume elements of metamaterials have observed their exceptional properties, these models are impractical when it comes to the real implementation of these advanced materials. Other models that deal with the material's continuity in relation to its microstructural phenomena are needed and more practical. Therefore, we strongly recommend the microcontinuum theories to model the mechanics of advanced materials. These advanced materials are, by definition, multiscale materials. The microcontinuum theories contain the necessary measures to relate the material's properties to its microstructural deformations and degrees of freedom. More interesting, the microcontinuum theories can reveal topological characteristics of metamaterials, and hence can be used to design mechanical insulators (Shaat, 2020a). Nonetheless, approaches that can relate the 'so-many' material coefficients of these microcontinuum theories to the material's microstructure are not fully established yet. As previously mentioned, this is an open forum to investigate by future studies on the mechanics of advanced materials.

5. Nonlocal mechanics of single-scale materials

Single-scale materials are the ones that do not require a multiscale modeling approach to reveal their mechanics. In other words, the mechanics of a single-scale material is generally independent of its microstructure. Noncrystalline materials, e.g., amorphous materials, are generally without microstructures and can be considered single-scale materials. Nonetheless, materials with microstructures, e.g., polycrystalline materials, could be considered single-scale materials as long as the search is for a material property that is independent of the material's microstructure. Generally speaking, the material can be considered a single-scale material if the material's mechanics of interest is concerned with only one material scale – which frequently is the macroscopic scale.

From the continuum mechanics point of view, a single-scale material can be modeled as a continuum without microstructure. Thus, a single-scale material is a composition of an infinite number of material particles each of which is a mass point that only moves with neither deformations nor rotations. Accordingly, a continuum model for single-scale materials can be obtained from the micromorphic theory by eliminating the micro-deformation tensor, i.e., $\boldsymbol{\psi} = \mathbf{0}$ where $\mathbf{0}$ is the zero tensor, from equation (13), as follows:

1: Particle's DOFs:
$\mathbf{u} = \boldsymbol{\delta}_i u_i$ Displacement Vector (3-Components)
2: Kinematical Variables:
$\boldsymbol{\gamma} = \nabla \mathbf{u}$ Relative Deformation Tensor (9-Components)
3: Equations of Motion:
$\nabla \cdot \boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho \ddot{\mathbf{u}}(\mathbf{x}) \quad \forall \mathbf{x} \in V$ (3-Equations)
$\boldsymbol{\tau}^{\mathbf{A}}(\mathbf{x}) = \mathbf{0}$ where $\boldsymbol{\tau}^{\mathbf{A}}$ is the skew-symmetric part of $\boldsymbol{\tau}$, i.e., $\boldsymbol{\tau}^{\mathbf{A}} = \tau_{jk}\left(\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k\right)$
4: Boundary Conditions:
$\mathbf{n} \cdot \boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x})$ or $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \quad \forall \mathbf{x} \in S$ (3-BCs)
5: Strain Energy Density Function:
$W(\mathbf{x}, \mathbf{x}') = W\left(\boldsymbol{\gamma}(\mathbf{x}), \boldsymbol{\gamma}(\mathbf{x}')\right) = a_{ijkl}(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{kl}(\mathbf{x})$
For homogeneous-isotropic materials:
$W(\mathbf{x}, \mathbf{x}') = \lambda(\mathbf{x}, \mathbf{x}')\gamma_{ii}(\mathbf{x}')\gamma_{jj}(\mathbf{x}) + 2\mu(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}')\gamma_{ij}(\mathbf{x})$
6: Constitutive Equations: $\forall \mathbf{x} \in V$
$\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \gamma_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V \left(a_{ijkl}(\mathbf{x}, \mathbf{x}')\gamma_{kl}(\mathbf{x}')\right) dV(\mathbf{x}')$
For homogeneous-isotropic materials:
$$\tau_{ij}(\mathbf{x}) = \int_V \left(\lambda(\mathbf{x}, \mathbf{x}')\gamma_{ll}(\mathbf{x}')\delta_{ij} + 2\mu(\mathbf{x}, \mathbf{x}')\gamma_{ij}(\mathbf{x}')\right) dV(\mathbf{x}') \tag{69}$$

The continuum model as presented in equation (69) is similar to the common nonlocal theory developed by Eringen (1972) and Eringen and Edelen (1972). The Cauchy stress tensor, $\boldsymbol{\tau}$, in equation (69) is a general tensor of 9-independent components. According to the second equation of motion, i.e., $\boldsymbol{\tau}^{\mathbf{A}} = \mathbf{0}$, the skew-symmetric part of the Cauchy stress tensor vanishes. Therefore, a version of the nonlocal theory (equation (69)) that depends on a symmetric stress tensor can be written, as follows:

1: Particle's DOFs:
$\mathbf{u} = \boldsymbol{\delta}_i u_i$ Displacement Vector (3-Components)
2: Kinematical Variables:
$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u}\nabla)$ Macrostrain Tensor (6-Components)
3: Equations of Motion:
$\nabla \cdot \boldsymbol{\tau}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \rho \ddot{\mathbf{u}}(\mathbf{x}) \quad \forall \mathbf{x} \in V$ (3-Equations)
4: Boundary Conditions:
$\mathbf{n} \cdot \boldsymbol{\tau}(\mathbf{x}) = \mathbf{T}(\mathbf{x})$ or $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) \quad \forall \mathbf{x} \in S$ (3-BCs)
5: Strain Energy Density Function:
$W(\mathbf{x}, \mathbf{x}') = W\left(\boldsymbol{\gamma}(\mathbf{x}), \boldsymbol{\gamma}(\mathbf{x}')\right) = a_{ijkl}(\mathbf{x}, \mathbf{x}')\varepsilon_{ij}(\mathbf{x}')\varepsilon_{kl}(\mathbf{x})$
For homogeneous-isotropic materials:
$W(\mathbf{x}, \mathbf{x}') = \lambda(\mathbf{x}, \mathbf{x}')\varepsilon_{ii}(\mathbf{x}')\varepsilon_{jj}(\mathbf{x}) + 2\mu(\mathbf{x}, \mathbf{x}')\varepsilon_{ij}(\mathbf{x}')\varepsilon_{ij}(\mathbf{x})$
6: Constitutive Equations: $\forall \mathbf{x} \in V$
$\tau_{ij}(\mathbf{x}) = \int_V \frac{\partial W(\mathbf{x}, \mathbf{x}')}{\partial \varepsilon_{ij}(\mathbf{x})} dV(\mathbf{x}') = \int_V \left(a_{ijkl}(\mathbf{x}, \mathbf{x}')\varepsilon_{kl}(\mathbf{x}')\right) dV(\mathbf{x}')$
For homogeneous-isotropic materials:
$$\tau_{ij}(\mathbf{x}) = \int_V \left(\lambda(\mathbf{x}, \mathbf{x}')\varepsilon_{ll}(\mathbf{x}')\delta_{ij} + 2\mu(\mathbf{x}, \mathbf{x}')\varepsilon_{ij}(\mathbf{x}')\right) dV(\mathbf{x}') \tag{70}$$

Now, the nonlocal theory as presented in equation (70) is the one developed by Eringen (1972) and Eringen and Edelen (1972). This nonlocal theory – which is now called Eringen's nonlocal theory – is identical to the classical theory of continuum mechanics except the formulation of the Cauchy stress tensor $\boldsymbol{\tau}$, which is commonly called the nonlocal stress. The nonlocal stress is formed depending on both local and nonlocal interactions, and this is what imparts the global balance requirement to the nonlocal theory.

Based on the preceding formulation, we can conclude that the commonly used nonlocal theory is a special model of the nonlocal micromorphic theory, and it can only be used to model single-scale materials. The nonlocal theory can capture phenomena associated with the material's macroscopic scale. In addition, it can only capture acoustic properties of materials, e.g., acoustic phonons. However, optical phonons are unpredictable using the nonlocal theory. In addition, the nonlocal theory can reflect some size-dependent effects, which are only related to the material's acoustic dispersions. However, it cannot capture any size-dependent phenomenon that is related to the material's microstructure, because it completely disregards the material's microstructure.

5.1. Strain and rotation gradient theories

Various strain gradient theories have been developed. Mindlin (1965) developed a second-order strain gradient theory in which the strain energy is a function of the strain and its first and second gradients. Then, Mindlin and Eshel (1968) reduced this theory to a first-order strain gradient theory in which the strain energy is a function of the strain and its first gradient only. Subsequently, a simplified version of the first strain gradient theory has been developed (Altan and Aifantis, 1992). Another version – called the modified strain gradient theory – was also proposed (Lam et al., 2003).

In addition to the strain gradient theories, continuum theories that depend on the gradients of the rotation tensor have been formulated. The classical one is the couple stress theory, which was developed by Mindlin and Tiersten (1962) and Toupin (1962). The strain energy function of the classical couple stress theory depends on the strain and the gradient of the rotation tensor. The rotation gradient models the body couples acting on the material particles. This gives the couple stress theory some similarity to the micropolar theory but with no difference between the micro-rotations and the macro-rotations. In 2002, Yang et al. (2002) claimed that the skew-symmetric part of the rotation gradient tensor has no contribution to the material's deformation. Therefore, they developed the modified couple stress theory that depends on the symmetric part of the rotation gradient. Subsequently, Hadjesfandiari and Dargush (2011) claimed that the material deformation depends only on the skew-symmetric part of the rotation gradient tensor and proposed the consistent couple stress theory. Recently, it was demonstrated that the modified and the consistent couple stress theories are physically infeasible (Shaat, 2015a; Neff et al., 2016). It was demonstrated that all the components of the rotation gradient tensor contribute to the material's deformation.

The strain gradient and the couple stress theories can be generally derived from the nonlocal theory presented in equation (69) by expanding the displacement gradient tensor, $\nabla\mathbf{u}(\mathbf{x}')$, about point $\mathbf{x}$, as follows (Shaat, 2017):

$$\nabla\mathbf{u}(\mathbf{x}') = \left(\mathbf{I} + (\mathbf{x}-\mathbf{x}').\,\nabla + \frac{1}{2}((\mathbf{x}-\mathbf{x}').\,\nabla)^2 + \frac{1}{6}((\mathbf{x}-\mathbf{x}').\,\nabla)^3 + \frac{1}{24}((\mathbf{x}-\mathbf{x}').\,\nabla)^4 + \dots\right)\nabla\mathbf{u}(\mathbf{x}) \tag{71}$$

Then, the displacement gradient tensor, $\nabla\mathbf{u}(\mathbf{x})$, is decomposed into its symmetric part (macrostrain tensor $\boldsymbol{\varepsilon}(\mathbf{x})$) and skew-symmetric part (rotation tensor $\boldsymbol{\theta}(\mathbf{x})$), which gives:

$$\begin{aligned}\nabla\mathbf{u}(\mathbf{x}') = &\left(\boldsymbol{\varepsilon}(\mathbf{x}) + (\mathbf{x}-\mathbf{x}').\,\nabla\boldsymbol{\varepsilon}(\mathbf{x}) + \frac{1}{2}((\mathbf{x}-\mathbf{x}').\,\nabla)^2\boldsymbol{\varepsilon}(\mathbf{x}) + \frac{1}{6}((\mathbf{x}-\mathbf{x}').\,\nabla)^3\boldsymbol{\varepsilon}(\mathbf{x}) + \frac{1}{24}((\mathbf{x}-\mathbf{x}').\,\nabla)^4\boldsymbol{\varepsilon}(\mathbf{x}) + \dots\right)\\ &+\left(\boldsymbol{\theta}(\mathbf{x}) + (\mathbf{x}-\mathbf{x}').\,\nabla\boldsymbol{\theta}(\mathbf{x}) + \frac{1}{2}((\mathbf{x}-\mathbf{x}').\,\nabla)^2\boldsymbol{\theta}(\mathbf{x}) + \frac{1}{6}((\mathbf{x}-\mathbf{x}').\,\nabla)^3\boldsymbol{\theta}(\mathbf{x}) + \frac{1}{24}((\mathbf{x}-\mathbf{x}').\,\nabla)^4\boldsymbol{\theta}(\mathbf{x}) + \dots\right)\end{aligned} \tag{72}$$

A strain gradient theory that depends on the strain gradients can be obtained by dropping the high-order gradients of the rotation tensor from equation (72) and substituting the result into equation $(69)_6$ and then into equation $(69)_3$, which gives the following field equation for a homogeneous-isotropic linear elastic material (Shaat, 2017):

$$\left[\lambda_0 + 2\mu_0 + \sum_{n=3}^{\infty}(a_n + b_n)\nabla^{(n-1)}\right](\nabla\nabla\cdot\mathbf{u}) - \frac{1}{2}\left[2\mu_0 + \sum_{n=3}^{\infty} b_n\nabla^{(n-1)}\right](\nabla\times\nabla\times\mathbf{u}) + \mathbf{f} = \rho\ddot{\mathbf{u}}\ \textit{i. e.}\,,\ n = 3{,}5, \dots \tag{73}$$

where

$$\begin{aligned}a_n &= \frac{\lambda_0}{\prod(n-1)}\int_V((\mathbf{x}-\mathbf{x}'))^{(n-1)}\lambda(\mathbf{x}-\mathbf{x}')dV(\mathbf{x}')\ \textit{for}\ n = 3{,}5, \dots\\ b_n &= \frac{2\mu_0}{\prod(n-1)}\int_V((\mathbf{x}-\mathbf{x}'))^{(n-1)}\mu(\mathbf{x}-\mathbf{x}')dV(\mathbf{x}')\ \textit{for}\ n = 3{,}5, \dots\end{aligned} \tag{74}$$

where λ_0 and μ_0 are the Lamé constants of the classical mechanics, and a_n and b_n are additional material coefficients.

To derive the couple stress theory, the high-order gradients of the strain tensor are eliminated from equation (72) and the result is sequentially substituted into equation $(69)_6$ and $(69)_3$, and the following field equation is obtained:

$$(\lambda_0 + 2\mu_0)(\nabla\nabla\cdot\mathbf{u}) - \frac{1}{2}\left[2\mu_0 + \frac{1}{2}\sum_{n=3}^{\infty} b_n\nabla^{(n-1)}\right](\nabla\times\nabla\times\mathbf{u}) + \mathbf{f} = \rho\ddot{\mathbf{u}}\quad \textit{i. e.}\,,\ n = 3{,}5, \dots \tag{75}$$

It follows from the preceding formulations that the strain gradient and the couple stress theories are nonlocal-type theories. These theories utilize length scales/material coefficients to implicitly express the nonlocal field (Kunin, 1968, 1982, 1983, 1984; Shaat, 2017; Maugin, 2017; Tuna and Trovalusci, 2020; Tuna et al., 2020). In general, theories that depend on gradients of the kinematical variables, parameters of the coupling between different degrees of freedom, and multiple length scales reveal nonlocal effects implicitly (Kunin, 1984). In comparison to the explicit nonlocal theories, which depend on nonlocal kernels, the implicit nonlocal theories (e.g., strain gradient and couple stress theories) are of weak nonlocal characters (Kunin, 1968, 1982, 1983, 1984; Shaat, 2017; Tuna and Trovalusci, 2020). Nonetheless, these implicit nonlocal theories provide simplifications over the explicit nonlocal theories. The gradient theories yield differential equations of motion with physically equivalent solutions to the explicit nonlocal theories (Maugin, 1979). On the one hand, the natural boundary conditions of the explicit nonlocal theories depend on integral operators of the strain fields at points near the boundary surface. On the other hand, the natural boundary conditions of the gradient theories depend on the strain field at the boundary surface and its derivatives. The natural boundary conditions of the gradient theories can be obtained by substituting equation (72) into equation $(69)_6$ then equation $(69)_4$. Nonetheless, the formulation of the gradient theories based on variational principles is highly recommended to explore the natural boundary conditions in relation to the essential boundary conditions (Polizzotto, 2003a, 2003b; Madeo et al., 2016). Whereas the boundary conditions of the gradient theories can be easy formed, the physics behind these conditions is still an open venue of research.

Table 1
Review on the implementations of the strain gradient and couple stress theories for the size-dependent mechanics of micro/nano-structures.

Scope of study	Nonlocal-Type Theories		References
Static Bending	Couple Stress Theory	Beam	Park and Gao (2006); Asghari et al. (2010); Şimşek and Reddy (2013a); Şimşek et al. (2013); Farokhi and Ghayesh (2015); Attia and Emam (2018)
		Plate	Tsiatas (2009); Reddy and Berry (2012); Shaat et al. (2014); Li and Pan (2015); He et al. (2015)
		Shell	Arefi (2018); Thanh et al. (2019)
	Strain Gradient Theory	Beam	Papargyri-Beskou et al. (2003a); Kong et al. (2009); Wang et al. (2010); Ansari et al. (2013); Zhang et al. (2013); Challamel (2013); Joseph et al. (2017)
		Plate	Lazopoulos (2009); Wang et al. (2011a); Li et al. (2014); Shaat and Abdelkefi (2016b)
		Shell	Balobanov et al. (2019)
Structure Stability Analysis	Couple Stress Theory	Beam	Nateghi et al. (2012); Abbasnejad et al. (2013); Şimşek and Reddy (2013b); Mohammad Abadi and Daneshmehr (2014); Shaat and Mohamed (2014); Shaat and Abdelkefi (2015a,b, 2017a); Attia and Emam (2018)
		Plate	Jung et al. (2014); He et al. (2015); Nguyen et al. (2017)
		Shell	Lou et al. (2016); Gholami et al. (2016); Zeighampour and Shojaeian (2019)
	Strain Gradient Theory	Beam	Papargyri-Beskou et al. (2003a); Ansari et al. (2013); Challamel (2013); Li and Hu (2015); Gholami et al. (2016)
		Plate	Wang et al. (2011a); Jamalpoor and Hosseini (2015); Mirsalehi et al. (2017); Mirkalantari et al. (2017)
		Shell	Papargyri-Beskou and Beskos (2009); Gholami et al. (2016)
Free Vibration	Couple Stress Theory	Beam	Ma et al. (2010); Asghari et al. (2010); Şimşek and Reddy (2013a); Chen and Li (2013); Attia and Emam (2018)
		Plate	Yin et al. (2010); Jomehzadeh et al. (2011); Ke et al. (2012); Li and Pan (2015); He et al. (2015); Nguyen et al. (2017)
		Shell	Tadi Beni et al. (2015); Razavi et al. (2017); Wang et al. (2020)
	Strain Gradient Theory	Beam	Papargyri-Beskou et al. (2003b); Wang et al. (2010); Ansari et al. (2013); Zhang et al. (2013); Challamel (2013)
		Plate	Wang et al. (2011a); Ansari et al. (2013); Shaat and Abdelkefi (2016b)
		Shell	Zhang et al. (2015); Gholami et al. (2016); Zeighampour and Shojaeian (2019)

These nonlocal theories would find so many important implementations like modeling the mechanics of various single-scale materials. Nonetheless, the utilization of the nonlocal theories in the literature, to a great extent, was confined to modeling the mechanics of micro/nano-structures. The nonlocal theories are very simple when compared to the microcontinuum theories. These theories use lower number of equations of motion and lower number of material coefficients than the microcontinuum theories. In addition, some of the size-dependent mechanics of micro/nano-structures can be captured using the nonlocal theories. Therefore, the nonlocal theories were intensively used in the field of mechanics of micro/nano-structures. Table 1 summarizes some important implementations of the strain gradient and couple stress theories in the field of the mechanics of micro/nano-structures. It follows from Table 1 that extensive interest of using the strain gradient and couple stress theories has been restricted to beam-like and plate-like structures. A few studies have been carried out on the mechanics of micro/nano-shells using the strain gradient and couple stress theories.

6. Eringen's nonlocal model: paradoxes and limitations

Recently, several questions have been raised about the applicability and limitations of the current form of the nonlocal theory, which has been developed and extended by Eringen (1972, 1983). In addition, certain paradoxical results in existing nonlocal solutions of nonlocal boundary value problems have been discussed in the literature. In this section, a comprehensive survey of the limitations and paradoxes of Eringen's nonlocal model is presented.

6.1. Nonlocal field problems and the global balance requirement

A continuum is in equilibrium if and only if its body forces are balanced globally with the external surface tractions and its inertia forces. In classical mechanics, the global balance requirement automatically postulates a local balance. In the classical mechanics, a balance law is valid not only for the entire elastic body but also for each point belongs to the elastic domain (Eringen, 1972; Eringen and Edelen, 1972). However, the local balance would be violated in some occasions, e.g., continua under nonlocal residual fields, and yet the continuum is in equilibrium only if it is under a global balance. Because of a nonlocal residual field, a global balance is an essential requirement for the equilibrium of continua. In nonlocal mechanics, the balance laws are postulated to be global (nonlocal) and violated locally (Eringen, 1972; Eringen and Edelen, 1972; Huang, 2004). In the current version of the nonlocal theory as developed by Eringen, the equilibrium equations are similar to those of the classical theory, and the nonlocal residuals are manifested in the constitutive equations. Therefore, Eringen's nonlocal constitutive equations are functionals that account for the nonlocal residuals (or the long-range effect). The global balance requirement of nonlocal mechanics can be understood by considering the discrete system of particles in Fig. 1. The local balance of a particle n, $\sum_{m=1}^{N_m}\mathbf{F}_{nm}-\bar{m}\ddot{\mathbf{u}}\neq 0$, is violated due to the nonlocal residual field, $\sum_{j=1}^{N_j}\mathcal{F}_{nj}$ (see equation (6)).

Based on this discussion, the global balance is an essential requirement that should be achieved by any nonlocal continuum theory. In other words, a nonlocal continuum theory is valid only and only if its equations of motions and constitutive equations satisfy the global balance requirement. Based on this observation, some of the recently proposed models of the nonlocal mechanics have to be revised, as these models violate the global balance requirement.

6.2. Solutions of nonlocal field problems

The original nonlocal elasticity developed by Eringen (called nonlocal integral elasticity) utilizes nonlocal integral constitutive equations where the stress is obtained from the integral of the strain field at a point times a kernel function over the whole body. The direct use of the nonlocal integral constitutive equations in boundary value problems results in integro-partial differential field equations, which are difficult to be analytically solved, especially for mixed boundary value problems (Eringen, 1983). In addition, the difficulties of applying the natural boundary conditions in the context of a nonlocal integral model were discussed in some studies (e.g., Polizzotto, 2001; Shaat, 2015b). Therefore, semi-analytical and computational methods were used to solve boundary value problems formed by the integral nonlocal constitutive equations. Here, these methods are briefly reviewed.

Pisano and Fuschi (2003) developed a closed form solution for nonlocal elastic bars under tension by transforming the Fredholm integral equation of the integral nonlocal model to a Volterra integral equation. A similar problem was considered by Zhu and Dai (2012) using both semi-analytical and numerical solutions. Consistent nonlocal finite element procedures for the nonlocal integral elasticity of non-homogeneous elastic continua were also developed (Marotti de Sciarra, 2008, 2009, 2013). In addition, two-dimensional nonlocal finite element models for homogeneous and non-homogeneous materials were developed (Pisano et al., 2009). Abdollahi and Boroomand (2013) developed low-residual approximate solutions for one- and two-dimensional nonlocal integral problems. Then, they introduced a boundary layer method for the nonlocal integral elasticity (Abdollahi and Boroomand, 2014). Khodabakhshi and Reddy (2015) provided a general finite element formulation for a two-phase integro-differential form of Eringen's nonlocal model. An iterative nonlocal residual approach was used to study the static bending of nano-sized Kirchhoff plates having both essential and natural boundary conditions (Shaat, 2015b). The buckling, bending, vibration, and dynamics of nanobeams using finite element models of the integral nonlocal theory were investigated in various studies (e.g., Koutsoumaris et al., 2017; Eptaimeros et al., 2018; Naghinejad and Ovesy, 2018, 2019). Tuna and Kirca (2016a, 2016b) claimed exact analytical solutions of the integral nonlocal models of the static and vibration of Euler-Bernoulli and Timoshenko beams. Approximate solutions of Eringen's nonlocal beam models were derived based on finite element formulations of the static bending, buckling and free vibration of nanobeams (Tuna and Kirca, 2017a). In addition, a perturbation approach was implemented to derive approximate solutions of the one-dimensional elastostatic nonlocal problem (Eroglu, 2020). Furthermore, various computational methods such as boundary element method (Schwartz et al., 2012), nonlocal Ritz method (Faroughi et al., 2017) and mesh-free method (Abdollahi and Boroomand, 2019) have been used to solve one-, two- and three-dimensional nonlocal integral elasticity problems.

In 1983, Eringen showed that it is possible to convert the integral nonlocal equations into an equivalent differential nonlocal equations. The differential nonlocal field equations are partial differential equations of the displacement field. Therefore, the differential nonlocal model was adopted in a long list of studies to investigate the mechanics of micro/nano-structures, e.g., beams, plates, and shells. Many of the studies that used the differential nonlocal model were reviewed by Rafii-Tabar et al. (2016), Farajpour et al. (2018), Wu and Yu (2019).

It was revealed that the nonlocal differential model and the nonlocal integral model are consistent and give equivalent results over unbounded continuous domains. Nonetheless, some paradoxes and inconsistencies were observed between the results of these two nonlocal models[1] when applied for bounded continuous domains. Next, the limitations and the complications of finding exact solutions of Eringen's nonlocal models for both unbounded and bounded domains are discussed.

6.2.1. Unbounded domains: material dispersions & limitations

Propagation of waves within various materials is normally dispersive. This means that each component of wave travels with a different velocity. Generally, waves with smaller wavenumbers travel faster than waves with larger wavenumbers (Askes et al., 2008; Jakata and Every, 2008). The dispersion of waves in materials would be due to the microstructural defects, the heterogeneities, the deformations, or the discrete structure of the material (Shaat, 2019). A non-neighbor interaction is another factor that would cause a wave dispersion (Shaat, 2019).

Various atomistic- and continuum mechanics-based models have been utilized to predict the dispersive phonons of materials. The applicability of each one of these models robustly depends on its ability to reflect the exact dispersive phonons of the considered material (Shaat, 2017). The classical continuum mechanics gives the acoustic dispersion curves non-dispersive, i.e., the phase velocity is constant and does not depend on the wavelength. This is not true for all materials when operate at high frequencies, as the envelope of an acoustic wave depends on the wavelength and is distorted as the wave travels. Therefore, the dispersion characteristics of materials can be better captured by the nonlocal theory.

In various studies, Eringen used the nonlocal theory to model the dispersion of materials. Utilizing an attenuation function of the type Green function of a linear differential operator, Eringen derived the dispersion relations in the form (Eringen, 2002):

$$\omega_L = \sqrt{\frac{(\lambda_0+\mu_0)k^2}{\rho(1+\xi_1^2k^2+\xi_2^4k^4)}}$$
$$\omega_T = \sqrt{\frac{\mu_0 k^2}{\rho(1+\xi_1^2k^2+\xi_2^4k^4)}} \tag{76}$$

where ω_L and ω_T are the frequencies of the longitudinal and transvers acoustic waves, λ_0 and μ_0 are the Lamé constants, ρ is the mass density, ξ_1 and ξ_2 are the nonlocal parameters, and k is the wavenumber.

Whereas equation (76) gives the frequencies of acoustic waves, which are dispersive as long as $\xi_1 > 0$ and $\xi_2 > 0$, limitations were observed when using the nonlocal dispersion relations defined in equation (76). Shaat (2017) and Shaat and Abdelkefi (2017b) revealed that the dispersion relations of some materials, e.g., silicon, gold, and platinum, cannot be captured by Eringen's nonlocal model (equation (76)). Eringen's nonlocal model assumed the same attenuation function for all material moduli, and, therefore, it cannot achieve the simultaneous fitting of the longitudinal and transverse dispersion curves of many materials.

In addition to its limitations to model the dispersion curves of some materials, Eringen's nonlocal model (equation (76)) has limitations to reveal the nonlocal effects on the material's Poisson's ratio. Poisson's ratio is generally non-constant and dependent on the material dispersion, as follows (Shaat, 2019):

$$\nu = \frac{1-2R^2}{2(1-R^2)} \tag{77}$$

where $R = \frac{\omega_T}{\omega_L}$. The experiment gave R non-constant and function of the wavenumber, k (Shaat, 2019). However, the substitution of Eringen's dispersion relations (76) into equation (77) gives Poisson's ratio by $\nu = \frac{\lambda_0}{2(\lambda_0+\mu_0)}$, which is the same Poisson's ratio as defined based on the classical theory of linear elasticity. This indicates that Poisson's ratio is independent of the nonlocal field in the context of Eringen's nonlocal model, which contradicts with the recent observations on the size-dependence of Poisson's ratio (Lakes, 2016; Ahadi and Melin, 2016; Faroughi and Shaat, 2018; Shaat and El Dhaba, 2019).

To overcome the aforementioned limitations of Eringen's nonlocal model, Shaat (2017) developed the general nonlocal theory, which assumes independent nonlocal fields for the different material coefficients. The general nonlocal theory gave better ability to model the material dispersion characteristics. In the following subsection, we briefly review the general nonlocal elasticity.

6.2.1.1. General nonlocal theory. The general nonlocal theory (Shaat, 2017) differs from Eringen's nonlocal theory in using different nonlocal kernels for the different material moduli. In the context of the general nonlocal theory, an isotropic-linear elastic material exhibits two different nonlocal fields for its normal and shear strains. With this, the general nonlocal theory can simultaneously fit both the longitudinal and transverse acoustic dispersion curves of various materials (Shaat, 2017; Shaat and Abdelkefi, 2017b). In addition, it effectively models the nonlocal character of Poisson's ratio of auxetic and non-auxetic materials (Faroughi and Shaat, 2018; Shaat, 2018c). The general nonlocal theory has a further ability to capture both softening and hardening behaviors of materials (Shaat, 2017, 2018c). This theory has

[1] We criticize dealing with the integral and different forms of the nonlocal theory as two different nonlocal models. They are – in fact – represent the same nonlocal model but with two different representations of the field equations. Despite this believe, here we described these two forms as it is done in the literature.

shed light on many aspects of nonlocal mechanics, especially when it has been used for some engineering applications (Faroughi and Shaat, 2018; Shaat, 2019). The research on the general nonlocal theory is still in the primary stages, and it is hoped that it will expand in the near future.

6.2.2. Bounded domains: paradoxes and exact solutions

There is a long-standing debate about whether the nonlocal boundary value problems admit or do not admit exact solutions (Romano et al., 2017a; Tuna and Kirca, 2017b). An exact solution of the nonlocal boundary problem requires solving the constitutive equation $(70)_6$ for the strain field after determining the stress field from the equilibrium equation $(70)_3$. On the one hand, the stress field is obtained a linear or a nonlinear field from the equilibrium equation $(70)_3$. For example, for elastostatic problems with no body forces, the stress field is a linear field. On the other hand, the stress field is a nonlinear field according to the constitutive equation $(70)_6$. For any given strain field, ε, the stress field, τ, is nonlinear due to the integration. Because of the inconsistency between the stress field obtained from the equilibrium equation and the stress field obtained from the constitutive equation, the nonlocal boundary value problem (equation (70)) admits no exact solution. This intuitive result comes in agreement with the recent conclusions in Ref. (Romano et al., 2017a). They attempted the exact solution of the nonlocal field problem, which required – with no other option – converting the nonlocal integral model (equation (70)) to an equivalent nonlocal differential model. This nonlocal differential model is the same as Eringen's nonlocal differential model (Eringen, 1983) but with extra constitutive boundary conditions (the constitutive boundary conditions were first represented by Benvenuti and Simone, 2013). They showed some cases of inconsistencies between the natural boundary conditions of the nonlocal equilibrium equation and the boundary conditions of the constitutive equation (Romano et al., 2017a).

The nonlocal integral field problem (equation (70)) can be numerically solved using finite element models (Marotti de Sciarra, 2008, 2009, 2013) or iterative methods (Shaat, 2015b, 2018c,d; Eroglu, 2020). Numerical methods can handle the integral form of the nonlocal model with no need to transform it into a differential form. These methods, however, give approximate solutions of the nonlocal boundary value problems. The analytical-exact solutions of the nonlocal boundary value problems require – with no choice – converting the nonlocal integral model to an equivalent nonlocal differential model. In an attempt to derive analytical solutions of various nonlocal boundary value problems, the nonlocal differential model was implemented in a long list of articles.

In recent years, several inconsistencies and paradoxes between results of the analytical solutions and numerical solutions of the nonlocal boundary value problems have been observed. In 2003, the first attempt to analytically solve the static bending of nonlocal beams was carried out by Peddieson et al. (2003). This analytical solution required implementing Eringen's differential nonlocal model (Eringen, 1983). The results of the nonlocal deflections of cantilever beams subjected to concentrated forces were obtained identical to the local ones and do not depend on the nonlocal field. These paradoxes were first discussed by Challamel and Wang (2008) and Li et al. (2015). In addition, the analytical solutions of the nonlocal model were obtained identical to those of the classical model for many nonlocal boundary value problems including clamped-clamped beams subjected to uniform or linear distributed loads, cantilever beams subjected to concentrated couples and/or concentrated forces, and fixed-fixed rods subjected to uniform and/or linear distributed axial loads (Benvenuti and Simone, 2013; Barretta et al., 2014). Furthermore, discrepancies were observed in the analytical solutions of the free vibration of nonlocal beams, especially cantilever beams (Lu et al., 2006; Tashakorian et al., 2018). It was observed that the fundamental frequency of a cantilever beam increases due to the nonlocal field while the other higher-mode frequencies decrease due to the nonlocal field. Further paradoxes were discussed by Benvenuti and Simone (2013) and Fernández-Sáez et al. (2016).

Various attempts have been made during the last decade to propose approaches that would eliminate the paradoxes of Eringen's nonlocal models. For instance, a mixed local-nonlocal constitutive equation was suggested to eliminate the ill-posedness and the paradoxes of Eringen's nonlocal models (Challamel and Wang, 2008; Romano et al., 2017a; Zhu and Li, 2017; Meng et al., 2018). In addition, a discrete microstructural model was proposed to properly derive the boundary conditions of cantilever beams (Challamel et al., 2014). For special cases of loading and boundary conditions, and with some reasoning approximations, solutions of the nonlocal field problems could be derived. However, some of these approaches have been recently discredited, and it was demonstrated that Eringen's nonlocal model is ill-posed and admits no exact solutions (Romano et al., 2017a). This has been revealed by observing the inconsistencies between the natural boundary conditions of the nonlocal equilibrium equation and the boundary conditions of the constitutive model, which generate upon the transformation of the integral nonlocal model to a differential nonlocal model (Benvenuti and Simone, 2013; Fernández-Sáez et al., 2016; Romano et al., 2017b; Shaat, 2018d). Nonetheless, some studies on the subject have proposed well-posed nonlocal models that showed consistencies between the natural boundary conditions of the nonlocal equilibrium equation and the boundary conditions of the constitutive model (Romano et al., 2017a; Maneshi et al., 2020; Evgrafov and Bellido, 2019; Patnaik et al., 2020). As a matter of fact, Eringen's integral nonlocal models – in general – have no exact solutions. More precisely, the exact solutions of the integral nonlocal models are generally unpredictable. These exact solutions exist at the limit of some special functions that evaluate the nonlocal field (Tuna and Kirca, 2017b). This can be understood by comparing the stress field obtained from the constitutive law for a given strain field and the stress field obtained from the equilibrium equation for the same given strain field. These two stress fields are not exactly the same, but one can evaluate and approximate the other after some carful and reasoning analytical manipulations (see Tuna and Kirca, 2017b).

Given the aforementioned paradoxes and complications of finding analytical-exact solutions of the nonlocal boundary value problems, approximate solutions using numerical methods are strongly recommended. The exact solution of Eringen's nonlocal boundary value problem can be approached analytically for special loading and boundary conditions by reasoning manipulations of the Dirac-delta and Heaviside functions at the boundary (Tuna and Kirca, 2017b). In addition, finite element nonlocal models gave consistent and feasible solutions of various nonlocal field problems. Furthermore, models that depend on the iterative nonlocal residual elasticity are computationally effective (Shaat, 2018c). Using the iterative nonlocal residual elasticity, solutions of the nonlocal field problem can be obtained by solving its local field counterpart and correcting the local field for a nonlocal residual field (Shaat, 2015b, 2018c,d). The iterative nonlocal residual elasticity was proposed to solve the paradoxes of the nonlocal models, and it has been used to solve various nonlocal field problems including static bending, vibration, and buckling of beams and plates (Shaat, 2015b, 2018c; Shaat et al., 2017c).

6.2.3. Fractional calculus for nonlocal mechanics

At the end of this review, it is worthy to refer the readers to the implementation of the fractional calculus in the nonlocal mechanics. The fractional calculus is an effective mathematical approach that can describe physical phenomena in continuous matter using fractional derivatives, which map some nonlocal influence (Oldham and Spanier, 1974; Poldubny, 1999; Sumelka, 2017; Sidhardh et al., 2020). The fractional calculus enables obtaining the stress and strain fields in a fractional-space/domain, and up on the transformation of these fractional-fields into the original-space/domain, the nonlocal fields are obtained. Applications of the fractional calculus into the nonlocal continuum mechanics were early introduced by Gubenko (1957), Rostovtsev (1959) and recently by Lazopoulos (2006), Di Paola et al. (2009a,b), Cottone et al. (2009a,b), Drapaca and Sivaloganathan (2012), Challamel et al. (2013), Atanackovic et al. (2014a,b), Tarasov (2014), Sumelka and Blaszczyk (2014), Carpinteri et al. (2014) and Lazopoulos and Lazopoulos (2016, 2020a,b). For instance, 1D and 2D nonlocal elasticity problems have been formulated adopting the Λ-fractional derivative and the corresponding Λ-space (Lazopoulos and Lazopoulos, 2020a,b). The Λ-fractional beam bending, buckling, and post-buckling as well as the Λ-fractional 2D plane elasticity fields were formulated in Λ-domain, and the corresponding nonlocal fields were, then, determined by transforming the fractional fields into the original domain (Lazopoulos and Lazopoulos, 2020a,b). These studies have showed the potential of the fractional calculus for solving nonlocal field problems; the subject that would find many important applications and implementations in various fields of nonlocal mechanics.

7. Open questions

Notwithstanding the undoubted success and the inherent potential of nonlocal mechanics, it should be remarked that some fundamentals still remain unclear. One important question, which is relevant to the physics of the nonlocal theory, is how the nonlocal field affects the material stiffness. Some researchers assert that the nonlocal effect reduces the stiffness while others predicted opposite trends for some materials. Whereas the developed nonlocal kernels would have some origins, the connections of these kernels to various materials is still unrevealed. In addition, the determination of the various material coefficients of the microcontinuum theories is still an open question. Furthermore, research should be carried out to come up with experimental models that would be used for determining these material coefficients for various materials.

As for the mechanics of multiscale-complex materials, studies that comprehensively show the formulations and the physical insights into the implementations of the nonlocal microcontinuum theories to these materials are needed. Nonetheless, studies on the molecular approaches for the multiscale modeling of complex materials would be considered the bases of the prospective studies on these implementations (e.g., Trovalusci, 2014; Capecchi et al., 2011; Gurtin, 2000; Capriz, 1989; Maugin, 1993).

There is still a lot that should be done on the mathematics of nonlocal mechanics. One important issue is the existence and uniqueness of solutions of nonlocal boundary value problems. The description of the boundary conditions with the existence of nonlocal fields is still ambiguous. In addition, analytical and computational methods should be developed to solve boundary value problems formed by various nonlocal microcontinuum theories. The implementations of these theories were limited for using their simple versions for unbounded domains or one-dimensional bounded domains. However, implementations of various microcontinuum theories for bounded domains are scarce. In line with this, research also should be focused on the description of the nonclassical boundary conditions of the microcontinuum theories. Inverse nonlocal problems of calibration of nonlocal kernels and delineation of the nature of nonlocal interactions from experimental observations are also essential subjects of both theoretical and practical interests. The required mathematical topics will pave the way for the implementation of the nonlocal microcontinuum theories to applications such as nonlocal viscoelasticity and nonlocal hyperelasticity.

8. Concluding remarks and future prospects

Nonlocal mechanics is a very long history because it would play more significant roles in multiscale and stochastic modeling of various materials and structures. It is useful to understand how nonlocal or long-range interactions arise and how they should be properly formulated. As a result, various theories of nonlocal mechanics have attracted extensive attention in the mechanics of materials community. Regarding to importance of this topic, in this review, we have presented an overview of the various theories of nonlocal mechanics and discussed the physics behind them. In addition, the applicability of each one of these theories for single-scale and multiscale materials was discussed. The mathematics of nonlocal mechanics was also interpreted. The nonlocal micromorphic theory was reviewed, and its simplifications to the nonlocal microstretch, nonlocal micropolar, the nonlocal relaxed micromorphic, the nonlocal reduced micromorphic, Eringen's nonlocal, strain gradient, and couple stress theories were demonstrated. Moreover, the existing complications of solving nonlocal field problems were reviewed. The various efforts to overcome these complications were also collected and discussed. Despite the huge development of nonlocal mechanics, there are still important questions to be understood. Therefore, some major unsolved questions were presented.

The preparation of this review was made possible owing to the recent theoretical and computational results obtained from the works of several research groups involved in the field of nonlocal mechanics. We have tried to provide a comprehensive review. However, there is little doubt that some papers and reports have not come to our attention and have been missed out in this review. This is in no way a judgment on the quality of those studies that have not been included in our review. Despite that, we have intentionally ignored some other works, which – as we believe – are questionable and should be revised.

Authors contributions

M. Shaat: Conceptualization and Writing - Original Draft, Review & Editing. E. Ghavanloo: Writing - original draft, Review & Editing. S. A. Fazelzadeh: Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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