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Research paper

# Input-to-state stability for impulsive switched systems with incommensurate impulsive switching signals

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ABSTRACT

This work mainly studies the input-to-state stability *(ISS)* property of impulsive switched systems with incommensurate impulsive switching signals, where the switching instant is different from the impulse jump instant. With the help of Lyapunov method and average dwell-time *(ADT)* approach, some sufficient conditions are provided to cope with the problem of *ISS* for impulsive switched system, where the hybrid effects of stabilizing impulses and destabilizing impulses are taken into account. It is shown that when all of the modes are *ISS*, a switched system under an *ADT* scheme is *ISS* even if there exist hybrid impulses. Moreover, when none of the modes is *ISS*, we show that *ISS* still can be achieved under the designed *ADT* scheme couples with hybrid impulses. Furthermore, when some of modes are not *ISS*, a relationship such that impulsive switched system is *ISS* can be established among *ADT* scheme, impulses, and total dwell-time between *ISS* and non-*ISS* modes. Two illustrative examples are presented, with their numerical simulations, to demonstrate the effectiveness of main results.



## 1. Introduction

Over the past decades, hybrid systems have become a large and growing interdisciplinary area of research due to their numerous applications in various fields of sciences and engineering, such as biological systems [1], automobiles and locomotives with different gears [2], manufacturing processes [3], and shrimp harvesting methods [4]. As a special class of hybrid systems, switched systems comprise of two components: a family of subsystems and a certain switching rule. Specially, we often call each subsystem a mode. The switching rule determines which one of subsystems being active at every instant of time [5–7]. In general, a switched system does not inherit properties of individual subsystems, a well-known example is that switching among globally exponentially stable subsystems could lead to instability, see [8]. Another important subclass of hybrid systems are impulsive systems, which are composed of three elements: a continuous dynamic, which governs the continuous evolution of system between impulses; a discrete dynamic, which governs the way that the system states are changed at impulse instants; and an impulsive law for determining when the impulses occur [9–11]. The interaction of continuous-time and discrete-time dynamics in a impulsive system can lead to rich dynamical behavior and phenomena

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that are not encountered in purely continuous- or discrete- systems and hence bring difficulties and challenges to the stability analysis and control design [12–17]. However, in the real world, impulse effect and switching phenomenon always exist simultaneously in system [18–20]. Thus, a more comprehensive model named impulsive switched system was constituted, which cannot be well described by purely continuous or purely discrete models. A fundamental analysis of the properties of impulsive switched system can be found in [21], and there are some latest research results, see [22–26].

When investigating the dynamics of a nonlinear system, it is necessary to characterize the effects of external inputs. Since its first introduction by Sontag [27], *ISS* has received much research attention owing to its capability in characterizing the effects from exogenous disturbances on dynamical systems [11,28–30]. Roughly speaking, *ISS* means that no matter what the size of the initial state is, the state will eventually approach a neighborhood of the origin whose size is proportional to the magnitude of the input. Originally introduced for estimating dynamical behavior of continuous systems, later it was also developed to many different types of discontinuous systems, such as impulsive systems [11,29,35], impulsive switched systems [26,31,32], stochastic systems [33–36], and so on. Especially, Hespanha et al. [11] extended the notion of *ISS* for impulsive system and provided a set of Lyapunov-based sufficient conditions for *ISS* property by controlling the frequency of impulse occurrence; *ISS* characterization for a class of impulsive systems where jump map depends on time was studied in [29]; In [35], some Razumikhin-type theorems on *ISS* for time-varying impulsive stochastic delay differential system were obtained by employing Razumikhin technique and *ADT* method.

For the problem of *ISS* for impulsive switched system, some interesting results have been introduced in recent years. For example, Liu et al. [31] investigated *ISS* and *iISS* of impulsive and switching hybrid systems with time-delay by using the method of multiple Lyapunov-Krasovskii functionals. In [37], with the help of Lyapunov and *ADT* methods, some sufficient conditions for *IOSS* of impulsive switched system were obtained, where both types of impulses, stabilizing impulses and destabilizing impulses, were considered. However, It should be mentioned that a common assumption existing in the aforementioned results is that the impulse time sequence is same as the switching time sequence, which is quite unpractical. In fact, in many actual operations, impulses and switchings cannot occur at same instant. Thus, it is necessary to study the impulsive switched system with incommensurate impulsive switching signals, which is the main motivation and novelty of this paper.

In this paper, we investigate the *ISS* property of impulsive switched system with incommensurate impulsive switched signals. With the help of Lyapunov method and *ADT* scheme, some sufficient conditions ensuring *ISS* property are obtained. The main contributions of this paper lie in two aspects: (i) We explore the *ISS* property of impulsive switched system where the impulse jump instant is different from the switching instant, which means that the impulses and switchings do not occur at same instant. (ii) We consider the impulsive switched system with hybrid impulses, i.e., we fully consider the hybrid effects of stabilizing impulses and destabilizing impulses. The rest of this paper is organized as follows. In Section 2, the problem is formulated, and some notations and definitions are given. In Section 3, the main results are presented. Two examples are given in Section 4, and conclusions are given in Section 5.

## 2. Preliminaries

**Notations.** Let $\mathbb{Z}_+$ denote the set of positive integer numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the $n$-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. $a\vee b$ and $a\wedge b$ are the maximum and minimum of $a$ and $b$, respectively. $\mathcal{P}=\{1,2,\dots,m\}, m\in\mathbb{Z}_+$ is an index set. For $\mathcal{J}\subseteq\mathbb{R}$ and $\mathcal{S}\subseteq\mathbb{R}^k$ with $1\le k\le n$, let $\mathcal{C}(\mathcal{J},\mathcal{S})=\{\varphi:\mathcal{J}\to\mathcal{S},\varphi\in\mathcal{C}_0\}$ and $\mathcal{G}=\{\varphi:[t_0,\infty)\to\mathcal{P},\varphi\in\mathcal{G}_0\}$, where $\mathcal{C}_0$ is the set of continuous functions, and $\mathcal{G}_0$ is the set of piecewise constant functions. Given a constant $c\in\mathbb{R}$, define $\mathcal{S}_c=\{\eta\in\mathbb{R}:\eta>c,\eta c>0\}$. Let $\mathcal{K}_\infty=\{\alpha\in C(\mathbb{R}_+,\mathbb{R}_+)\mid\alpha(0)=0,\alpha(r)$ is strictly increasing in $r$, and $\alpha(r)\to\infty$ as $r\to\infty\}$, $\mathcal{KL}=\{\beta\in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)\mid\beta(r,t)$ is in class $\mathcal{K}$ w.r.t. $r$ for each fixed $t\ge0$, and $\beta(r,t)$ is strictly decreasing to 0 as $t\to\infty$ for each fixed $r\ge0\}$.

Consider the following impulsive switched system

$$\begin{cases}\dot{x}(t)=f_{\sigma(t)}(x,u), t\neq t_n, n\in\mathbb{Z}_+,\\ x(t)=g_n(x(t^-)), t=t_n, n\in\mathbb{Z}_+,\end{cases}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the system state, $u\in\mathbb{R}^m$ is a measurable locally bounded disturbance input, and $\dot{x}(t)$ denotes the right-hand derivative of $x$. $\{o_l, l\in\mathbb{Z}_+\}$ is a strictly increasing sequence during interval $[t_0,\infty)$, which is called the switching time sequence satisfying $o_l\to\infty$ as $l\to\infty$. It determines when the switching occurs. $\sigma\in\mathcal{P}$ is called a switching signal. When $\sigma(t)=i, 1<i<m,$ the mode $\dot{x}=f_i$ is activated. $\{t_n, n\in\mathbb{Z}_+\}$ is a strictly increasing sequence on $[t_0,\infty)$, which is called the impulse time sequence, and determines when the impulse occurs. To prevent the occurrence of accumulation points, such as Zeno phenomenon, we always assume that $0\le t_0<t_1<\ldots<t_n\to\infty$ as $n\to\infty$, and we denote such kind of impulse time sequence by set $\mathcal{F}_0$ for later use. In addition, we assume that $f_{\sigma(t)}\in\mathcal{C}(\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R}^n)$ is locally Lipschitz, $g_n\in\mathcal{C}(\mathbb{R}^n,\mathbb{R}^n)$, and $f_{\sigma(t)}(0,0)=g_n(0)\equiv0, \forall\sigma\in\mathcal{P}, \forall n\in\mathbb{Z}_+$. All signals in this paper, including the state $x$ and the input $u$, are assumed to be right continuous and to have left limits at all instants. In view of this, we denote by $(\cdot)^-$ and $(\cdot)^+$ the left-limit and right-limit operators, respectively, i.e., $x(t^-)=\lim_{s\nearrow t}x(s)$ and $x(t^+)=\lim_{s\searrow t}x(s)$. We assume that the solution of system (1) with initial condition $x(t_0)=x_0$ uniquely exists on $[t_0,\infty)$.

In the literature, it is often assumed that the impulse occurs when and only when the switching is activated, see for recent works [23,24,31,37]. That is, the switching instant is also the impulse jump instant, which is a strongly restrictive condition. Our aim of this paper is to study impulsive switching laws in which switching instant is different from the impulse jump instant. To do this, we assume that $\{o_l\} \cap \{t_n\} = \emptyset$.

In the following, we introduce some definitions.

**Definition 1** [38]**.** The switching signal $\sigma(t)$ is said to have average dwell time $\tau_\sigma$ if there exist $N_0$, $\tau_\sigma > 0$ such that

$$N_\sigma(T, t) \le N_0 + \frac{T - t}{\tau_\sigma}, \ \forall T \ge t \ge t_0, \tag{2}$$

where $N_0 > 0$ is called the chatter bound, $\tau_\sigma$ denotes the switching *ADT* constant, and $N_\sigma(T, t)$ is the number of switchings occurring on the interval $[t, T)$.

Denote the switching instants on the interval $[t_0, T)$ by $o_1, o_2, \ldots, o_{N_\sigma(t,t_0)}$, and the index of the mode that is active on the interval $[o_l, o_{l+1})$ by $\sigma(o_l)$. The sequence $(\{t_n\}, \sigma)$ composed of impulse instants and switching signals is called an impulsive switching signal. Let $\mathcal{F}[\{t_n\}, \sigma]$ denote a class of impulsive switching signals, where the switching signal $\sigma$ satisfies *ADT* condition (2) and the impulse time sequence $\{t_n\} \in \mathcal{F}_0$. In addition, the number of impulse signals on the interval $[t, T)$ is denoted by $N_{imp}(T, t)$.

**Definition 2** [31]**.** Given an impulsive switching signal $(\{t_n, \sigma\})$, system (1) is said to be *ISS* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition $(t_0, x_0)$ and measurable locally bounded disturbance input $u$, the corresponding solution of (1) satisfies

$$|x(t)| \le \beta(|x_0|, t - t_0) + \gamma(\|u\|_{[t_0,t]}), t \ge t_0. \tag{3}$$

Furthermore, it is said to be *uniform ISS (UISS)* over a given class impulsive switching signals $\mathcal{F}[\{t_n\}, \sigma]$ if the *ISS* property expressed by the above inequality holds for every sequence in $\mathcal{F}[\{t_n\}, \sigma]$, with functions $\beta$ and $\gamma$ that are independent of the choice of the sequence.

Given a locally Lipschitz function $V_i : \mathbb{R}^n \to \mathbb{R}_+, i \in \mathcal{P}$, the upper right-hand Dini derivative of $V_i$ with respect to the $i$th mode of system (1) is define by

$$D^+V_i(x) = \limsup_{h \to 0^+} \frac{1}{h}[V_i(x + hf_i(x, u)) - V_i(x)].$$

## 3. Main results

In this section, we will show under what Lyapunov-like conditions *UISS* for impulsive switched system (1) can be established. We start with the situation in which every mode is *ISS* or non-*ISS*, and then the mixed case will be explored.

**Theorem 1.** *Suppose that there exist functions $\alpha_1$, $\alpha_2$, $\varphi \in K_\infty$, locally Lipschitz functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, constants $M \ge 0, \mu \ge 1, c \in \mathbb{R}, \eta \in \mathcal{S}_c$, and bounded sequence $\{d_n\}, n \in \mathbb{Z}_+$, such that for all $i, j \in \mathcal{P}, \forall x \in \mathbb{R}^n$,*

$$\begin{cases} \alpha_1(|x|) \le V_i(x) \le \alpha_2(|x|), & \text{(a)} \\ D^+V_i(x) \le cV_i(x), \text{whenever} \quad |x| \ge \varphi(|u|), & \text{(b)} \\ V_i(g_n(x)) \le \exp(d_n)V_i(x), \quad \forall n \in \mathbb{Z}_+, & \text{(c)} \\ V_i(x) \le \mu V_j(x). & \text{(d)} \end{cases} \tag{4}$$

*Then system (1) is UISS over the class $\mathcal{F}^\star[\{t_n\}, \sigma]$, where $\mathcal{F}^\star[\{t_n\}, \sigma]$ denotes a class of impulsive switching signals in $\mathcal{F}[\{t_n\}, \sigma]$ satisfying*

$$\left( \sum_{n \in N_{imp}(t,s)} d_n \right) + \left( \frac{\ln \mu}{\tau_\sigma} + \eta \right)(t - s) \le M, \ \forall t \ge s \ge t_0. \tag{5}$$

**Proof.** Let $t_0 \ge 0$ be arbitrary. For $t \ge t_0$, define the ball around the origin

$$B(t) := \left\{ x \mid |x| \le \varphi(\|u\|_{[t_0,t]}) \right\},$$

and

$$\mathcal{X}\left(\|u\|_{[t_0,t]}\right) := \alpha_1\Big(\exp(M)\alpha_2\Big(\alpha^{-1}\Big(\exp(d)\alpha_2\Big(\varphi\Big(\|u\|_{[t_0,t]}\Big)\Big)\Big)\Big)\Big),$$

where $d = \max_{n \in \mathbb{Z}_+}\{d_n \vee 0\} < \infty$. Furthermore, define

$$\gamma\left(\|u\|_{[t_0,t]}\right) = \omega\mathcal{X}\left(\|u\|_{[t_0,t]}\right) + \varphi(\|u\|_{[t_0,t]}).$$

Note that $\varphi$ is a non-decreasing function, and therefore $\mathcal{X}$ and $\gamma$ are non-decreasing function, $B$ has non-decreasing volume. Assume that $x(t) = x(t, t_0, x_0)$ be the solution of system (1) with the initial value $(t_0, x_0)$ and the impulsive switching signal $(\{t_n\}, \sigma) \in \mathcal{F}^\star[\{t_n\}, \sigma]$. Let $u$ be a given input function. Firstly, we begin the discussion for the following two cases:

**Case (a)**: If $|x| \ge \varphi(\|u\|_{[t_0,t]})$ during some time interval $[t_1', t_1'')$, then suppose that there exist impulse time sequence $\{t_n\}$ and switching time sequence $\{o_l\}$ such that

$$t_1' < t_n < \cdots < t_{n+h} < o_l < \cdots < t_{n+h+s} < t_1''.$$

When $N_{imp}(t_1^{'}, o_l) = 0$, i.e., there is no impulse on the interval $[t_1^{'}, o_l)$, it follows from (4b) that

$$V_{\sigma(t)}(x(t)) \leq \exp\Big(c(t-t_1^{'})\Big)V_{\sigma(t_1^{'})}(x(t_1^{'})), \ \ \forall t \in [t_1^{'}, o_l). \tag{6}$$

While $N_{imp}(t_1^{'}, o_l) > 0$, it follows that

$$V_{\sigma(t)}(x(t)) \leq \exp\Big(c(t-t_1^{'})\Big)V_{\sigma(t_1^{'})}(x(t_1^{'})), \ \ \forall t \in [t_1^{'}, t_n).$$

For $t \in [t_n, t_{n+1} \wedge o_l)$, we have according to (4b) and (4c)

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq \exp\Big(c(t-t_n)\Big)V_{\sigma(t_n)}(x(t_n)) \\ &\leq \exp\Big(c(t-t_n)\Big)\exp(d_n)V_{\sigma(t_n^-)}(x(t_n^-)) \\ &\leq \exp\Big(d_n + c(t-t_1^{'})\Big)V_{\sigma(t_1^{'})}(x(t_1^{'})), \end{aligned}$$

which implies that

$$V_{\sigma(t)}(x(t)) \leq \exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + c(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})), \ \ \forall t \in [t_1^{'}, o_l), \tag{7}$$

where $d_0 = 0$. Combining (6) and (7), it is easy to get

$$V_{\sigma(t)}(x(t)) \leq \exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + c(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})), \ \ \forall t \in [t_1^{'}, o_l). \tag{8}$$

Similarly, for $t \in [o_l, o_{l+1} \wedge t_{n+h+1})$, it holds that

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq \exp\Big(c(t-o_l)\Big)V_{\sigma(o_l)}(x(o_l)) \\ &\leq \exp\Big(c(t-o_l)\Big)\mu V_{\sigma(o_l^-)}(x(o_l^-)) \\ &\leq \exp\Big(c(t-o_l)\Big)\mu \exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + c(o_l-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})) \\ &\leq \mu \exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + c(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})). \end{aligned} \tag{9}$$

Repeating the similar argument as (9), we finally obtain that

$$V_{\sigma(t)}(x(t)) \leq \mu^{N_\sigma(t,t_1^{'})}\exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + c(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})), \ \ \forall t \in [t_1^{'}, t_1^{''}),$$

which, together with (2) and (5), yields that

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq \exp\left(\left(N_0 + \frac{t-t_1^{'}}{\tau_\sigma}\right)\ln\mu\right)\exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + c(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})) \\ &= \exp\Big(N_0\ln\mu\Big)\exp\left(\left(\sum_{n\in N_{imp}(t,t_1^{'})} d_n\right) + (c+\frac{\ln\mu}{\tau_\sigma})(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})) \\ &\leq \omega\exp\Big(M - \lambda(t-t_1^{'})\Big)V_{\sigma(t_1^{'})}(x(t_1^{'})), \ \ \forall t \in [t_1^{'}, t_1^{''}), \end{aligned} \tag{10}$$

where $\omega = \exp(N_0\ln\mu) > 1, \lambda = \eta - c > 0$.

**Case (b)**: If $|x| \geq \varphi(\|u\|_{[t_0,t]})$ during some time interval $[t_2^{'}, t_2^{''})$, then suppose that there is no switching instant. Applying the same argument as (8), one obtains that

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq \exp\left(\left(\sum_{n\in N_{imp}(t,t_2^{'})} d_n\right) + c(t-t_2^{'})\right)V_{\sigma(t_2^{'})}(x(t_2^{'})) \\ &\leq \exp\Big(M - \lambda(t-t_2^{'})\Big)V_{\sigma(t_2^{'})}(x(t_2^{'})), \ \ \forall t \in [t_2^{'}, t_2^{''}). \end{aligned} \tag{11}$$

Furthermore, we denote the first time when $x(t) \in B(t)$ by $\check{t}_1$, i.e., $\check{t}_1 := \inf\left\{t \geq t_0 : |x| \leq \varphi(\|u\|_{[t_0,t]})\right\}$. If $\check{t}_1 = \infty$, then it follows from (10) that

$$|x(t)| \leq \beta(|x_0|, t - t_0), \ \forall t \in [t_0, \infty),$$

where $\beta(s, t - t_0) = \omega\alpha_1^{-1}\big(\exp(M - \lambda(t - t_0))\alpha_2(s)\big)$. For the case that $\check{t}_1 < \infty$, there are two cases: $\check{t}_1 \leq o_1$ and $\check{t}_1 > o_1$. If $\check{t}_1 \leq o_1$, whether there exists impulse on the interval $[t_0, \check{t}_1)$ or not, then we always get from (11) that

$$V_{\sigma(t)}(x(t)) \leq \exp\big(M - \lambda(t - t_0)\big)V_{\sigma(t_0)}(x_0), \ \forall t \in [t_0, \check{t}_1).$$

If $\check{t}_1 > o_1$, similarly, then it follows from (10) that

$$V_{\sigma(t)}(x(t)) \leq \omega\exp\big(M - \lambda(t - t_0)\big)V_{\sigma(t_0)}(x_0), \ \forall t \in [t_0, \check{t}_1).$$

Combining above two cases, it can be deduced that

$$\begin{aligned} |x(t)| &\leq \omega\alpha_1^{-1}\big(\exp(M - \lambda(t - t_0))\alpha_2(|x_0|)\big) \\ &= \beta(|x_0|, t - t_0), \ \forall t \in [t_0, \check{t}_1). \end{aligned} \tag{12}$$

For $t \geq \check{t}_1$, we denote the first time when $x(t)$ moves out of $B(t)$ by $\hat{t}_1$, i.e., $\hat{t}_1 := \inf\left\{t \geq \check{t}_1 : |x| > \varphi(\|u\|_{[t_0,t]})\right\}$. If $\hat{t}_1 = \infty$, then $|x(t)| \leq \varphi(\|u\|_{[t_0,t]})$ for all $t \geq \check{t}_1$. It then follows from the discussion of (12) that $|x(t)| \leq \beta(|x_0|, t - t_0) + \varphi(\|u\|_{[t_0,t]})$, $t \in [t_0, \infty)$, which implies that (3) holds. If $\hat{t}_1 < \infty$, then it is obvious that $|x(t)| \leq \varphi(\|u\|_{[t_0,t]})$ for $\check{t}_1 \leq t < \hat{t}_1$. Combining with the case that $\hat{t}_1 = \infty$, we have $|x(t)| \leq \beta(|x_0|, t - t_0) + \varphi(\|u\|_{[t_0,t]})$, $t \in [t_0, \hat{t}_1)$. Furthermore, defining $\check{t}_2 := \inf\left\{t \geq \hat{t}_1 : |x| \leq \varphi(\|u\|_{[t_0,t]})\right\}$ to denote the second time when $x(t) \in B(t)$. We consider two cases that $\check{t}_2 = \infty$ and $\check{t}_2 < \infty$. For the former case, we need to discuss whether there is switching time on the interval $[\hat{t}_1, \infty)$ or not. If there exists switching time on the interval $[\hat{t}_1, \infty)$, then it follows from (10) that

$$V_{\sigma(t)}(x(t)) \leq \omega\exp\big(M - \lambda(t - \hat{t}_1)\big)V_{\sigma(\hat{t}_1)}(x(\hat{t}_1)), \ \forall t \in [\hat{t}_1, \infty). \tag{13}$$

If $\hat{t}_1$ is an impulse instant, then one obtains that

$$|x(\hat{t}_1)| \leq \alpha_1^{-1}\Big(\exp(d_{k(\hat{t}_1)})\alpha_2\big(\varphi(\|u\|_{[t_0,\hat{t}_1]})\big)\Big),$$

where $k(\hat{t}_i) = \{n \in \mathbb{Z}_+ : \hat{t}_i = t_n\}$.

If $\hat{t}_1$ is not an impulse instant, then we get

$$|x(\hat{t}_1)| = \alpha_1^{-1}\Big(\alpha_2\big(\varphi(\|u\|_{[t_0,\hat{t}_1]})\big)\Big).$$

Thus, we can deduce that $|x(\hat{t}_1)| \leq \alpha_1^{-1}\Big(\exp(d)\alpha_2\big(\varphi(\|u\|_{[t_0,\hat{t}_1]})\big)\Big)$. Together with (13), one can obtain that

$$\begin{aligned} |x(t)| &\leq \omega\alpha_1^{-1}\Big(\exp(M)\alpha_2\Big(\alpha_1^{-1}\Big(\exp(d)\alpha_2\big(\varphi(\|u\|_{[t_0,t]})\big)\Big)\Big)\Big) \\ &= \omega\mathcal{X}\big(\|u\|_{[t_0,t]}\big), \ \forall t \in [\hat{t}_1, \infty). \end{aligned}$$

If there is no switching time on the interval $[\hat{t}_1, \infty)$, then it follows from (11) that

$$V_{\sigma(t)}(x(t)) \leq \exp\big(M - \lambda(t - \hat{t}_1)\big)V_{\sigma(\hat{t}_1)}(x(\hat{t}_1)).$$

According to the fact that $|x(\hat{t}_1)| \leq \alpha_1^{-1}\Big(\exp(d)\alpha_2\big(\varphi(\|u\|_{[t_0,\hat{t}_1]})\big)\Big)$, it can be deduced that

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}\Big(\exp(M)\alpha_2\Big(\alpha_1^{-1}\Big(\exp(d)\alpha_2\big(\varphi(\|u\|_{[t_0,t]})\big)\Big)\Big)\Big) \\ &= \mathcal{X}\big(\|u\|_{[t_0,t]}\big), \ \forall t \in [\hat{t}_1, \infty). \end{aligned}$$

After the above discussion, for $t \in [\hat{t}_1, \infty)$, we have

$$|x(t)| \leq \omega\mathcal{X}\big(\|u\|_{[t_0,t]}\big),$$

which implies that

$$|x(t)| \leq \gamma\big(\|u\|_{[t_0,t]}\big), \forall t \in [\check{t}_1, \infty).$$

Similarly, for the case that $\check{t}_2 < \infty$, it is easy to see that

$$|x(t)| \leq \gamma\big(\|u\|_{[t_0,t]}\big), \ \forall t \in [\check{t}_1, \check{t}_2).$$

Arguing in the same way, we finally arrive at

$$|x(t)| \leq \gamma\big(\|u\|_{[t_0,t]}\big), \ \forall t \in [\check{t}_1, \infty).$$

After the above discussion, we conclude that

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma\big(\|u\|_{[t_0,t]}\big), \ \forall t \in [t_0, \infty).$$

Note that the functions $\beta$ and $\gamma$ are independent of the choice of the impulsive switching signals in $\mathcal{F}^{\star}[\{t_n\},\sigma]$. Thus, impulsive switched system (1) is *UISS* over the class $\mathcal{F}^{\star}[\{t_n\},\sigma]$. The proof is completed. □

**Remark 1.** In [37], the authors considered the impulsive switched systems with two types of impulses, where constant $\mu \in (0, \infty)$ determined how the impulses effect the dynamical behavior of system. However, it should be noted that $\mu$ was studied as a constant, which means that there is one and only one type of impulses, stabilizing impulses or destabilizing impulses. While in the result of our paper, we present a sufficient condition for *ISS* in which hybrid impulses, i.e., both of stabilizing and destabilizing impulses, are involved. In [24,31,37], various switched laws for impulsive switched system in which the impulse signals and the switching signals occur synchronously were studied. In fact, in many actual operations, impulses and switchings cannot occur at the same time. All of the results obtained this paper are based on the fact that the impulse time sequence is totally different from the switching time sequence, which indicates that our presented results have wider adaptive range.

**Remark 2.** Theorem 1 presents some Lyapunov-based sufficient conditions of *ISS* for impulsive switched system (1), where condition (4b) governs the continuous dynamics between two impulses of each mode, condition (4c) governs the way that the discrete dynamics to be changed at impulse instants, condition (4d) establishes the relationship between two different modes, and condition (5) establishes the relationship between impulse and switching signals. It is worth mentioning that, the hybrid effects of destabilizing impulses and stabilizing impulses are fully considered in this paper. Especially, when $c > 0$, condition (4b) implies that it is possible that each of the continuous dynamics is not *ISS*. Nevertheless, condition (4c) implies that there exist stabilizing impulses and destabilizing impulses simultaneously in system (1) with $d_n \in \mathbb{R}$. In this case, condition (5) shows that *ISS* can be achieved under the designed *ADT* scheme couples with stabilizing impulses. Similarly, when $c < 0$, condition (5) shows that system (1) under an *ADT* scheme is *ISS* even if there exist destabilizing impulses. In other words, the *ISS* property can be established whether the continuous dynamic of the system is *ISS* or not. In particular, if we strengthen some restrictions on condition (4b) and (4c), then the following results can be derived.

**Corollary 1.** *Under conditions in Theorem 1, suppose that there exist cosntants $\mu \geq 1$, $c < 0$, $M \geq 0$, and bounded sequence $\{d_n\}, n \in \mathbb{Z}_+$, then system (1) is UISS over the class $\mathcal{F}^{\bullet}[\{t_n\},\sigma]$, where $\mathcal{F}^{\bullet}[\{t_n\},\sigma]$ is a class of impulsive switching signals in $\mathcal{F}[\{t_n\},\sigma]$ satisfying*

$$\begin{cases} \displaystyle\sum_{n \in N_{imp}(t,s)} d_n \leq M, \ \forall t > s \geq t_0, \\ \frac{\ln \mu}{\tau_\sigma} + c < 0. \end{cases} \tag{14}$$

**Remark 3.** In Corollary 1, we consider a special case that the rate coefficients $c < 0$ and $d_n \in \mathbb{R}$. It shows a fact that it is possible that the continuous dynamics of each mode are *ISS*, but the effect of each impulse is different (stabilizing or destabilizing impulses). Since the hybrid impulses and the switching may destroy the *ISS* property, we require that condition (14) holds such that the total weight of impulses and the average dwell time $\tau_\sigma$ can be balanced. According to (14), it is easy to see that the *ADT* constant satisfies $\tau_\sigma > \frac{\ln \mu}{-c}$, which indicates that the dwell time on every *ISS* mode cannot be overly short.

By the ideas proposed by Hespanha [11], let $\mathcal{F}^+$ denote a class of impulse time sequences satisfying the *ADT* condition given by

$$N_{imp}^+ \leq N_0^+ + \frac{t-s}{\tau^+}, \ \forall t \geq s \geq t_0,$$

where $N_0^+ > 0$ is called the chatter bound, $\tau^+$ denotes the *ADT* constant, and $N_{imp}^+$ is the number of destabilizing impulses occurring on the interval $[s, t)$. Similarly, let $\mathcal{F}^-$ denote a class of impulse time sequences satisfying the reverse *ADT* condition given by

$$N_{imp}^- \geq -N_0^- + \frac{t-s}{\tau^-}, \ \forall t \geq s \geq t_0,$$

where $N_0^- > 0$ is called the chatter bound, $\tau^-$ denotes the reverse *ADT* constant, and $N_{imp}^-$ is the number of stabilizing impulses occurring on the interval $[s, t)$.

Let $\mathcal{F}^* = \mathcal{F}[N_0^+, N_0^-, \tau^+, \tau^-, \tau_\sigma, N_\sigma]$ denote a class of impulsive switching signals, where the switching signal $\sigma$ satisfies condition (2), the destabilizing impulse time sequence $\{t_n\}^+ \in \mathcal{F}^+$, and the stabilizing impulse time sequence $\{t_n\}^- \in \mathcal{F}^-$. Then the following corollary for *ADT* scheme is derived.

**Corollary 2.** *Under conditions (4a)–(4d) in Theorem 1 with $\mu \geq 1, c \in \mathbb{R}, d^+ > 0, d^- > 0$, and $\eta \in \mathcal{S}_c$, suppose that there exist constants $\theta, \gamma \in \mathbb{R}$. **(i)** When $\frac{\ln \mu}{\tau_a} + \eta < 0$ and $\theta > 1$, system (1) is UISS over the class $\mathcal{F}^*$, where*

$$\tau^+ \geq \frac{d^+}{-\theta\left(\frac{\ln \mu}{\tau_a} + \eta\right)}, \quad N_0^+ > 0 \ \text{ and } \ \tau^- \leq \frac{d^-}{(1-\theta)\left(\frac{\ln \mu}{\tau_a} + \eta\right)}, \quad N_0^- > 0.$$

***(ii)*** *When* $\frac{\ln\mu}{\tau_a}+\eta>0$ *and* $\gamma<0$, *system (1) is UISS over the class* $\mathcal{F}^*$, *where*

$$\tau^+\geq\frac{d^+}{-\gamma\left(\frac{\ln\mu}{\tau_a}+\eta\right)},\quad N_0^+>0\quad\text{and}\quad\tau^-\leq\frac{d^-}{(1-\gamma)\left(\frac{\ln\mu}{\tau_a}+\eta\right)},\quad N_0^->0.$$

**Remark 4.** Corollary 2 considers a special case that $d_n\equiv d^+$ if $d_n>0$ and $d_n\equiv -d^-$ if $d_n<0$, $n\in\mathbb{Z}_+$. Then the *ADT*-like conditions in Corollary 2 are derived, where case (i) considers the case that $\eta<-\frac{\ln\mu}{\tau_\sigma}$, which implies that all of modes are *ISS*; while case (ii) considers the case that $\eta>-\frac{\ln\mu}{\tau_\sigma}$, which implies that all of modes are *ISS* when $0>\eta>-\frac{\ln\mu}{\tau_\sigma}$ and all of modes are non-*ISS* when $\eta>0$. It is easy to see that whether the continuous dynamic of each mode is *ISS* or not, the destabilizing impulses cannot occur too frequently due to their negative effects on the stability of system. On the other hand, in order to ensure the *ISS*, the stabilizing impulses should be applied sufficiently frequently, which implies that the interval between two stabilizing impulses must not be overly long.

In the following, we will consider the mixed case in which not all the modes is *ISS*. Suppose that a mode with $\mathcal{P}_1\subseteq\mathcal{P}$ is *ISS* and a mode with $\mathcal{P}_2\subseteq\mathcal{P}$ is non-*ISS*. Let $\mathcal{P}_1\bigcup\mathcal{P}_2=\mathcal{P}$ such that $\mathcal{P}_1\bigcap\mathcal{P}_2=\varnothing$. Denote by $\pi(t)$ the total activation time of the system in $\mathcal{P}_2$.

**Assumption 1.** Assume that there exist constants $\rho$, $T_0\geq 0$ such that $\pi(t)\leq T_0+\rho(t-t_0)$ for $t\geq t_0$.

**Theorem 2.** *Suppose that Assumption 1 holds and there exist functions* $\alpha_1,\alpha_2,\varphi\in\mathcal{K}_\infty$, *locally Lipschitz functions* $V_i:\mathbb{R}^n\to\mathbb{R}_+$, *constants* $c_1,c_2\in\mathbb{R}_+$, $\mu\geq 1$, *and bounded sequence* $\{d_n\}$, $n\in\mathbb{Z}_+$, *such that conditions (4a), (4c) and (4d) hold for all* $x\in\mathbb{R}$, $i,j\in\mathcal{P}$ *and furthermore, the following holds:*

$$|x(t)|\geq\varphi(|u|)\Rightarrow\begin{cases}D^+V_i(x)\leq -c_1V_i(x), & \forall i\in\mathcal{P}_1,\\ D^+V_i(x)\leq c_2V_i(x), & \forall i\in\mathcal{P}_2.\end{cases}\tag{15}$$

*Then system (1) is UISS over the class* $\mathcal{F}^\triangle[\{t_n\},\sigma]$, *where* $\mathcal{F}^\triangle[\{t_n\},\sigma]$ *denotes a class of impulsive switching signals in* $\mathcal{F}[\{t_n\},\sigma]$ *satisfying that* $\Sigma_{n\in N_{imp}(t,s)}d_n$ *is uniformly bounded for all* $t\geq s\geq 0$ *and*

$$\begin{cases}0\leq\rho<\frac{c_1}{c_1+c_2},\\ \tau_\sigma>\frac{\ln\mu}{c_1-(c_1+c_2)\rho}.\end{cases}$$

**Proof.** Consider the function $W(t):=\exp(c_1t)V_{\sigma(t)}(x(t))$, then it follows from (15) that

$$\begin{cases}D^+W(t)\leq 0 \quad\text{if}\quad \sigma(t)\in\mathcal{P}_1,\\ D^+W(t)\leq(c_1+c_2)W(t)\quad\text{if}\quad\sigma(t)\in\mathcal{P}_2.\end{cases}$$

Similar to the argument in Theorem 1, if $|x|\geq\varphi(\|u\|_{[t_0,t]})$ during some time interval $[t_1^{'},t_1^{''})$, then suppose that there exist impulse time sequence $\{t_n\}$ and switching time sequence $\{o_l\}$ such that

$$t_1^{'}<t_n<\cdots<t_{n+h}<o_l<\cdots<t_{n+h+s}<t_1^{''}.$$

Obviously, for any $t\in[t_1^{'},t_1^{''})$, one sees that

$$W(t)\leq\mu^{N_\sigma(t,t_1^{'})}\exp\left(\left(\sum_{n\in N(t,t_1^{'})}d_n\right)+(c_1+c_2)\pi(t,t_1^{'})\right)W(t_1^{'}),$$

which, together with (2) and the Assumption 1, yields that

$$\begin{aligned}&V_{\sigma(t)}(x(t))\\ &\leq\exp\left(\left(N_0+\frac{t-t_1^{'}}{\tau_\sigma}\right)\ln\mu\right)\exp\left(\left(\sum_{n\in N(t,t_1^{'})}d_n\right)+(c_1+c_2)\pi(t,t_1^{'})-c_1(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'}))\\ &=\exp\left(N_0\ln\mu+(c_1+c_2)T_0\right)\exp\left(\left(\sum_{n\in N(t,t_1^{'})}d_n\right)+((c_1+c_2)\rho+\frac{\ln\mu}{\tau_\sigma}-c_1)(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'}))\\ &=\varrho_1\exp\left(\left(\sum_{n\in N(t,t_1^{'})}d_n\right)+((c_1+c_2)\rho+\frac{\ln\mu}{\tau_\sigma}-c_1)(t-t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})),\end{aligned}\tag{16}$$

where $\varrho_1 = \exp\left(N_0 \ln \mu + (c_1 + c_2)T_0\right)$. Note that $\Sigma_{n\in N_{imp}(t,s)} d_n$ is uniformly bounded, then there exists a positive constant $M$ such that $\Sigma_{n\in N_{imp}(t,s)} d_n \le M$ for all $t \ge s \ge t_0$. In view of $\rho < \frac{c_1}{c_1+c_2}$ and $\tau_\sigma > \frac{\ln \mu}{c_1-(c_1+c_2)\rho}$, we can establish the following inequality:

$$(c_1 + c_2)\rho + \frac{\ln \mu}{\tau_\sigma} - c_1 < 0.$$

Together with (16), it is easy to get that

$$V_{\sigma(t)}(x(t)) \le \varrho_1 \exp\left(M - \lambda_1(t - t_1^{'})\right)V_{\sigma(t_1^{'})}(x(t_1^{'})), \quad \forall t \in [t_1^{'}, t_1^{''}),$$

where $\lambda_1 = -((c_1 + c_2)\rho + \frac{\ln \mu}{\tau_\sigma} - c_1) > 0$. Similarly, if $|x| \ge \varphi(\|u\|_{[t_0,t]})$ during some time interval $[t_2^{'}, t_2^{''})$, then suppose that there is no switching. It is obvious that

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\le \exp\left(\left(\sum_{n\in N(t,t_2^{'})} d_n\right) + (c_1 + c_2)\pi(t, t_2^{'}) - c_1(t - t_2^{'})\right)V_{\sigma(t_2^{'})}(x(t_2^{'})) \\ &\le \exp\left((c_1 + c_2)T_0\right)\exp\left(\left(\sum_{n\in N(t,t_2^{'})} d_n\right) + ((c_1 + c_2)\rho - c_1)(t - t_2^{'})\right)V_{\sigma(t_2^{'})}(x(t_2^{'})) \\ &\le \varrho_2 \exp\left(M - \lambda_2(t - t_2^{'})\right)V_{\sigma(t_2^{'})}(x(t_2^{'})), \forall t \in [t_2^{'}, t_2^{''}), \end{aligned}$$

where $\varrho_2 = \exp((c_1 + c_2)T_0)$, $\lambda_2 = -((c_1 + c_2)\rho - c_1) > 0$.

Then repeating the arguments used in Theorem 1, one can obtain that system (1) is *UISS* over the class $\mathcal{F}^{\triangle}[\{t_n\}, \sigma]$. The proof is completed. □

**Remark 5.** It is worth mentioned that, in [37], the authors also considered the case that not all of the modes is *ISS*. In order to ensure the *ISS*, Theorem 2 in [37] required that the $\mu \in (0, 1)$, which implies that there only exist stabilizing impulses. Note that in our result of Theorem 2, we consider the hybrid effects of stabilizing impulses and destabilizing impulses, and *ISS* still can be achieved even if there exist destabilizing impulses due to the effects of stabilizing impulses and switching.

**Remark 6.** Recently, some interesting results on impulsive switched system have been reported, such as [25,39,40] in references of the paper. Although many papers dealt with the problem of *ISS* for impulsive switched system, there is little work on studying impulsive switched system with incommensurate impulsive switching signals. In this paper, we discuss the *ISS* property of impulsive switched system, where Theorem 1 is based on the fact that all modes are *ISS* or non-*ISS* and Theorem 2 is based on the fact that some modes are not *ISS*. However, all the results in this paper are all based on the fact that the impulse time sequence is totally different from the switching time sequence, which is the main motivation and novelty of this paper. Furthermore, rate coefficients $d_n \in \mathbb{R}$ imply that there exist stabilizing impulses and the destabilizing impulses in system simultaneously. That is to say, the hybrid effects of two types of impulses are fully considered. In this sense, the theoretical results proposed in this paper enrich the study on the *ISS* property of impulsive switched system.

## 4. Applications

In this section, two numerical examples will be presented to demonstrate the effectiveness of our results.

**Example 1.** Consider the family of two modes given by

$$\dot{x}(t) = f_1(x, u) = \begin{pmatrix} -x_1 + x_1x_2 + \tanh(x_2 - x_1) \\ -x_2 - x_1^2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$\dot{x}(t) = f_2(x, u) = \begin{pmatrix} -x_1 + x_2^3 + \sin(x_2 - x_1) \\ -2x_2 - 2x_1x_2^2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with impulse functions

$$g_{3n-2}(x) = 0.5x, g_{3n-1}(x) = 1.2x, g_{3n}(x) = 1.5x, n \in \mathbb{Z}_+,$$

where $\mathcal{P} = \{1, 2\}$, and $sat(x)$ is the saturation function defined by $sat(x) = 1/2(|x + 1| - |x - 1|)$. Next, we design the impulsive switching law such that system is *UISS*. Choose $V_1 = x_1^2 + x_2^2$ and $V_2 = x_1^2 + 0.5x_2^2$ as *ISS*-Lyapunov functions for the two modes. It is obvious that condition (4a) holds and condition (4d) is satisfied with $\mu = 2$. Let $\varphi(s) = 4s, s \in \mathbb{R}_+$. Thus, if $|x| \ge \varphi(|u|)$, then one sees that $D^+V_1 \le -\frac{1}{2}V_1(x)$ and $D^+V_2 \le -\frac{1}{2}V_2(x)$, which implies that condition (4b) is satisfied with $c = -\frac{1}{2}$. For all $i \in \mathcal{P}$, we have

$$V_i\left(g_{3n-2}(x)\right) \le 0.25V_i(x), V_i\left(g_{3n-1}(x)\right) \le 1.44V_i(x), V_i\left(g_{3n}(x)\right) \le 2.25V_i(x),$$

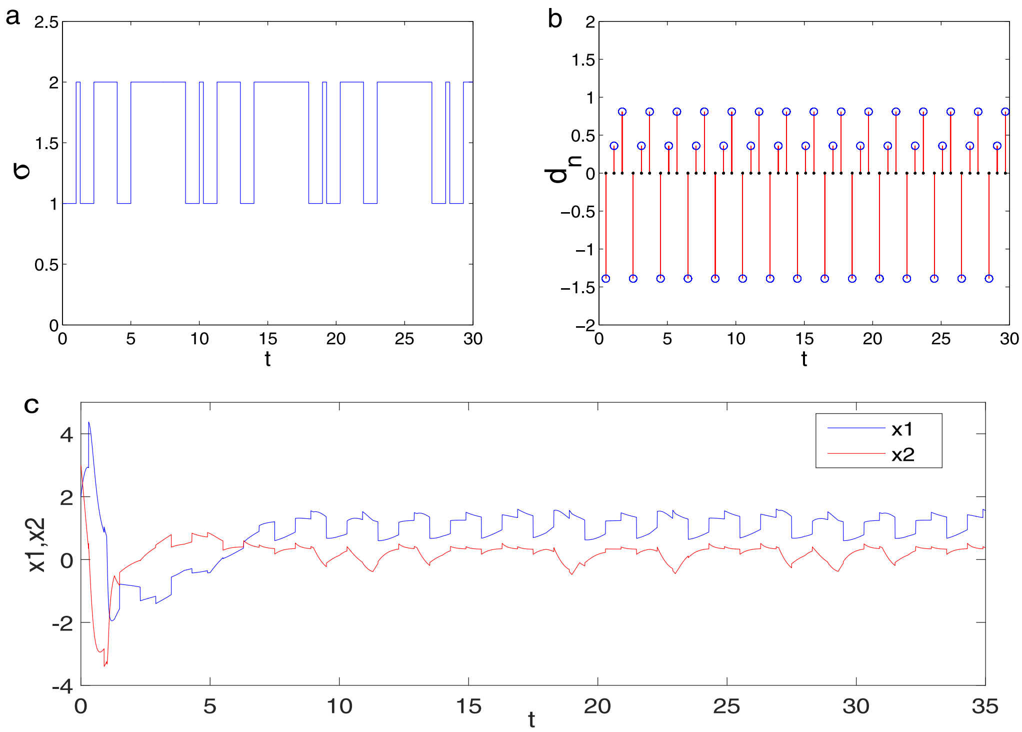


**Fig. 1.** Simulation results for Example 1.

which implies that condition (14) is satisfied with $d_{3n-2} = -1.39, d_{3n-1} = 0.36, d_{3n} = 0.81$, and $M = 1.2$. Then it follows from Corollary 1 that system (1) is *UISS* over the class $\mathcal{F}^{\bullet}[\{t_n\}, \sigma]$ with $N_0 > 0$ and *ADT* constant $\tau_\sigma > \frac{\ln \mu}{-c} \approx 1.38$. In particular, the impulse time sequence is given by $t_{3n-2} = 2n - 1.5, t_{3n-1} = 2n - 0.9, t_{3n} = 2n - 0.3, n \in \mathbb{Z}_+$, the switching time sequence is given by $\tau_{6n-5} = 9n - 8, \tau_{6n-4} = 9n - 7.7, \tau_{6n-3} = 9n - 6.7, \tau_{6n-2} = 9n - 5, \tau_{6n-1} = 9n - 4, \tau_{6n} = 9n, n \in \mathbb{Z}_+$, which implies that $\tau_\sigma = 1.5$ and the switching time is different from the impulse jump time. Meanwhile, Let $x_1(0) = 2, x_2(0) = 3, u_1 = sat(x_1)$ and $u_2 = \cos(x_2)$, then the Fig. 1(a)–(c) illustrate the switching signals, the weights of impulses, and the state trajectory of system (1), respectively.

**Example 2.** Consider the family of two modes given by

$$\begin{cases} \dot{x} = f_1(x, u) = x + u_1, & \text{(a)} \\ \dot{x} = f_2(x, u) = -x + u_2, & \text{(b)} \end{cases} \tag{17}$$

with impulse functions

$$g_{3n-2}(x) = 0.6x, g_{3n-1}(x) = 1.3x, g_{3n}(x) = 1.2x, n \in \mathbb{Z}_+,$$

where $\mathcal{P} = \{1, 2\}$. Next, we design the impulsive switching law between the *ISS* modes and non-*ISS* modes such that the system is *UISS*. We choose $V_1 = x^2$ and $V_2 = 0.5x^2$ as *ISS*-Lyapunov functions for the two modes. That is to say, condition (4a) holds and condition (4d) is satisfied with constant $\mu = 2$. Let $\varphi(s) = 2s, s \in \mathbb{R}_+$. Note that when $|x| \ge \varphi(|u|)$, we have $D^+V_1(x) \le 3V_1$, which implies that subsystem (17a) is non-*ISS*. By the way, it is easy to see that if $|x| \ge \varphi(|u|)$, then $D^+V_2(x) \le -V_2$ holds, which implies that the subsystem (17b) is *ISS*. Thus, condition (15) is satisfied with $c_1 = 1, c_2 = 3$. For all $i \in \mathcal{P}$, we have

$$V_i\big(g_{3n-2}(x)\big) \le 0.36V_i(x), V_i\big(g_{3n-1}(x)\big) \le 1.69V_i(x), V_i\big(g_{3n}(x)\big) \le 1.44V_i(x),$$

which implies that condition (4c) is satisfied with $d_{3n-2} = -1.02, d_{3n-1} = 0.52, d_{3n} = 0.36, n \in \mathbb{Z}_+$, and furthermore, $\Sigma_{n \in N_{imp}(t,s)} d_n$ is uniformly bounded for all $t \ge s \ge 0$. Thus, if we choose $\rho = 1/8 < \frac{c_1}{c_1 + c_2} = 1/4$, then it follows from Theorem 2 that system (1) is *UISS* over the class $\mathcal{F}^{\triangle}[\{t_n\}, \sigma]$ with $N_0 > 0$, and *ADT* constant $\tau_\sigma > \frac{\ln \mu}{c_1 - (c_1 + c_2)\rho} \approx 1.39$. In particular, the impulse time sequence is given by $t_{3n-2} = 0.3n - 0.19, t_{3n-1} = 0.3n - 0.11, t_{3n-2} = 0.3n - 0.07, n \in \mathbb{Z}_+$, and the switching time sequence is given by $\tau_{6n-5} = 8.4n - 7.4, \tau_{6n-4} = 8.4n - 6.9, \tau_{6n-3} = 8.4n - 5.9, \tau_{6n-2} = 8.4n - 3.4, \tau_{6n-1} = 8.4n - 1.9, \tau_{6n} = 8.4n, n \in \mathbb{Z}_+$. That is, the switching time is different from the impulse jump time and the *ADT* constant $\tau_\sigma = 1.4$. Meanwhile, if we choose $x(0) = 1$ and $u_1 = u_2 = sat(x)$, then the Fig. 2(a)–(c) illustrate the switching signals, the weights of impulses, and the state trajectory of system (1), respectively.

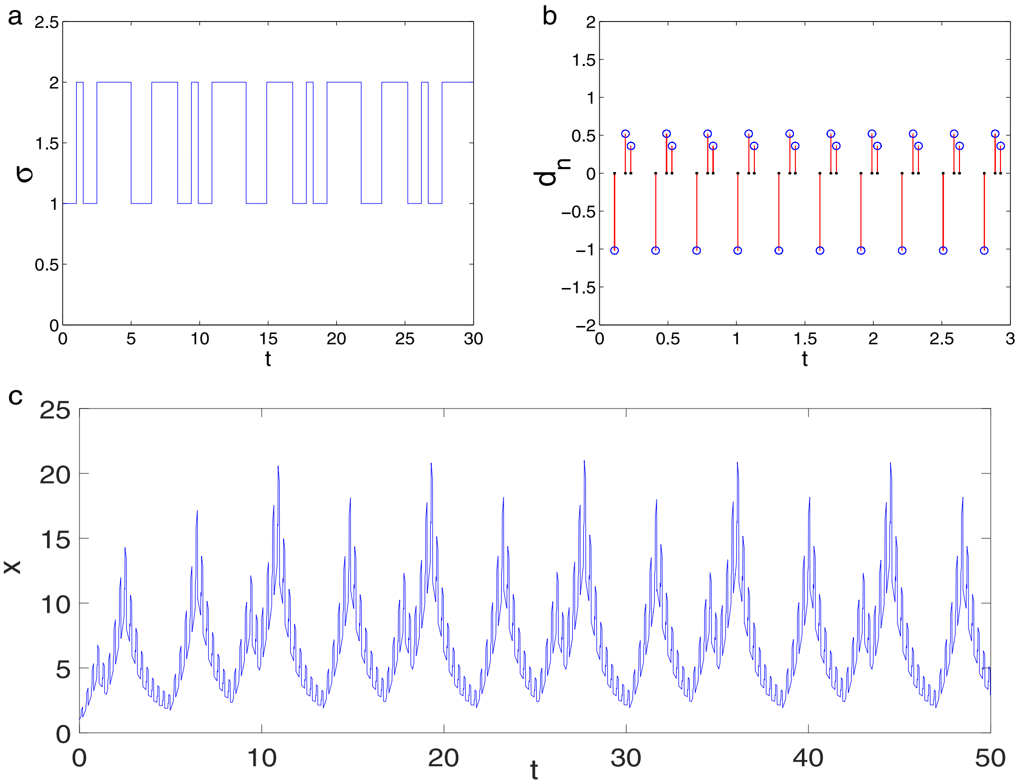


**Fig. 2.** Simulation results for Example 2.

## 5. Conclusion and future work

In this paper, the problem of *ISS* for impulsive switched system with incommensurate impulsive switching signals has been investigated, where the switching time is different from the impulse jump time. Based on Lyapunov method and *ADT* approach, some sufficient conditions have been proposed, where the hybrid effects of two types of impulses are taken into account. We have shown that when all of the modes are *ISS*, a switched system under an *ADT* scheme is *ISS* even if there exist destabilizing impulses. Moreover, when none of the modes is *ISS*, *ISS* still can be achieved under the designed *ADT* scheme couples with stabilizing impulses. In particular, we have also shown that the sufficient condition ensuring *ISS* property can be found when the impulsive switched system is composed of some *ISS* modes and some non-*ISS* modes. However, It is worth noting that our results only apply for a class of impulsive switched systems without time delay. Hence, an open question to be investigated in the future is the design of impulsive switching law such that the impulsive switched system with incommensurate impulsive switching signals is *ISS*, where the effects of hybrid delayed impulses are considered. In addition, another interesting direction for future research is to consider the *IOSS* property of impulsive switched system where the instants of stabilizing impulses are determined by an event-triggered mechanism *(ETM)*.

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