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Photon-added deformed spin coherent states and bipartite entanglement

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ABSTRACT

In present manuscript, we build the photon-added deformed spin coherent states (PA-DSCSs) according to deformed spin algebra (DSA). We present the way for solving the unity operator problem and we give the solution for some particular cases. The obtained states are developed through the Holstein-Primakoff (HP) realization of the DSA. The statistical features of the obtained coherent states are examined through the variation of the Mandel's parameter. By using the PA-DSCSs, we investigate the bipartite entanglement with respect to different parameters of these states.

Introduction

In many areas of physics, coherent states (CSs) have been played a significant role and provided an incredibly rich structure. By exploiting quantum harmonic oscillator states, Schrödinger was the first to present these states and demonstrate a link between quantum and classical formulations [1]. The CSs are characterized by the generators of the Heisenberg Lie algebra (LA) which are specified by the annihilation (A-) and creation (C-) operators of the quantum harmonic oscillator. These CSs have been employed by Glauber's to offer a novel explanation of the optical light effects by considering the eigenkets of the A-operator [2]. A. Klauder has reintroduced and developed the same states while taking into account a collection of continuous states for every Lie group. The unifying factor between these states is that they are all related to the Heisenberg LA for harmonic oscillators. Several significant physical applications are provided by these expansions. Later, Perelomov [3,4] and Gilmore [5] proposed the CSs connected to any Lie group. Spin (SCSs) associated to the $SU(2)$ group and Peremelov and Barut-Giradello CSs related to the $SU(1,1)$ group are two examples of these states that are for quantum harmonic oscillators. These states provide various applications in theoretical physics [6–11] by describing a vast collection of quantum systems.

On the other hand, the concept of CSs has been expanded to include the situation of q-deformations [12–17] via quantum groups, which were first proposed as a mathematical description of deformed LAs. They were first proposed as a logical development of the idea of CSs. The q-boson A- and C- operators were used to define a q-deformed harmonic oscillator [18,19], with the latter operator fulfilling the crucial Heisenberg-Weyl algebra commutation relations [18–20]. The eigenstates of the q-boson A-operator are the q-deformed CSs that Biedenharn and MacFarlane introduced. Such states have been thoroughly explored [21–24] and widely used in quantum optics and mathematical physics [25–35]. In order to produce CSs associated with the $su_q(2)$ quantum algebra (QA), deformed SCSs were also created [36,37]. These states have recently attracted a great deal of interest because of their potential applicability in many fields of physics [38–40]. The realizations of some representations for the $su_q(2)$ QA, such as the Schwinger boson realization [13,14] and the HP realization [41], are then constructed using the oscillators that are associated to the q-boson operators. A vast class of quantum systems established from various potentials, such as Poschl-Teller, infinite and Morse potentials, etc., have been described using these deformed CSs (by selecting the q-deformed parameter (DP) appropriately, see Ref. [42]). Numerous nonclassical features, such as squeezing, antibunching effect and sub-Poissonian photon statistics are noticed in physical systems described by CSs (for a discussion see Ref. [43]). Bunched and antibunched light from a real laser have been experimentally proven to have statistics for the number of photons that can be either super- or sub-Poissonian. The deformed CSs are of significance from a physical standpoint because they provide the preferable description of non-ideal physical devices like lasers [44]. The q-DP then serves as a tunable parameter, indicating how far from the ideal the implemented device is. Recent research by Chakrabarti and Vasan [45] has explored entanglement via CSs for $su_q(1,1)$ QA of bipartite quantum systems. Founded on the non-commutativity of the coproduct structure of the QA generators, they

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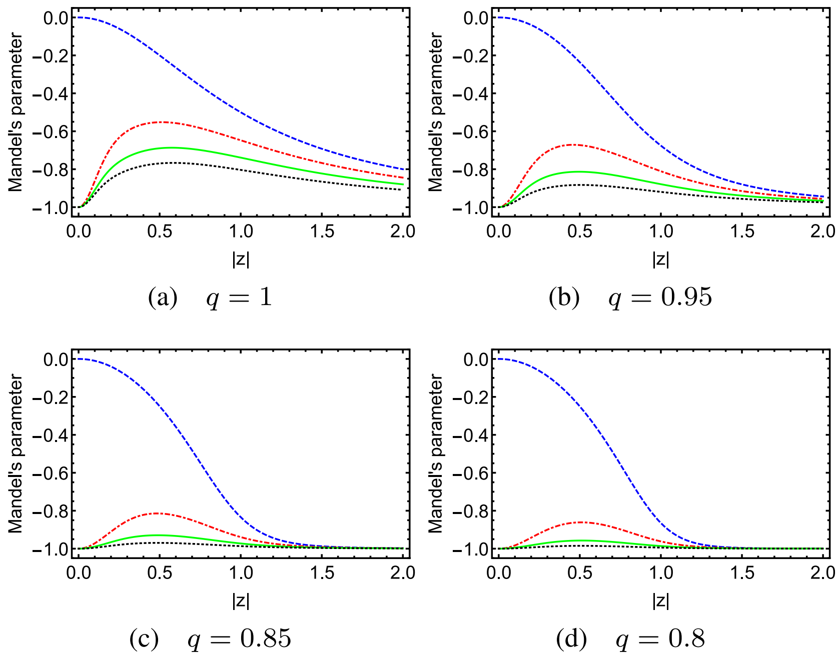


Fig. 1. The MP of the PA-DSCSs, defined by Eq. (39), in terms of $|\zeta|$ for different values of q and m with spin number $j = 5$. (a) is displayed for the case of $q = 1$, (b) is displayed for the case of $q = 0.95$, (c) is displayed for the case of $q = 0.85$, and (d) is displayed for the case of $q = 0.8$. dashed (Blue line): PA-DSCSs for $m = 0$; dashed-dotted (red line): PA-DSCSs for $m = 1$; solid (green line): PA-DSCSs for $m = 2$; dotted (black line): PA-DSCSs for $m = 3$. Q_q takes negative values for various values of $|\zeta|$, q and m, displaying a sub-Poissonian distribution of PA-DSCSs. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

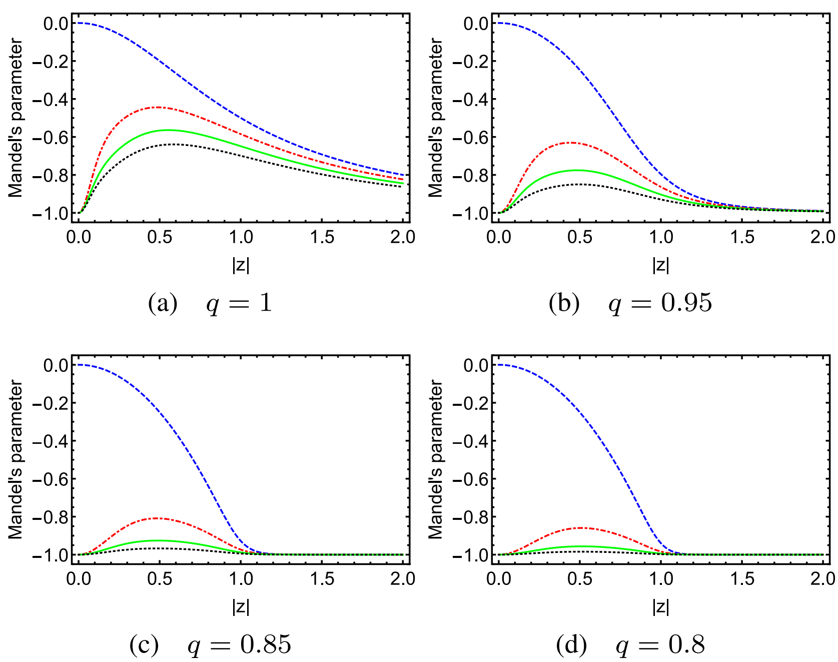


Fig. 2. The MP of the PA-DSCSs, defined by Eq. (39), in terms of $|\zeta|$ for different values of q and m with spin number $j = 10$. (a) is displayed for the case of $q = 1$, (b) is displayed for the case of $q = 0.95$, (c) is displayed for the case of $q = 0.85$, and (d) is displayed for the case of $q = 0.8$. dashed (Blue line): PA-DSCSs for $m = 0$; dashed-dotted (red line): PA-DSCSs for $m = 1$; solid (green line): PA-DSCSs for $m = 2$; dotted (black line): PA-DSCSs for $m = 3$. Q_q takes negative values for various values of $|\zeta|$, q and m, displaying a sub-Poissonian distribution of PA-DSCSs. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

determined the presence of a naturally entangled structure from the value of the q-DP. Several efforts on a number of topics have been inspired by the realization of the $su_q(2)$ QA in the context of q-harmonic oscillators.

Numerous quantum information processing and transmission activities necessitate the establishment of entangled states, which has received much study and is a crucial step. Beam splitters [46,47], Cavity QED [48], NMR systems [49], and other devices have recently been suggested and experimentally implemented to produce quantum entanglement. Kim et al. [50] conducted an extensive study on the entanglement produced by a beam splitter. They demonstrated that at least one of the input fields must be nonclassical in order to generate entanglement at output states of the beam splitter device. Ivan et al. have demonstrated that the output state will be entangled if the input state is defined a product state of a field with a sub-Poissonian photon statistics in one input mode and the vacuum in the other mode [51].

The following three definitions can be used to identify coherent states. The CSs are eigenstates for the A-operator. The second definition asserts that CSs are the result of the displacement operator acting on the initial ground state. We can also define the CSs as quantum states that verify the minimal Heisenberg uncertainty. The PA-CSs for the harmonic oscillator were described for the first time by Agarwal and Tara [52]. These states generated a lot of interest and offered several physical applications [53–56]. Due to their potential applications, many generalizations have been introduced [57–63]. Therefore, CSs of this type could be valuable. In present manuscript, we construct the PA-DSCSs associated to DSA. We present the way for solving the problem of the unity operator and we give the solution for some particular cases. The obtained states are developed through the HP realization of the DSA. The statistical features of these deformed CSs are examined via the variation of the Mandel's parameter. By using the PA-DSCSs, we investigate the bipartite entanglement with respect to different parameters of these states.

The manuscript is outlined as follows. In Section "Photon added spin coherent states" we introduce the PA-DSCSs with the minimal set condition of Klauder's. In Section "Generation of the PA-DSCS" deals with the generation of the PA-DSCSs. Section "Entangled PA-DSCS" describes the entangled PA-DSCSs in the framework of DSA and give the conditions for which these bipartite states are maximally entangled states. Finally, we summary the work in the last section.

Photon added spin coherent states

Let us start by discussing the CSs associated to $su_q(2)$ LA. We start with the $su_q(2)$ algebra which is characterized by three generators $\hat{J}^q_\pm$ and $\hat{J}^q_z$ verifying the relations of commutation

$$[\hat{J}^q_z, \hat{J}^q_\pm] = \pm\hat{J}^q_\pm; \qquad [\hat{J}^q_+, \hat{J}^q_-] = 2\hat{J}^q_z. \tag{1}$$

The generators obey also the following relations

$$(\hat{J}^q_z)^+ = \hat{J}^q_z, \quad (\hat{J}^q_-)^+ = \hat{J}^q_+, \quad (\hat{J}^q_+)^+ = \hat{J}^q_-. \tag{2}$$

By using a nonnegative and half-integer parameter j, the unitary irreducible representation of the DSA is indexed. The orthogonal basis of the representation space is denoted by $|j, \mathrm{M}\rangle_q$, with $\mathrm{M} = j, j-1, \ldots, -j$. The DSA operators acting on this basis as,

$$\hat{J}_z|j, \mathrm{M}\rangle_q = \mathrm{M}|j, \mathrm{M}\rangle_q, \tag{3}$$

$$\hat{J}^q_\pm|j, \mathrm{M}\rangle = \left[[j \pm \mathrm{M} + 1]_q[j \mp \mathrm{M}]_q\right]^{\frac{1}{2}} |j, \mathrm{M} \pm 1\rangle_q. \tag{4}$$

The DSA operators can be written as a function of the deformed oscillator operators $(\hat{A}^q_+, \hat{A}^q)$ in several realizations [41,64]. The DSCSs given in terms of one-mode deformed C- and A-operators are associated with the HP realization form of the DSA [41]. This realization is the operators

$$\hat{J}^q_+ = \hat{A}^q_+\sqrt{[2j-\hat{n}]_q}, \ \hat{J}^q_- = \sqrt{[2j-\hat{n}]_q}\hat{A}^q, \ \hat{J}^q_z = [\hat{n}]_q - j, \tag{5}$$

where $\hat{A}^q$ and $\hat{A}^q_+$ are respectively the deformed A- and C-operator. The states $|j, \mathrm{M}\rangle_q$ depend on the deformed Fock states $|n\rangle_q$ according to $|j, \mathrm{M}\rangle_q \sim |n\rangle_q$ with $n = j + \mathrm{M}$ and that $n = 0, 1, \ldots, 2j$ for a fixed value of j. In this way, the Hilbert space associated to deformed operators is introduced [65]. The usual operators $\{\mathbb{1}; \hat{\mathrm{A}}; \hat{\mathrm{A}}_+\}$ are utilized for describing and constructing the so-called QA of the Heisenberg-Weyl group associated to quantum harmonic oscillators. The deformed version of quantum oscillators is obtained via the deformed LA constructed by using the deformed operators $\{\mathbb{1}; \hat{\mathrm{A}}^q; \hat{\mathrm{A}}^q_+\}$ that obey the relations of commutation:

$$\hat{\mathrm{A}}^q\hat{\mathrm{A}}^q_+ - q^2\hat{\mathrm{A}}^q_+\hat{\mathrm{A}}^q = \mathbb{1}, \ [\hat{\mathrm{A}}^q, \hat{\mathrm{A}}^q_+] = \mathbb{1} + (q^2-1)\hat{\mathrm{A}}^q_+\hat{\mathrm{A}}^q = \mathbb{1} + (q^2-1)\hat{\mathbb{N}}^q \tag{6}$$

where $\hat{\mathbb{N}}^q = \hat{A}^q_+\hat{A}^q$ and $0 < q < 1$ is a real, and $\hat{\mathrm{A}}^q$ and $\hat{\mathrm{A}}^q_+$ acting on the deformed Fock state as

$$\hat{\mathrm{A}}^q_+|n\rangle_q = \left([n+1]_q\right)^{\frac{1}{2}}|n+1\rangle_q, \hat{\mathrm{A}}^q|n\rangle_q = \left([n-1]_q\right)^{\frac{1}{2}}|n-1\rangle_q,$$

$$\hat{\mathbb{N}}^q|n\rangle_q = [n]_q|n\rangle_q. \tag{7}$$

The Fock space corresponds to deformed states is determined by construction

$$|n\rangle_q = \frac{\left(\hat{\mathrm{A}}^q_+\right)^n}{\sqrt{[n]_q!}}|0\rangle \tag{8}$$

where $[n]_q$ is considered as

$$[n]_q = \frac{1-q^{2n}}{1-q^2}, \tag{9}$$

and that

$$[x]_q! = [x]_q[x-1]_q \cdots [1]_q; \quad [0]_q! = 1. \tag{10}$$

The function $[x]_q$ depends on the parameter q and we can recover the usual QA when the parameter q reaches the value 1.

By using the deformed raising operator J^q_+ that acts on the ground state, A DSCS $|z, j\rangle_q$ is defined as

$$|\zeta, j\rangle_q = \frac{1}{(1+|z|^2)^j_q}\exp_q\left(\zeta\hat{J}^q_+/\hbar\right)|0\rangle_q. \tag{11}$$

which is given by

$$\exp_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} .$$

Expanding the function $\exp_q(x)$, the DSCS becomes

$$|\zeta, j\rangle_q = \mathcal{N}_q\left(|\zeta|^2\right)\sum_{n=0}^{2j}\begin{bmatrix}2j\\n\end{bmatrix}_q^{\frac{1}{2}}\zeta^n|n\rangle_q. \tag{12}$$

Since for $m = 2j$, we have $\hat{J}^q_+|m\rangle = 0$ the summation goes up to $2j$. The factor of normalization is written as

$$\mathcal{N}_q\left(|\zeta|^2\right) = \frac{1}{\sqrt{(1+|\zeta|^2)^{2j}_q}} , \tag{13}$$

where, we have utilized the deformed Newton's binomial formula:

$$(x+y)^n_q := \sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q x^{x-m}y^m. \tag{14}$$

Here, the deformed binomial function is,

$$\begin{bmatrix}n\\m\end{bmatrix}_q = \frac{[n]!}{[n]_q![n-m]_q!} \qquad \text{for} \qquad n \geq m . \tag{15}$$

In the present manuscript, we construct a considerable set of deformed states through the addition of photons to the traditional DSCS $|\zeta, j\rangle_q$.

According to Klauder's minimal conditions [6], a quantum state $|\zeta\rangle$ to be a CS should satisfy:

• Normalization condition:

$$\langle\zeta|\zeta\rangle = 1; \tag{16}$$

• Continuity property:

$$|\zeta - \zeta'| \to 0 \quad \Rightarrow \quad ||\zeta\rangle - |\zeta'\rangle| \to 0; \tag{17}$$

• Resolution of unity:

$$\int\int_C d^2\zeta|\zeta\rangle\langle\zeta|W(|\zeta|^2) = \mathbb{1}, \tag{18}$$

where $W(|\zeta|^2)$ is a positive function. The validation of the third condition depends on the quantum system, while the first two conditions are easy to be verified.

The PA-DSCSs $|z, j, m\rangle_q$ can be constructed through repeated application of the operator $\hat{J}^q_+$ to DSCSs $|\zeta, j\rangle_q$

$$|\zeta, j, m\rangle_q = N_{q,m}(|\zeta|^2)(\hat{J}^q_+)^m|\zeta, j\rangle_q \tag{19}$$

where m is a nonnegative integer representing the number of added photons. By using Eq. (4), we get

$$(J^q_+)^m|n\rangle_q = \left[\frac{[n+m]_q![2j-n]_q!}{[n]_q![2j-n-m]_q!}\right]^{\frac{1}{2}}|n+m\rangle_q. \tag{20}$$

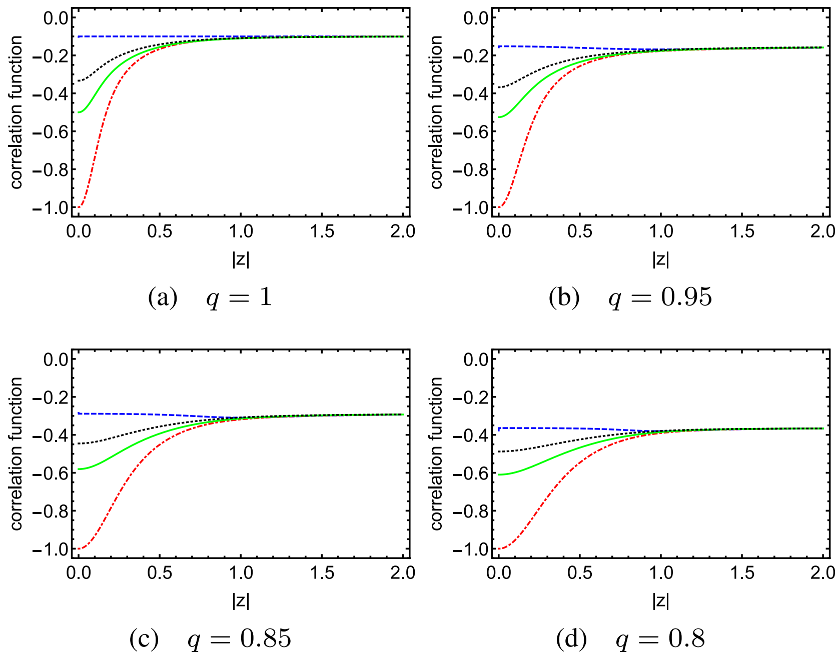


Fig. 3. The SOCF $G^{(2)}(0)$ for the PA-DSCSs in terms of $|\zeta|$ for different values of q and m with spin number $j = 5$. (a) is displayed for the case of $q = 1$, (b) is displayed for the case of $q = 0.95$, (c) is displayed for the case of $q = 0.85$, and (d) is displayed for the case of $q = 0.8$. dashed (Blue line): PA-DSCSs for $m = 0$; dashed-dotted(red line): PA-DSCSs for $m = 1$; solid (green line): PA-DSCSs for $m = 2$; dotted (black line): PA-DSCSs for $m = 3$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

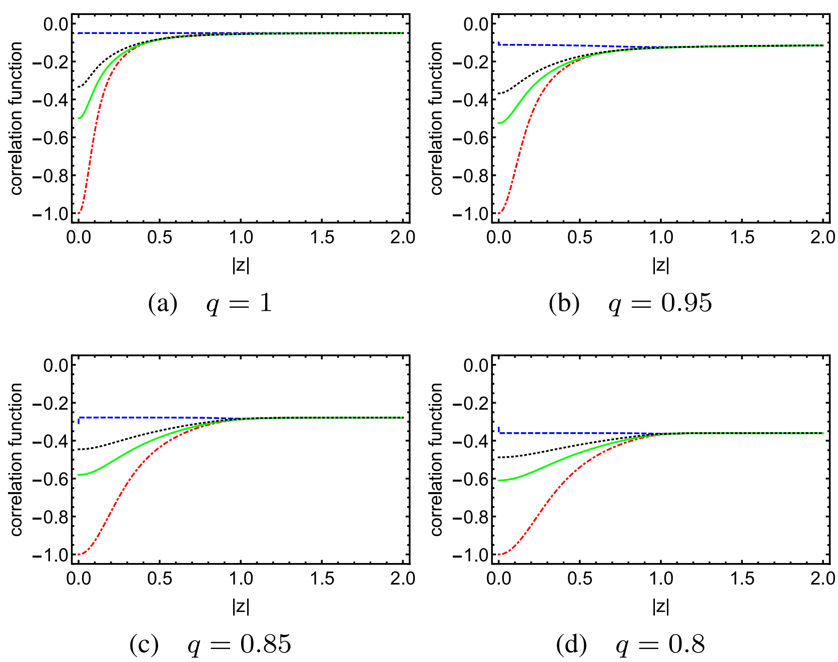


Fig. 4. The SOCF $G^{(2)}(0)$ for the PA-DSCSs in terms of $|\zeta|$ for different values of q and m with spin number $j = 10$. (a) is displayed for the case of $q = 1$, (b) is displayed for the case of $q = 0.95$, (c) is displayed for the case of $q = 0.85$, and (d) is displayed for the case of $q = 0.8$. dashed (Blue line): PA-DSCSs for $m = 0$; dashed-dotted (red line): PA-DSCSs for $m = 1$; solid (green line): PA-DSCSs for $m = 2$; dotted (black line): PA-DSCSs for $m = 3$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Substituting (20) into (19), we find that

$$|\zeta,j,m\rangle_q = N_{q,m}(|\zeta|^2)\sum_{n=0}^{2j-m}\left[\frac{[2j]_q![n+m]_q!}{([n]_q!)^2[2j-n-m]_q!}\right]^{\frac{1}{2}}\zeta^n|n+m\rangle_q \tag{21}$$

where the factor of normalization is

$$N_{q,m}(|\zeta|^2) = \left(\sum_{n=0}^{2j-m}\left[\frac{[2j]_q![n+m]_q!}{([n]_q!)^2[2j-n-m]_q!}\right]|\zeta|^{2n}\right)^{-\frac{1}{2}}. \tag{22}$$

It is clear that, if in the obtained result when the number m tends to zero, we can recover the DSCS $|z,j\rangle_q$.

Let us now present the problem of the resolution of unity which is given in the following form

$$\int\int_C d^2\zeta|\zeta,j,m\rangle_q W_{q,m}\left(|\zeta|^2\right)\ {}_q\langle\zeta,j,m| = \mathbb{1}_{2j-m}. \tag{23}$$

i.e., we try to obtain a positive $W_{q,m}\left(|\zeta|^2\right)$ obeying Eq. (23). Substitute

$$\pi \sum_{n=0}^{2j-m} |n+m\rangle_q\, {}_q\langle n+m| \left[\frac{[2j]_q![n+m]_q!}{([n]_q!)^2[2j-n-m]!} \int_0^{\infty} dx N_{q,m}^2(x) W_{q,m}(x)\, x^n \right] = \mathbb{1}_{2j-m} \tag{24}$$

Box I.

(21) into Eq. (23), use $d^2\zeta = 1/2d\,(|\zeta|)\,d\varphi$, $(\zeta = |\zeta|e^{i\varphi})$, and execute the angular integration. We can obtain for $|\zeta|^2 = x$: Eq. (24) is given in Box I.

or the condition on $W_{q,m}(x)$

$$\pi \int_0^{\infty} dx N_{q,m}^2(x) W_{q,m}(x)\, x^n = \frac{([n]_q!)^2[2j-n-k]_q!}{[2j]_q![n+m]_q!}. \tag{25}$$

Introducing the function $\hat{W}_{q,m}(x) = \pi N_{q,m}^2(x) W_{q,m}(x)$, Eq. (25) becomes

$$\int_0^{\infty} x^n \hat{W}_{q,m}(x)\, dx = \frac{([n]_q!)^2[2j-n-m]_q!}{[2j]_q![n+m]_q!}, \quad \forall n = 0, 1, \dots, \tag{26}$$

which represents the Stieltjes power-moment problem [66] for the function $\hat{W}_{q,m}(x)$. Here, the main problem is to provide a positive solution $\hat{W}_{q,m}(x)$ in Eq. (26).

The Eq. (26) can be interpreted as the inverse Mellin transform problem [66], by setting $n \to s-1$ and rewriting Eq. (26) as

$$\begin{aligned} \int_0^{\infty} x^{s-1} \hat{W}_{q,m}(x)\, dx &= \frac{([s-1]_q!)^2[2j+1-s-m]_q!}{[2j]_q![s+m-1]_q!} \\ &\equiv \mathcal{M}\left[\hat{W}_k(x)\,; s\right], \end{aligned} \tag{27}$$

and that

$$\begin{aligned} \hat{W}_{q,m}(x) &= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{([s-1]_q!)^2[2j+1-s-m]_q!}{[2j]_q![s+m-1]_q!} x^{-s} ds \\ &\equiv \mathcal{M}^{-1}\left[\frac{([s-1]_q!)^2[2j+1-s-m]_q!}{[2j]_q![s+m-1]_q!}; x\right], \end{aligned} \tag{28}$$

where $\mathcal{M}$ and $\mathcal{M}^{-1}$ represent the Mellin transform and inverse Mellin transform, respectively.

The function $\hat{W}_{q,m}(x)$ can be found by considering the deformed gamma function [67] satisfying

$$\Gamma_q(s+1) = [s]_q \Gamma_q(s), \quad \text{with} \quad \Gamma_q(1) = 1. \tag{29}$$

In the $q \to 1$ limit, we obtain the usual gamma function

$$\lim_{q\to 1} \Gamma_q(s) = \Gamma(s), \tag{30}$$

and for a nonnegative integer n, we have

$$\Gamma_q(n+1) = [n]_q!. \tag{31}$$

Then, the measure function $\hat{W}_{q,m}(x)$ can analytically continued as

$$\hat{W}_{q,m}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\left[\Gamma_q(s)\right]^2 \Gamma_q(2j+2-m-s)}{[2j]_q!\Gamma_q(s+k)} x^{-s} ds. \tag{32}$$

The evaluation of the integral in Eq. (32) may be found in the framework of the residues theory.

To illustrate the approach mentioned above, we consider the usual case that corresponds to choosing $[n]_q = n$. In this case, Eq. (32) is given in terms of the usual gamma function as:

$$\hat{W}_{1,m}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{[\Gamma(s)]^2\, \Gamma(2j+2-m-s)}{[2j]!\Gamma(s+k)} x^{-s} ds. \tag{33}$$

This integration in equation can be obtained by using the following property of the Mellin transform [68]

$$\int_0^{\infty} F\left(\frac{x}{t}\right) G(t) \frac{1}{t} dt = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F^*(s) G^*(s) x^{-s} ds. \tag{34}$$

By choosing $F^*(s) = \Gamma(s)\Gamma(2j+2-m-s)$, and $G^*(s) = \Gamma(s)/\Gamma(s+m)$ which result

$$\mathcal{M}^{-1}\left[F^*(s); x\right] = \frac{\Gamma(2j+2-m)}{(1+x)^{2j+2-m}}, \tag{35}$$

$$\mathcal{M}^{-1}\left[G^*(s); x\right] = \frac{H(1-x)(1-x)^{m-1}}{\Gamma(k)} \tag{36}$$

where H represents the Heaviside function.

Therefore, the measure function is

$$\hat{W}_{1,m}(x) = \frac{\Gamma(c)}{\Gamma(c+m-1)\Gamma(m)} \int_0^1 \frac{t^{c-1}(1-t)^{m-1}}{(t+x)^c} dt \tag{37}$$

which is a nonnegative function of x with $c = 2j+2-m$. In the limiting case $q \to 1$ and $m \to 0$, the weighting function of the ordinary SCSs is given by

$$W_{1,0}(x) = \frac{2j+1}{\pi(1+x)^2}. \tag{38}$$

To analyze the statistical distribution of photons for the PA-DSCSs and determine whether the distribution is Poissonian, sub-Poissonian, or super-Poissonian, it is appropriate to utilize the Mandel's parameter (MP). The MP is defined as [69]

$$Q_q = \frac{{}_q\langle A_+^q A^q A_+^q A^q\rangle_q - {}_q\langle A_+^q A^q\rangle_q^2 - {}_q\langle A_+^q A^q\rangle_q}{{}_q\langle A_+^q A^q\rangle_q}. \tag{39}$$

The MP determines whether the PA-DSCSs states have a sub-Poissonian distribution ($-1 \le Q_q < 0$, nonclassical states), Poissonian distribution ($Q_q = 0$, standard coherent states) or super-Poissonian distribution ($Q_q > 0$, classical states).

Based on the numerical results, we explain the impact of spin number j, photon-added number m, amplitude $|\zeta|$ and DP q on the statistical properties for the PA-DSCSs. Figs. 1 and 2 show the behavior of the parameter Q_q versus $|\zeta|$ for different values of j, m and q. In general, the variation of the parameter Q_q demonstrates that the PA-DSCSs statistical properties have a different order according to the values of $|\zeta|$. We can observe that the parameter Q_q takes negative value for different values of $|\zeta|$, q, m and j, exhibiting the sub-Poissonian distribution of photons. Interestingly, we can observe that the increase of m leads to decrease the parameter Q_q and the PA-DSCSs become more quantum mechanical as m attains high values. In this context, more quantum mechanical refers to the states that get farther from the states with Poissonian distribution $Q_q = 0$. Furthermore, the parameter Q_q decreases as the q-DP gets farther from 1 and it attains the values -1 as $|\zeta|$ becomes significantly large. The dependence of the MP on the spin number j provides that the MP increases with decreasing m, showing that the PA-DSCSs are farther to classical states form small values of the spin number j. In this limit, the parameter Q_q attains the value -1 for small values of $|\zeta|$ whenever the q-DP gets farther from 1. We can conclude that the enhancement the non-classicality of PA-DSCSs can be made by a proper choice of the spin number j, DP q, excited number m and the amplitude ζ.

The second-order correlation function (SOCF), which is defined as follows, is largely used to examine the effects of bunching or antibunching [70]

$$G^{(2)}(\tau) = \frac{\langle :\, I(t)I(t+\tau)\, :\rangle}{\langle I(t)\rangle^2}, \tag{40}$$

where I represents the field intensity with $:\, I(t)I(t+\tau) := A_+^q(t)A_+^q(t+\tau)A^q(t+\tau)A^q(t+\tau)$. The function $G^{(2)}(\tau)$ is proportional to the probability

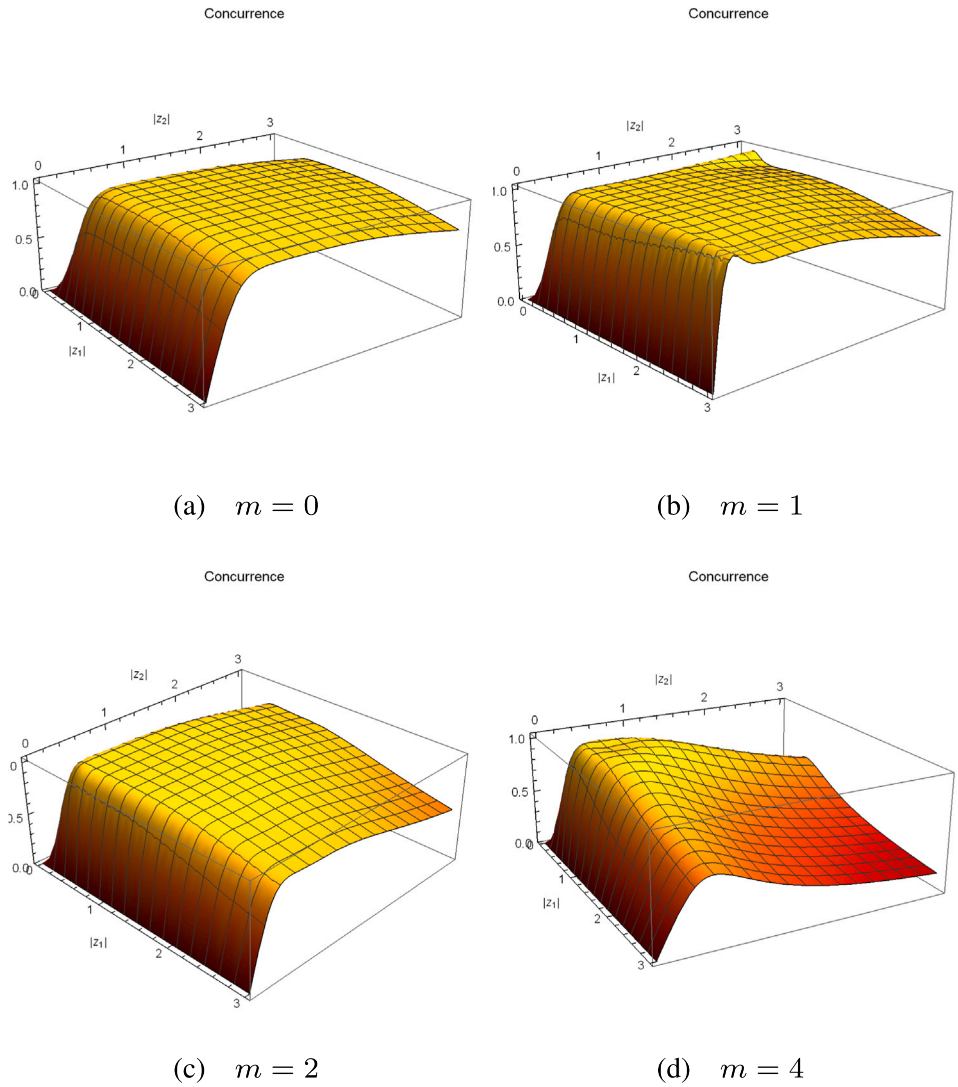


Fig. 5. (Color online) The concurrence, given by equation. (64), as a function of $|\zeta_1|$ and $|\zeta_2|$ for various values of m with $j_1 = j_2 = 3$ and $q = 0.95$. (a) is for $m = 0$, (b) is for $m = 1$, (c) is for $m = 2$, and (d) is for $m = 4$. It is clear that the amount of the entanglement of entangled PA-DSCSs can be controlled and enhanced by a considerable choice the physical parameters.

for detecting one photon at the time t and a second one at $t+\tau$. $G^{(2)}(0)$ is proportional to the probability for detecting two photons in the same time. For $G^{(2)}(\tau) < G^{(2)}(0)$, the probability of detecting a second photon after delay time τ decreases and this indicates bunching effect. On the other side, for $G^{(2)}(\tau) > G^{(2)}(0)$, the probability of detecting a second photon increases with the delay time and this indicates the antibunching effect. For coherent states $G^{(2)}(\tau) = 1$.

According to Figs. 3 and 4, it is visible that $G^{(2)}(0) < 1$, which indicates the sub-Poissonian statistics.

Generation of the PA-DSCS

To generate the PA-DSCS in (21) physically, we introduce the slab of excited two-level atoms (TLAs) via a cavity. Let $|\Phi(0)\rangle = |\zeta, j, m\rangle_q |e\rangle$ defines the TLA–field state at $t = 0$, where $|e\rangle$ represents the excited state of the TLA. The Hamiltonian of interaction takes the following configuration

$$H = \hbar\lambda \left(\sigma_+ \hat{J}_-^q + \hat{J}_+^q \sigma_-\right), \tag{41}$$

where σ_+ (σ_-) represents the raising (lowering) operator of atomic states. The state $|\Phi(0)\rangle$ evolves in times as

$$|\Phi(t)\rangle = \exp\left[-i\mu\left(\sigma_+ \hat{J}_-^q + \hat{J}_+^q \sigma_-\right)\right] |\Phi(0)\rangle, \tag{42}$$

where we have set $\mu = \hbar\lambda$ with λ defines the coupling constant. For $\mu \ll 1$, we have

$$|\Phi(t)\rangle \cong \left(1 - i\mu\left(\sigma_+ \hat{J}_-^q + \hat{J}_+^q \sigma_-\right)\right) |\zeta, j, m\rangle_q |e\rangle. \tag{43}$$

Then, the whole system (atom-field) state is given by

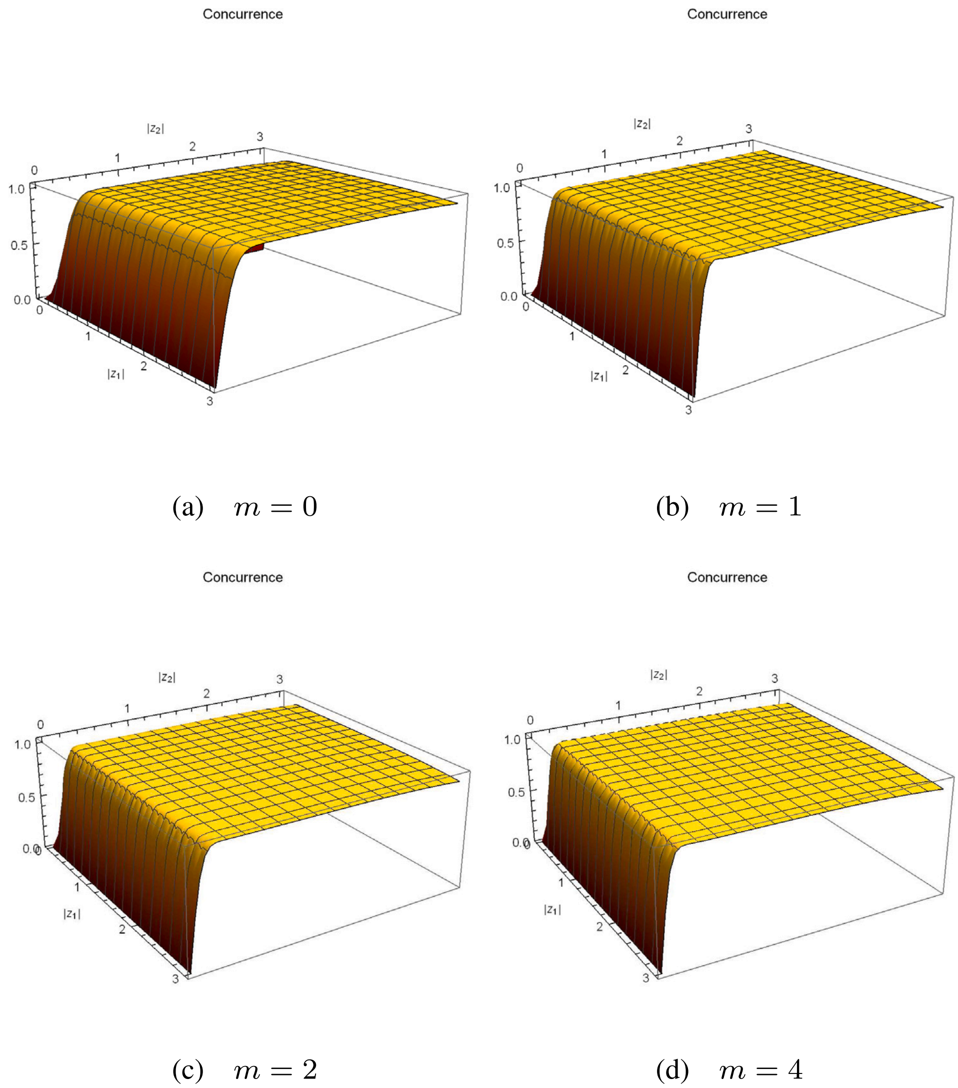
$$|\Phi(t)\rangle = |\zeta, j, m\rangle_q |e\rangle - i\mu \hat{J}_+^q |\zeta, j, m\rangle_q |g\rangle. \tag{44}$$

Consequently, if the TLA is detected in the ground state $|g\rangle$, the field state is transferred to $\hat{J}_+^q |\zeta, j, m\rangle_q$ which corresponds to the PA-DSCS $|\zeta, j, 1\rangle_q$. Hence, we can in principle generate the state $|\zeta, j, 1\rangle_q$. By generalizing the above process, we can easily generate PA-DSCSs with different values of the parameter m, by considering the Hamiltonian operator

$$H_m = \hbar\lambda \left(\sigma_+ \left(\hat{J}_-^q\right)^m + \left(\hat{J}_+^q\right)^m \sigma_-\right). \tag{45}$$

Entangled PA-DSCS

In present section, we give sufficient conditions that allow us to determine whether the bipartite PA-DSCSs are entangled, and we discuss the degree of entanglement of these quantum states in terms of j, m, q and $|\zeta|$.



(a) $m = 0$ (b) $m = 1$

(c) $m = 2$ (d) $m = 4$

Fig. 6. (Color online) The concurrence, given by equation. (64), as a function of $|\zeta_1|$ and $|\zeta_2|$ for various values of m with $j_1 = j_2 = 10$ and $q = 0.95$. (a) is for $m = 0$, (b) is for $m = 1$, (c) is for $m = 2$, and (d) is for $m = 4$. It is clear that the amount of the entanglement of entangled PA-DSCSs can be controlled and enhanced by a considerable choice the physical parameters.

Recently, it has been proposed two inequalities for the entanglement detection [71]. Let a composite system AB. The ket space for the bipartite system is $H = H_A \otimes H_B$ where H_A defines the ket space for subsystem A, and H_B corresponds to the space for subsystem B. We consider two operators $\hat{A}$ and $\hat{B}$ on H_A and H_B respectively. The state of bipartite system is entangled if it can be held the conditions

$$|\langle \hat{A}\hat{B}^{\dagger}\rangle|^2 > \langle \hat{A}^{\dagger}\hat{A}\hat{B}^{\dagger}\hat{B}\rangle \tag{46}$$

$$|\langle \hat{A}\hat{B}\rangle|^2 > \langle \hat{A}^{\dagger}\hat{A}\rangle\langle \hat{B}^{\dagger}\hat{B}\rangle.$$

Now, we explore their implications for the bipartite PA-DSCSs. In this case, we assume that each quantum subsystem is represented by a PA-DSCS as

$$|\psi\rangle = A|\zeta_1, j_1, m\rangle_q \otimes |\zeta_2, j_2, m\rangle_q + B|\zeta_1', j_1, m\rangle_q \otimes |\zeta_2', j_2, m\rangle_q. \tag{47}$$

In terms of the Fock state $|n_i\rangle$ (i = 1,2), we have

$$|\psi\rangle = \sum_{n_1=m}^{2j_1}\sum_{n_2=m}^{2j_2} \mathcal{A}^{j_1,j_2}_{n_1,n_2}|n_1\rangle \otimes |n_2\rangle, \tag{48}$$

where Eq. (49) is given in Box II.

where

$N_1 = n_1 - m$ and $N_2 = n_2 - m$.

Let $\hat{J}^q_{1-}$ and $\hat{J}^q_{2-}$ be the angular momentum lowering operators of the $su_{q_1}(2)$ and $su_{q_2}(2)$ QAs respectively, with raising operators $\hat{J}^q_{1+}$ and J^q_{2+}. The conditions of bipartite entanglement are

$$|\langle \hat{J}^q_{1-}\hat{J}^q_{2+}\rangle|^2 > \langle \hat{J}^q_{1+}\hat{J}^q_{1-}\hat{J}^q_{2+}\hat{J}^q_{2-}\rangle \tag{50}$$

$$|\langle \hat{J}^q_{1-}\hat{J}^q_{2-}\rangle|^2 > \langle \hat{J}^q_{1+}\hat{J}^q_{1-}\rangle\langle \hat{J}^q_{2+}\hat{J}^q_{2-}\rangle. \tag{51}$$

Using (48), we find that Eq. (52) is given in Box III.

$$\langle\psi|\hat{J}^q_{1^{\dagger}}\hat{J}^q_{1-}|\psi\rangle = \sum_{n_1=m}^{2j_1}\sum_{n_2=m}^{2j_2} |\mathcal{A}^{j_1,j_2}_{n_1,n_2}|^2 [n_1]_q[2j_1 - n_1 + 1]_q \tag{53}$$

and

$$\langle\psi|\hat{J}^q_{2^{\dagger}}\hat{J}^q_{2-}|\psi\rangle = \sum_{n_1=m}^{2j_1}\sum_{n_2=m}^{2j_2} |\mathcal{A}^{j_1,j_2}_{n_1,n_2}|^2 [n_2]_q[2j_2 - n_2 + 1]_q. \tag{54}$$

The entanglement condition is

$$|\langle\psi|\hat{J}^q_{1-}\hat{J}^q_{2-}|\psi\rangle|^2 > \langle\psi|\hat{J}^q_{1+}\hat{J}^q_{1-}|\psi\rangle\langle\psi|\hat{J}^q_{2+}\hat{J}^q_{2-}|\psi\rangle. \tag{55}$$

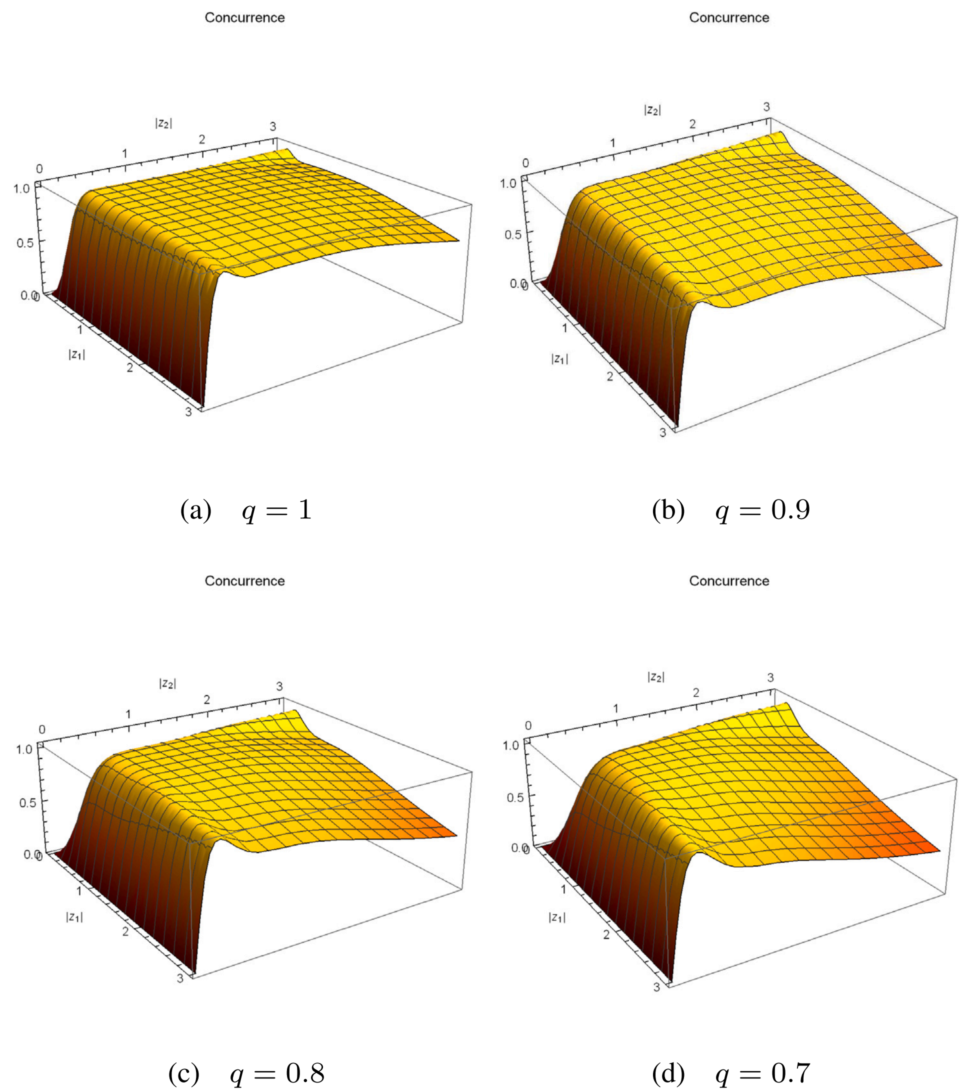


Fig. 7. (Color online) The concurrence, given by equation. (64), as a function of $|\zeta_1|$ and $|\zeta_2|$ for various values of q with $j_1 = j_2 = 3$ and $m = 1$. (a) is for $q = 1$, (b) is for $q = 0.9$, (c) is for $q = 0.8$, and (d) is for $q = 0.7$. It is clear that the amount of the entanglement of entangled PA-DSCSs can be controlled and enhanced by a considerable choice the physical parameters.

$$\mathcal{A}^{j_1,j_2}_{n_1,n_2} = \left[A N_{q,m}(|\zeta_1|^2) N_{q,m}(|\zeta_2|^2) \zeta_1^{N_1} \zeta_2^{N_2} + B N_{q,m}(|\zeta_1'|^2) N_{q,m}(|\zeta_2'|^2) {\zeta'}_1^{N_1} {\zeta'}_2^{N_2} \right] \left(\frac{[2j_1]_q![2j_2]_q![n_1]_q![n_2]_q!}{\left([N_1]_q![N_2]_q!\right)^2 [2j_1 - n_1]_q![2j_2 - n_2]_q!} \right)^{\frac{1}{2}}, \tag{49}$$

Box II.

$$\langle \psi | \hat{J}^q_{1-} \hat{J}^q_{2-} | \psi \rangle = \sum_{n_1=m+1}^{2j_1} \sum_{n_2=m+1}^{2j_2} \mathcal{A}^{*j_1,j_2}{}_{n_1,n_2} \mathcal{A}^{j_1,j_2}_{n_1-1,n_2-1} \left([n_1 - 1]_q [n_2 - 1]_q [2j_1 - n_1 + 2]_q [2j_2 - n_2 + 2]_q \right)^{\frac{1}{2}} \tag{52}$$

Box III.

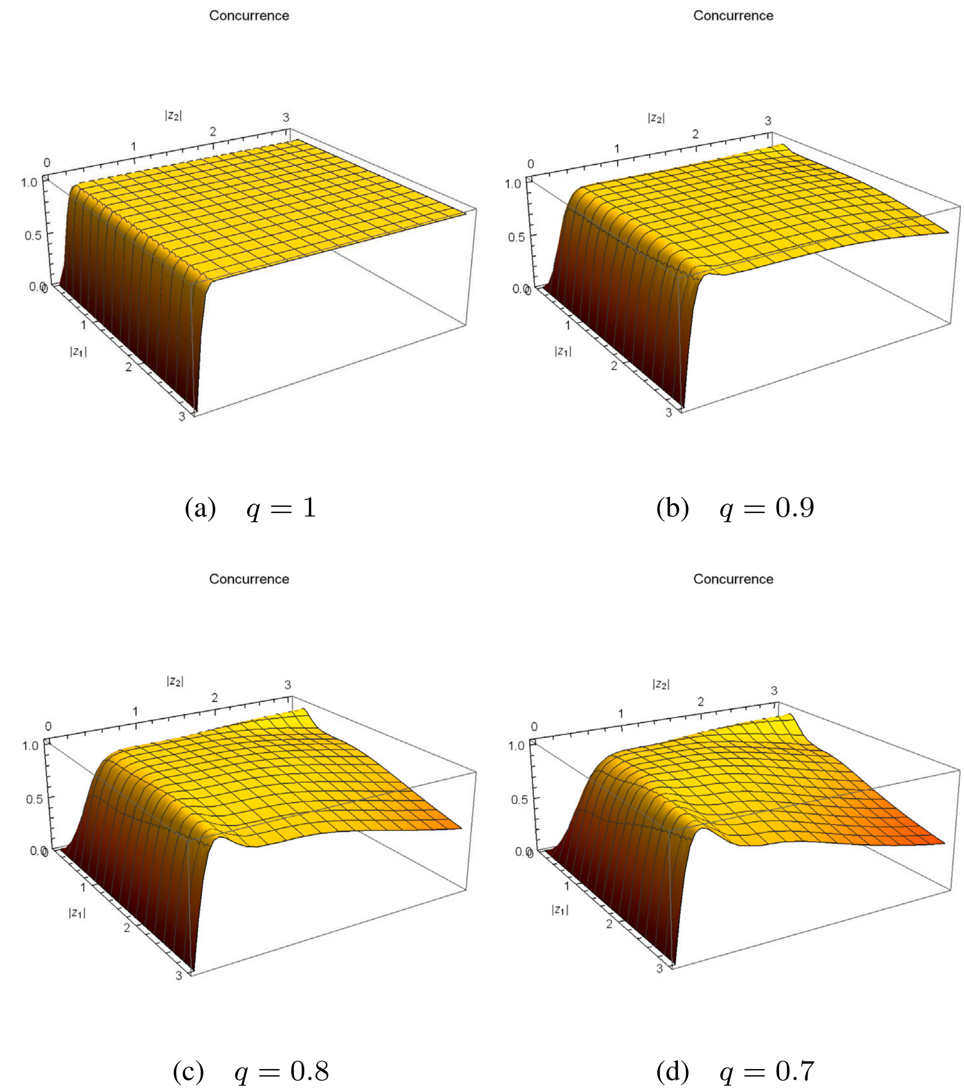


Fig. 8. (Color online) The concurrence, given by equation. (64), as a function of $|\zeta_1|$ and $|\zeta_2|$ for various values of q with $j_1 = j_2 = 10$ and $m = 1$. (a) is for $q = 1$, (b) is for $q = 0.9$, (c) is for $q = 0.8$, and (d) is for $q = 0.7$. It is clear that the amount of the entanglement of entangled PA-DSCSs can be controlled and enhanced by a considerable choice the physical parameters.

We note that the bipartite state (47) is factorizable if and only if one of the following is true: $A = 0$, $B = 0$, $|\zeta_1, j_1, m\rangle_q = \pm e^{i\phi_1}|\zeta_1', j_1, m\rangle_q$, or $|\zeta_2, j_2, m\rangle_q = \pm e^{i\phi_2}|\zeta_2', j_2, m\rangle_q$.

Now let us study the bipartite entanglement properties for entangled PA-DSCSs defined as follows

$$|\Psi\rangle = \mathcal{N}_\phi\Big(|\zeta_1, j_1, m\rangle_q \otimes |-\zeta_2, j_2, m\rangle_q + e^{i\phi}|-\zeta_1, j_1, m\rangle_q \otimes |\zeta_2, j_2, m\rangle_q\Big). \tag{56}$$

The quantum states $|\zeta_1, j_1, m\rangle_q$ and $|-\zeta_1, j_1, m\rangle_q$ (subsystem 1) and $|\zeta_2, j_2, m\rangle_q$ and $|-\zeta_2, j_2, m\rangle_q$ (subsystem 2) are spanned a 2-dimensional subspaces of the ket space and assumed to be linearly independent.

The normalization coefficient $\mathcal{N}_\phi$ is given by

$$\mathcal{N}_\phi = \frac{1}{\sqrt{2 + 2\cos\phi\langle\zeta_1, j_1, m| - \zeta_1, j_1, m\rangle_q\langle-\zeta_2, j_2, m|\zeta_2, j_2, m\rangle_q}}. \tag{57}$$

The state (56) represents the bipartite entanglement of a composite system in the framework of DSCSs.

Numerous quantum entanglement measures of bipartite states have suggested and examined. In this manuscript, we utilize the concurrence to examine the entanglement of PA-DSCSs. It has been shown that the concurrence of a system of two qubits $|\Psi\rangle$ can be defined as

$$C(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle|, \tag{58}$$

where the tilde designs the operation of spin-flip $|\tilde{\Psi}\rangle = \sigma_y \otimes \sigma_y|\Psi^*\rangle$. The ket $|\Psi^*\rangle$ represents the complex conjugate of the ket $|\Psi\rangle$, and σ_y is the pauli y−operator $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The function C ranges from 0 for factorizable states to 1 for maximally entangled states.

The bipartite quantum state (56) can be assumed as a state of two qubits by considering an orthogonal basis in the subspace spanned by $|\zeta_1, j_1, m\rangle_q$ and $|-\zeta_1, j_1, m\rangle_q$ and by $|-\zeta_2, j_2, m\rangle_q$ and $|\zeta_2, j_2, m\rangle_q$ as [72]

$$|0\rangle = |\zeta_1, j_1, m\rangle_q \qquad |1\rangle = \left(\frac{|-\zeta_1, j_1, m\rangle_q - \mathcal{V}_1|\zeta_1, j_1, m\rangle_q}{\mathcal{U}_1}\right) \quad \text{for subsystem 1} \tag{59}$$

$$|0\rangle = |\zeta_2, j_2, m\rangle_q \qquad |1\rangle = \left(\frac{|-\zeta_2, j_2, m\rangle_q - \mathcal{V}_2|\zeta_2, j_2, m\rangle_q}{\mathcal{U}_2}\right) \quad \text{for subsystem 2,} \tag{60}$$

where

$$\mathcal{U}_1 = \sqrt{1 - |\mathcal{V}_1|^2} \quad \text{and} \quad \mathcal{U}_2 = \sqrt{1 - |\mathcal{V}_2|^2}$$

with

$$\mathcal{V}_1 = {}_q\langle \zeta_1, j_1, m| - \zeta_1, j_1, m\rangle_q \tag{61}$$

and

$$\mathcal{V}_2 = {}_q\langle -\zeta_2, j_2, m|\zeta_2, j_2, m\rangle_q. \tag{62}$$

In this basis, the entangled PA-DSCSs (56) takes the form

$$|\Psi\rangle = \mathcal{N}_\phi e^{i\phi}\mathcal{U}_1|01\rangle + \mathcal{N}_\phi \mathcal{U}_2|10\rangle + \left(\mathcal{N}_\phi \mathcal{V}_2 + \mathcal{N}_\phi e^{i\phi}\mathcal{V}_1\right)|11\rangle. \tag{63}$$

The concurrence for the entangled PA-DSCSs can be expressed as

$$C(|\psi\rangle) = 2\mathcal{N}_\phi^2 \mathcal{U}_1\mathcal{U}_2. \tag{64}$$

The conditions for which the PA-DSCSs (56) are maximally entangled states depend on $\mathcal{B}_1, \mathcal{B}_2$ and Φ. In general, we have two cases: (1) $\mathcal{V}_1 \neq 0$ and $\mathcal{V}_2 \neq 0$ corresponding to the case of non-orthogonal states, the concurrence reaches its maximal value as $\langle \zeta_1, j_1, m| - \zeta_1, j_1, m\rangle_q = \langle \zeta_2, j_2, m| - \zeta_2, j_2, m\rangle_q$ and $\Phi = \pi$; (2) $\mathcal{V}_1 = 0$ and $\mathcal{V}_2 = 0$ corresponding to the case of orthogonal states, the PA-DSCSs become maximally entangled states ($C = 1$) for different values of ϕ. The conditions of maximum entanglement of bipartite states were discussed in the paper [73].

In Figs. 5 and 6, we display the variation of the measure of entangled PA-DSCSs as a function of ζ_1 and ζ_2 for various values m with a fixed spin number. Label (a) is for $m = 0$, label (b) is for $m = 1$, label (c) is for $m = 2$ and label (d) for $m = 4$. From the figures, we can find that the amount of the concurrence depends on the parameter m with respect to the variation of the parameters $|\zeta_1|$ and $|\zeta_2|$. On the other hand, large values of concurrence with accompanied by small values of $|\zeta_1|$ and $|\zeta_2|$ for which the concurrence takes the value 1. Furthermore, the quantum entanglement can be increased with the increase of the spin number j_1 and j_2. In Figs. 7 and 8, we show the variation of the concurrence of entangled PA-DSCSs as a function of ζ_1 and ζ_2 for different values of the q-DP with fixed values of spin number. Label (a) is for $q = 1$, label (b) is for $q = 0.9$, label (c) is for $q = 0.8$ and label (d) for $q = 0.7$. It is clear that the amount of the entanglement of entangled PA-DSCSs can be controlled and enhanced by a considerable choice the q-DP. We can obtain significant amount of entanglement for the entangled PA-DSCSs when the parameter q gets closer from 1 according to the physical parameters.

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In the $m \to 0$ limit, the set of orthogonal maximally entangled DSCSs is given by

$$\begin{cases} |\Psi_1\rangle = \frac{1}{\sqrt{2}}\left(\Big|\zeta_1, j_1, 0\Big\rangle\Big| - \zeta_2, j_2, 0\Big\rangle + \Big| - \zeta_1, j_1, 0\Big\rangle\Big|\zeta_2, j_2, 0\Big\rangle\right) \text{ for } \phi = 0 \\ |\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(\Big|\zeta_1, j_1, 0\Big\rangle\Big| - \zeta_2, j_2, 0\Big\rangle - \Big| - \zeta_1, j_1, 0\Big\rangle\Big|\zeta_2, j_2, 0\Big\rangle\right) \text{ for } \phi = \pi. \end{cases}$$

where

$$|\zeta_i| = \frac{1}{\sqrt{q^{2j_1 - 2k_1 + 1}}} \quad \text{with} \quad k_i \in \{1, 2, \ldots, 2j_i\} \text{ and } i = 1, 2.$$

In the case of non-orthogonal states, the maximally entangled DSCSs are given by

$$|\Psi\rangle = \mathcal{N}_\pi \left(|\zeta_1, j_1, 0\rangle_q \otimes |-\zeta_2, j_2, 0\rangle_q - |-\zeta_1, j_1, 0\rangle_q \otimes |\zeta_2, j_2, 0\rangle_q\right),$$

with condition

$$\langle \zeta_1, j_1, 0| - \zeta_1, j_1, 0\rangle_q = \langle \zeta_2, j_2, 0| - \zeta_2, j_2, 0\rangle_q.$$

In nutshell, by an algebraic approach, we have demonstrated that it is possible to attain maximum entanglement for PA-DSCSs.

Conclusion

We have introduced PA-DSCSs that are associated to DSA. We have presented the way for solving the problem of the unity operator and we gave the solution for some particular cases. The obtained states are developed through the HP realization of the DSA. We have found that the statistical features of photons of these CSs are sub- Poissonian in the context of deformed Fock space. Furthermore, we have introduced the way for generating the PA-DSCSs. On the other hand, we have considered bipartite entanglement phenomenon in the framework of deformed QA using PA-DSCSs. We have displayed the entanglement properties of the proposed bipartite states. In the present manuscript, we have only considered the bipartite quantum entanglement in the context of DSA. The more difficult task is to quantify the genuine multipartite entanglement, we expect that the obtained results can be expanded to the case of multipartite systems and show potential applications in various quantum information and optics contexts.

CRediT authorship contribution statement

K. Berrada: Writing – original draft, Writing – review & editing. **H. Eleuch:** Writing – review & editing.

Declaration of competing interest

The authors claim no conflict of interest statement.

Data availability

No data was used for the research described in the article.

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