Alexandria Engineering Journal 100 (2024) 260–267

Contents lists available at ScienceDirect

Alexandria Engineering Journal

journal homepage: www.elsevier.com/locate/aej

Original article

Entanglement and teleportation in thermal states of spin chains with nonlinear coupling

Nour Zidan [a,b,*], Ahmed Redwan [c], Tarek El-Shahat [c], Montasir Qasymeh [d], Mahmoud Abdel-Aty [b,e,f]

[a] Department of Mathematics, College of Science, Jouf University, Sakaka, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt
[c] Department of Mathematics, Faculty of Science, Al-Azhar University, 71524 Assiut, Egypt
[d] Department of Electrical Engineering, Abu Dhabi University, Abu Dhabi, 59911, United Arab Emirates
[e] Deanship of Graduate Studies and Research, Ahlia University, 10878 Manama, Bahrain
[f] Center for Photonics and Smart Materials, Zewail City of Science and Technology, Egypt

ARTICLE INFO

Keywords:
Entanglement
Nonlinear coupling
Teleportation

ABSTRACT

The impact of nonlinear coupling on entanglement and teleportation in a two-qubit spin system featuring Dzyaloshinskii–Moriya (DM) interaction was investigated while considering both homogeneous and inhomogeneous magnetic fields. Additionally, the dependencies of entanglement and teleportation on temperature and magnetic field for this specific interaction were analyzed. The findings demonstrate that the nonlinear coupling and DM interaction play significant roles in enhancing entanglement and teleportation. Furthermore, the results show that nonlinear coupling can positively affect the negativity and accuracy of teleportation, which are critical factors in quantum information processing. This suggests that nonlinear coupling could be valuable in developing quantum technologies.

1. Introduction

Quantum entanglement, as an impervious nonclassical correlation between spatially separated quantum systems, has played an essential role in quantum information processing [1]. Several investigations of entanglement in thermal equilibrium states of spin chains dependent on an external magnetic field at finite temperature have been made [2–4], where it is well known that an important emerging field is quantum entanglement in matter systems such as spin chains. Recently, exciting works have investigated the thermal quantum discord in different Heisenberg models [5,6]. The anisotropic antisymmetric exchange interaction is presented in phenomenological difference by Dzyaloshinskii [7] and in microscopic grounds by Moriya [8]. Many authors introduced detailed and comprehensive studies that have shown the dependence of entanglement and teleportation on various parameters, for example, interaction strength, temperature, single-ion anisotropy, and magnetic field [9–16]. The entanglement depends on the distance between spins and other physical parameters in the case of realistic systems, such as short-range entanglement between spins and charge degrees of freedom it is observed in quantum dots [17, 18],nanotubes [19] and molecules [20]. Some authors paid to discuss quantum teleportation by using thermal entangled states as entanglement resources [21,22]. The quantum teleportation based on the quantum nonlocal property can be activated with the Heisenberg spin chain as a quantum channel [23]. Based on the preceding discussion, exploring the relationship between wave solutions of nonlinear models and spin chains reveals that a spin chain is a distinct category of wave solution that arises in specific nonlinear partial differential equations (PDEs) [24–30]. Spin chains have been observed in various physical systems, including water waves [31], plasma waves [32], and optical waves [33], and have found applications in fields such as optical communications [34,35] and signal processing [36]. Many authors discussed various problems in various branches of science to investigate the nonlinear coupling in Heisenberg spin chain. There are some effective authors in this field see for example: The nonlinear spin excitations in one-dimensional, anisotropic Heisenberg spin chains have been discussed [37]. The nonlinear σ model is properly derived without spoiling the original spin degrees of freedom. The method shows that a single disordered phase continuously extends from a frustrated uniform regime to an unfrustrated distorted regime [38]. The nonlinear spin dynamics of (2 + 1)-dimensional Heisenberg ferromagnetic (FM) spin chains with bilinear and anisotropic interactions in

* Corresponding author at: Department of Mathematics, College of Science, Jouf University, Sakaka, Saudi Arabia.
E-mail address: nazidane@ju.edu.sa (N. Zidan).

https://doi.org/10.1016/j.aej.2024.05.026
Received 5 December 2023; Received in revised form 25 March 2024; Accepted 8 May 2024
Available online 22 May 2024

$$H = \begin{pmatrix} 4k\gamma^2 + 2B + J + k & 0 & 0 & 2(J+2k)\gamma \\ 0 & 2b - J + 5k & 2(-iD_z + J - 2k) & 0 \\ 0 & 2(iD_z + J - 2k) & -2b - J + 5k & 0 \\ 2(J+2k)\gamma & 0 & 0 & 4k\gamma^2 - 2B + J + k \end{pmatrix}$$

Box I.

the semiclassical limit using the coherent state was investigated [39]. In [40], the authors demonstrate that nonlinear magnetic solitary excitations (solitons) traveling through a Heisenberg spin chain may be used as a robust tool capable of coherent control of the qubit's state. The authors [41] analyze the domain-wall confinement in large-S spin-chain antiferromagnets with easy axis, singleion anisotropy. It is motivated by the observation of discrete energy levels in the anisotropic antiferromagnet $CaFe_2O_4$ a spin-5/2 system consisting of weakly coupled zigzag chains. Many experimental studies have shown that there is a plethora of materials, such as the edge-shared cuprates $LiCu_2O_2$ [42], $LiCuVO_4$ [43–45], Li_2ZrCuO_4 [46], $Ca_2Y_2Cu_5O_10$ [47] and $NaCu_2O_2$ [48], which can be adequately described by a chain model with ferromagnetic (FM) nearest-neighbor (NN) interaction and antiferromagnetic (AFM) next-nearest-neighbor (NNN) interaction.

In this work, we investigate the effect of the nonlinear coupling on the thermal entanglement in the two-qubit spin system with Dzyaloshinskii–Moriya (DM) interaction in the presence of the homogeneous and inhomogeneous magnetic field. We discuss the entanglement and teleportation via the quantum channel constructed by this system. We also investigate the temperature and magnetic field dependence of the thermal entanglement of this system for this interaction.

The objective of this paper is to investigate the impact of nonlinear coupling interaction effects of the entanglement and the process of teleportation. The structure of the paper is as follows. Section 2 describes a two-qubit spin system and gives the eigenvalues and the eigenvectors. In Section 3, we calculated the system's density matrix. We discussed the numerical modeling results, such as the negativity and fidelity of the system for the nonlinear coupling with different model parameters. Finally, in Section 4, we present our conclusions.

2. Model and solution for a two-qubit

The Hamiltonian H for a two-qubit anisotropic Heisenberg model with z-component interaction parameter D_z is

$$\begin{aligned} H &= J((1+\gamma)\sigma_1^x\sigma_2^x + (1-\gamma)\sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z) + k((1+\gamma)\sigma_1^x\sigma_2^x \\ &\quad + (1-\gamma)\sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z)^2 \\ &\quad + D_z(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x) + (B+b)\sigma_1^z + (B-b)\sigma_2^z \end{aligned} \tag{1}$$

where J is the linear coupling coefficient, k is the nonlinear coupling, γ is the anisotropic parameter, D_z is the z-component DM interaction parameter, $\sigma^i (i = x, y, z)$ are Pauli matrices, B is a homogeneous part of the magnetic field and b describes the inhomogeneity. All the parameters are dimensionless. In the standard basis ($|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$), the Hamiltonian has the following matrix form: (see the equation in Box I). A straightforward calculation gives the following eigenvalues and eigenvectors of the Hamiltonian H.

$$\begin{aligned} E_1 &= 2\sqrt{B^2 + \gamma^2J^2 + 4\gamma^2Jk + 4\gamma^2k^2} + J + 4\gamma^2k + k \\ E_2 &= -2\sqrt{B^2 + \gamma^2J^2 + 4\gamma^2Jk + 4\gamma^2k^2} + J + 4\gamma^2k + k \\ E_3 &= 2\sqrt{b^2 + D_z^2 + J^2 - 4Jk + 4k^2} - J + 5k \\ E_4 &= -2\sqrt{b^2 + D_z^2 + J^2 - 4Jk + 4k^2} - J + 5k \\ |\psi_1\rangle &= \frac{1}{\sqrt{1+\mu^2}}(\mu|11\rangle + |00\rangle) \end{aligned} \tag{2}$$

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{\sqrt{1+\nu^2}}(\nu|11\rangle + |00\rangle) \\ |\psi_3\rangle &= \frac{1}{\sqrt{1+\zeta\zeta^*}}(\zeta|10\rangle + |01\rangle) \\ |\psi_4\rangle &= \frac{1}{\sqrt{1+\xi\xi^*}}(\xi|10\rangle + |01\rangle) \end{aligned}$$

where

$$\begin{aligned} \mu &= \frac{\sqrt{B^2 + \gamma^2J^2 + 4\gamma^2Jk + 4\gamma^2k^2} + B}{\gamma(J+2k)} \\ \nu &= \frac{-\sqrt{B^2 + \gamma^2J^2 + 4\gamma^2Jk + 4\gamma^2k^2} + B}{\gamma(J+2k)} \\ \zeta &= -\frac{i\left(\sqrt{b^2 + D_z^2 + J^2 - 4Jk + 4k^2} + b\right)}{D_z - i(J-2k)} \\ \xi &= \frac{i\left(\sqrt{b^2 + D_z^2 + J^2 - 4Jk + 4k^2} - b\right)}{D_z - i(J-2k)} \end{aligned} \tag{3}$$

3. Thermal density matrix

The state of the two-qubit system at thermal equilibrium is determined by the density matrix

$$\rho(T) = \frac{1}{Z}\sum_{i=1}^{4}\left(\exp\left[\frac{-E_i}{T}\right]\right)|\psi_i\rangle\langle\psi_i|, \qquad where \quad Z = Tr(\rho(T)) \tag{4}$$

In the representation with basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the density operator for the system under consideration in thermal equilibrium can be worked out to be

$$\rho(T) = \frac{1}{Z}\begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{5}$$

has matrix elements given in Eq. (5) as follows: (see the equation in Box II).

3.1. Negativity of the system

In this subsection, we consider the negativity as a measure of entanglement and study its dependence on the system's parameters. Negativity measurement plays a crucial role in quantum information theory, helping to quantify the amount of entanglement between two quantum systems. It is also used in various tasks, such as quantum cryptography, teleportation, and error correction. Negativity was introduced by Vidal and Werner [49]. Certainly, negativity is the best instrument to define bipartite quantum correlations [50]. With the thermal density matrix ρ, we can evaluate the entanglement by using the negativity. This measure states that if λ_i represents the eigenvalues of ρ^{T_2}, then the negativity is given by [51,52], $N = \sum_{i=1}^{4}|\lambda_i| - 1$ where T_2 refers to the partial transposition for the second subsystem. By using this definition for our system, the negativity can be calculated explicitly as, $N = \max[0, -2\min[\lambda_i]]$. The values of N range from zero to one: For a maximally-entangled when $N = 1$, while for a unentangled state $N = 0$.

$$\rho_{11} = \frac{e^{-\frac{J+4\gamma^2 k+k}{T}}\left(\sqrt{B^2+\gamma^2(J+2k)^2}\cosh\left(\frac{2\sqrt{B^2+\gamma^2(J+2k)^2}}{T}\right) - B\sinh\left(\frac{2\sqrt{B^2+\gamma^2(J+2k)^2}}{T}\right)\right)}{\sqrt{B^2+\gamma^2(J+2k)^2}}$$

$$\rho_{14} = \rho_{41} = -\frac{\gamma(J+2k)e^{-\frac{J+4\gamma^2 k+k}{T}}\sinh\left(\frac{2\sqrt{B^2+\gamma^2(J+2k)^2}}{T}\right)}{\sqrt{B^2+\gamma^2(J+2k)^2}}$$

$$\rho_{22} = e^{\frac{J-5k}{T}}\left(\cosh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right) - \frac{b\sinh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right)}{\sqrt{b^2+D_z^2+(J-2k)^2}}\right)$$

$$\rho_{23} = \frac{(iD_z - J + 2k)e^{\frac{J-5k}{T}}\sinh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right)}{\sqrt{b^2+D_z^2+(J-2k)^2}}$$

$$\rho_{32} = -\frac{(iD_z + J - 2k)e^{\frac{J-5k}{T}}\sinh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right)}{\sqrt{b^2+D_z^2+(J-2k)^2}}$$

$$\rho_{33} = e^{\frac{J-5k}{T}}\left(\frac{b\sinh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right)}{\sqrt{b^2+D_z^2+(J-2k)^2}} + \cosh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right)\right)$$

$$\rho_{44} = \frac{e^{-\frac{J+4\gamma^2 k+k}{T}}\left(B\sinh\left(\frac{2\sqrt{B^2+\gamma^2(J+2k)^2}}{T}\right) + \sqrt{B^2+\gamma^2(J+2k)^2}\cosh\left(\frac{2\sqrt{B^2+\gamma^2(J+2k)^2}}{T}\right)\right)}{\sqrt{B^2+\gamma^2(J+2k)^2}}$$

$$Z = 2e^{-\frac{J+4\gamma^2 k+5k}{T}}\left(e^{\frac{2\left(J+2\gamma^2 k\right)}{T}}\cosh\left(\frac{2\sqrt{b^2+D_z^2+(J-2k)^2}}{T}\right) + e^{\frac{4k}{T}}\cosh\left(\frac{2\sqrt{B^2+\gamma^2(J+2k)^2}}{T}\right)\right)$$

Box II.

In Fig. 1(a) we plot the negativity N as a function of the nonlinear coupling k with different values DM interaction $D_z = 10$ dashed dot line, $D_z = 5$ dashed line, $D_z = 0$ solid line, with the absence of a homogeneous and nonhomogeneous magnetic field $B = b = 0$, with the linear coupling coefficient $J = -1$, the temperature $T = 1$ and the anisotropic parameter $\gamma = 2$. We note that the negativity conserve on complete entanglement for $D_z = 0$ more than $D_z = 10$. So, for $D_z = 10$ dashed dot line, we find that the initially unentangled state, there is a death of entanglement, we observed that the negativity starts from The highest value ($N = 1$) and then it go to down to zero, with the evolution value of k the negativity grows to the top quickly to reaches steady-going values at 1. For $D_z = 5$ dashed line, we find that the initially short entangled state, and there is a death of entanglement, Thus, the curve ascends to maximum value of entanglement. for $D_z = 0$ solid line, we find that the initially long entangled state and happen a death of entanglement, then we have a Permanently entanglement. The of the negativity are shown in Fig. 1(b) as the function of the nonlinear coupling constant k in the presence of a homogeneous magnetic field $B = 5, b = 0$, we find that the initially non-zero entanglement, then it gets the same behavior as the function in the previous Fig. 1(a), we see that the negativity starts from 0.9 and decay slowly before the revival and reaches to steady-going values at 1 as in Fig. 1(b). Fig. 1(c) shown that the effect of an inhomogeneous magnetic field $B = 1, b = 1$ with DM interaction for versus k. It notes that the negativity decays very fast before the revival and reaches to steady-going values. Happen a height faster increasing of the entanglement with increasing of the values DM interaction, so the negativity reaches to steady-going values. With increasing an inhomogeneous magnetic field $B = 1, b = 5$, there is a faster death before it reaches to steady-going values as in Fig. 1(d). In Fig. 2(a) we plot the negativity N as a function of temperature T with different values DM interaction in the absence of a homogeneous magnetic field $B = 0, b = 0$, with the linear coupling coefficient $J = -1$, the nonlinear coupling $k = -1$ and the anisotropic parameter $\gamma = 2$. We observed that negativity start from The highest value for the entanglement at 1 and sudden death occur. With increase of homogeneous magnetic field $B = 5, b = 0$, we see that the entanglement decay rapidly. For the entanglement begins from 0.75 and decay quickly with increase DM interaction as Fig. 2(b). In Fig. 2(c) we studied the effect of an inhomogeneous magnetic field $B = 1, b = 5$, with different values DM interaction, there is a homogeneous magnetic field on N versus T. We observed that N decay and occurs phenomenon sudden death. When increasing an inhomogeneous magnetic field, we see that N decay quickly with increase values of DM interaction as seen in Fig. 2(d).

Fig. 3(a) shown for $B = 1, b = 1$ the behaviors of the negativity as the function of temperature T and the nonlinear coupling constant k. We find that the negativity monotonically decreases with the increase of temperature. At the same time, we see that the critical temperature T_c does not improved with increasing k. Fig. 3(b) shows the negativity versus DM interaction and nonlinear coupling k. It is clear that the negativity decrease slowly with increasing DM interaction, while that N decreases quickly and increase to reaches constant values. In Fig. 3(c) we see that the negativity is always larger than zero for any value of k and a homogeneous magnetic field B, which means that the

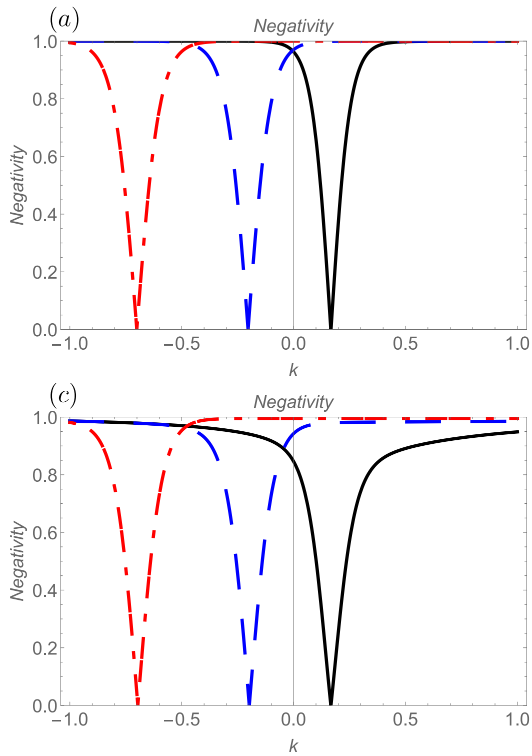

Fig. 1. The evolution of Negativity (N) in terms of k. The solid, dashed and dot-dashed curves are evaluated for $D_z = 0, 5, 10$, respectively, with ($\gamma = 2$, $J = -1$). We set ($T = 1, b = 0, B = 0$) in (a), ($T = 1, b = 0, B = 5$) in (b), ($T = 1, B = 1, b = 1$) in (c) and ($T = 1, B = 1, b = 5$) in (d).

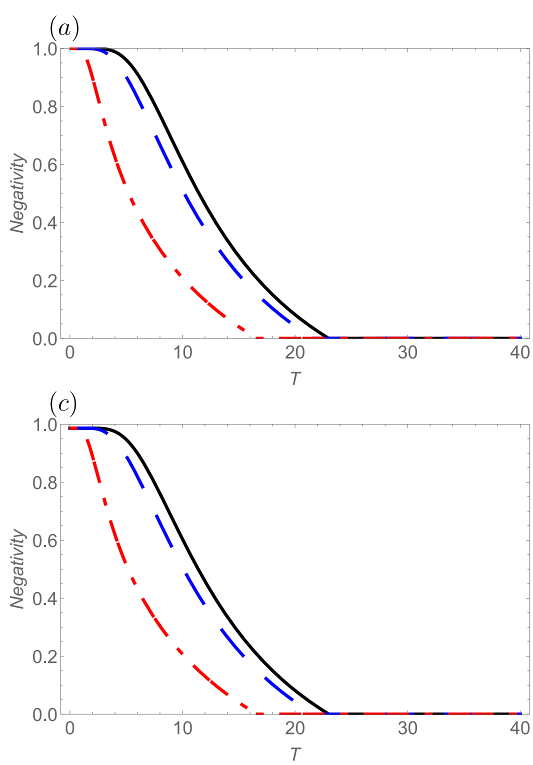

Fig. 2. The evolution of Negativity (N) in terms of T. The solid, dashed and dot-dashed curves are evaluated for $D_z = 0, 5, 10$, respectively, with ($\gamma = 2$, $J = -1$). We set ($k = -1, b = 0, B = 0$) in (a), ($k = -1, b = 0, B = 5$) in (b), ($k = -1, B = 1, b = 1$) in (c) and ($k = -1, B = 1, b = 5$) in (d).

anisotropic parameter γ does not completely of the entanglement but generates entanglement. Fig. 3(d) shows that the negativity versus b and nonlinear coupling k, we see that the negativity is always larger than zero for any value of k and an inhomogeneous magnetic field b, we see that the negativity slowly decreases with the increase of b, while the negativity has been increased with k increased.

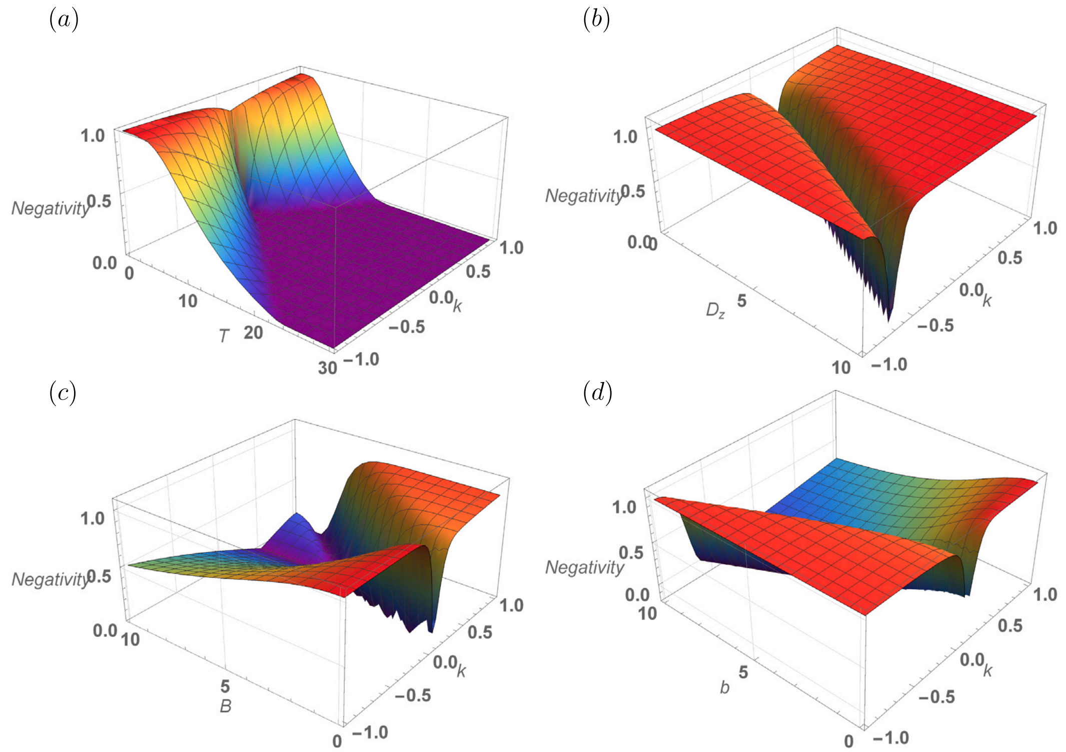

Fig. 3. The evolution of Negativity (N) against the different parameters. In (a) is plotted between T and k at $B = 1, b = 1, D_z = 1$, in (b) is plotted between D_z and k at $B = 1, b = 1, T = 1$, in (c) is plotted between B and k at $T = 1, D_z = 1, b = 1$, and in (d) is plotted between b and k at $B = 1, T = 1, D_z = 1$, with ($\gamma = 2$, $J = -1$).

3.2. The effect of initial state and inputting state on the fidelity

The fidelity between ρ_{in} and ρ_{out} characterizes the quality of the teleported state ρ_{out} [53,54]. By teleporting the state of one of the two entangled qubits, we consider partial teleportation of entanglement. We consider inputting a two qubits state in the special pure state

$$|\psi_{in}\rangle = \cos(\frac{\theta}{2})|01\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|10\rangle, \tag{6}$$

where $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. The output state is then given by [55]

$$\rho_{out}(t) = \sum_{ij} P_{ij}(\sigma_i \otimes \sigma_j)\rho_{in}(\sigma_i \otimes \sigma_j), \tag{7}$$

where $\sigma_i (i = 0, x, y, z)$ signify the unit matrix I and three components of the Pauli matrix respectively,

$$P_{ij} = tr[E^i\rho(T)]tr[E^j\rho(T)], \sum P_{ij} = 1. \tag{8}$$

Here $E^0 = |\psi^-\rangle\langle\psi^-|$, $E^1 = |\phi^-\rangle\langle\phi^-|$, $E^2 = |\phi^+\rangle\langle\phi^+|$, $E^3 = |\psi^+\rangle\langle\psi^+|$, and $\psi^\pm = \frac{1}{\sqrt{2}}|01\rangle \pm |10\rangle$, $\phi^\pm = \frac{1}{\sqrt{2}}|00\rangle \pm |11\rangle$. Fidelity is often a useful test of the quality of teleportation. It is between the teleported state ρ_{out} and the initial state to be teleported ρ_{in}, it describe as following

$$F(\rho_{in}, \rho_{out}) = \left[Tr\left(\sqrt{\sqrt{\rho_{in}}\rho_{out}\sqrt{\rho_{in}}}\right)\right]^2. \tag{9}$$

For $F(\rho_{in}, \rho_{out}) = 0$, it means that the information is completely destroyed in the transmission process, and the teleportation fails. While for $F(\rho_{in}, \rho_{out}) = 1$ it means that the final state is identical to the initial state, In common situation $0 < F < 1$ it results in distortions to the quantum information at a certain extent after being transmitted.

In Fig. 4(a) we plot the Fidelity versus the nonlinear coupling k with different values DM interaction in the absence of a homogeneous magnetic field. We observed that it happen long death, before happen revival quickly and reaches to steady-going values at 1. With increase DM interaction, occurs revival quickly at long periods. In the presence of a homogeneous magnetic field $B = 0$, there is the long death before occurs revival quickly and reaches to steady-going values at 1. With increase DM interaction $B = 5$, the long death occurs at long periods with increase DM interaction and occurs grows of Fidelity with the increase of k as seen in Fig. 4(b). In Fig. 4(c) we studied the effect of an inhomogeneous magnetic field with different values DM interaction when there is a homogeneous magnetic field $b = 1$ on Fidelity with versus k. The short death occurs at long periods with increase DM interaction, with increase of the inhomogeneous magnetic field, there is a short death before that revival quickly and reaches to the steady-going values at 1 as in Fig. 4(d).

In Fig. 5(a) we plot the Fidelity versus T with different values DM interaction, $D_z = 10$ dashed dot line, $D_z = 5$ dashed line, $D_z = 0$ solid line. In the absence of a homogeneous magnetic field $B = 0$, we observed that the Fidelity increased with increasing of T and it increased faster for large DM interaction. In the presence of a homogeneous magnetic field at $B = 5$, there is increased slowly of Fidelity by with increase DM interaction as see in Fig. 5(b). In Fig. 5(c) we studied the effect of an inhomogeneous magnetic field with different values DM interaction on the Fidelity against T with a homogeneous magnetic field $B = 1$. We observed that it increased slowly of fidelity with increase DM interaction. With increased of an inhomogeneous magnetic field $b = 5$, we found that faster grows of fidelity as seen in Fig. 5(d).

In Fig. 6(a) we plotted the fidelity as a function of nonlinear couplings k and temperature T, we find that the fidelity monotonically increases with the increase of temperature. At the same time, and the critical temperature T_c is improved with increasing k. Fig. 6(b) shows the fidelity versus DM interaction and nonlinear coupling k. It is clear to see that the fidelity decay by increasing DM interaction, while that k decreases quickly and increase before it reaches to a constant values. In Fig. 6(c) we see that the fidelity versus B and nonlinear coupling k. We observed that the fidelity quickly decreases with the increase of B, while increase with k. Fig. 6(d) shows that the fidelity versus b and nonlinear coupling k. We see that the fidelity go slowly decreases with the increase of b, while it increase with k before it reaches to a constant values.

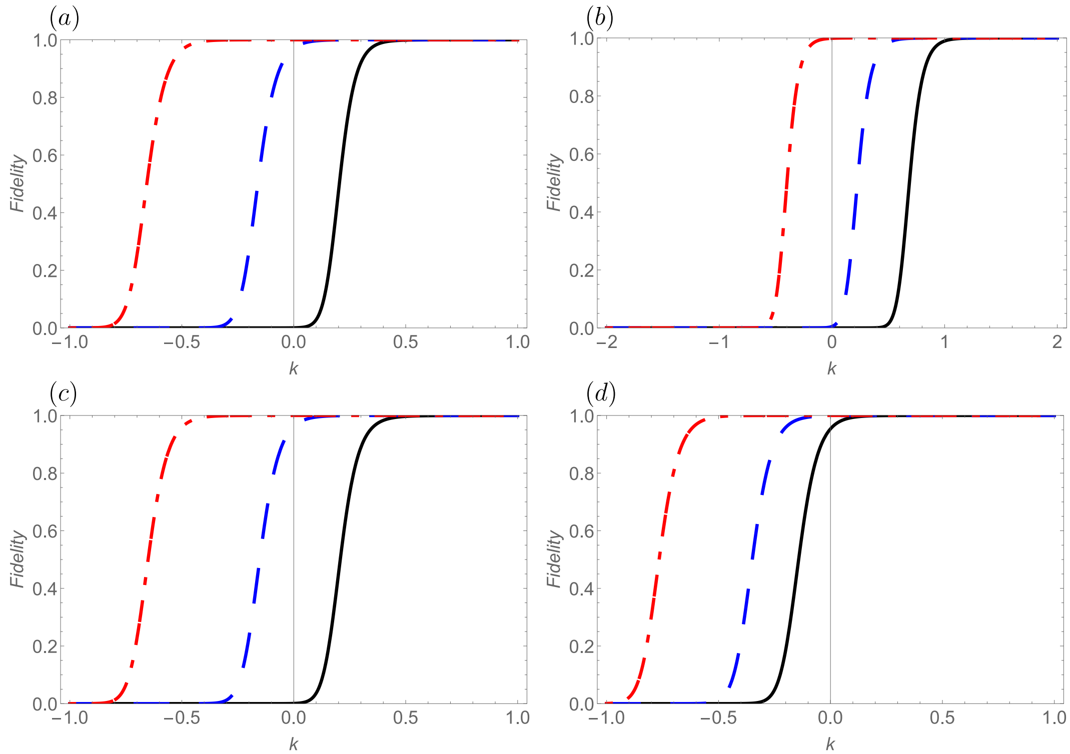

Fig. 4. The behavior of Fidelity in terms of k. The solid, dashed and dot-dashed curves are evaluated for $D_z = 0, 5, 10$, respectively, with ($\gamma = 2$, $J = -1$). We set $T = 1, b = 0, B = 0$ in (a), $T = 1, b = 0, B = 5$ in (b), $T = 1, B = 1, b = 1$ in (c) and $T = 1, B = 1, b = 5$ in (d).

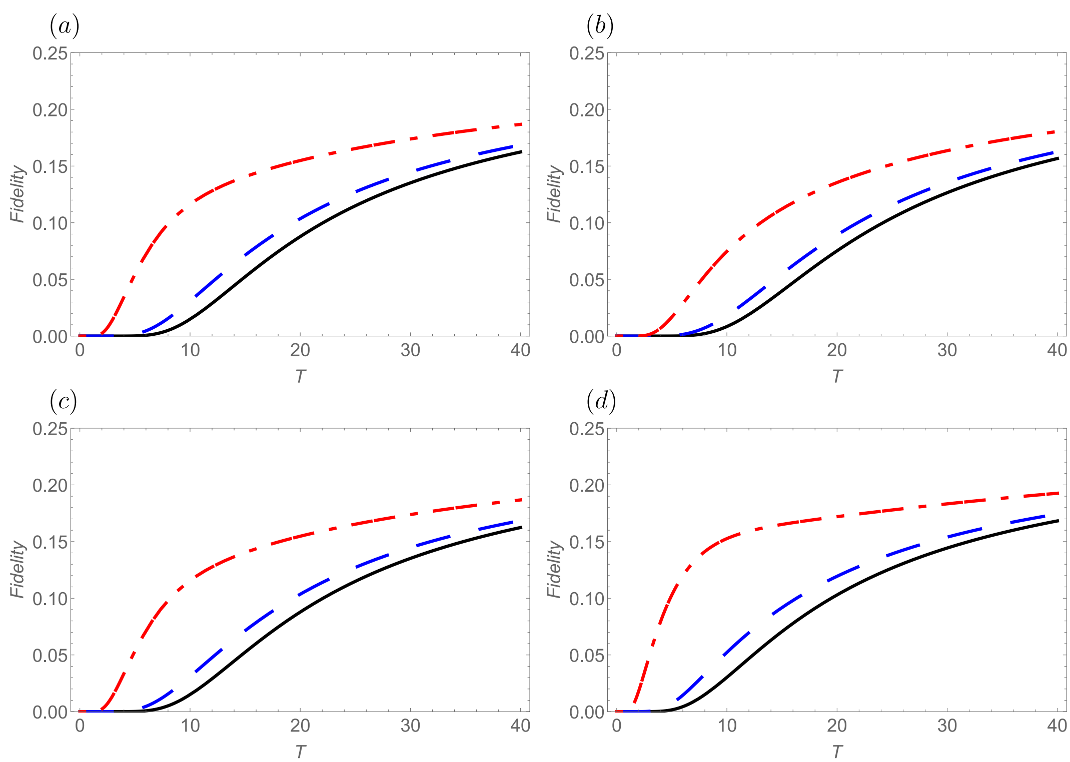

Fig. 5. The behavior of Fidelity in terms of T. The solid, dashed and dot-dashed curves are evaluated for $D_z = 0, 5, 10$, respectively, with ($\gamma = 2$, $J = -1$). We set $k = -1, b = 0, B = 0$ in (a), $k = -1, b = 0, B = 5$ in (b), $k = -1, B = 1, b = 1$ in (c) and $k = -1, B = 1, b = 5$ in (d).

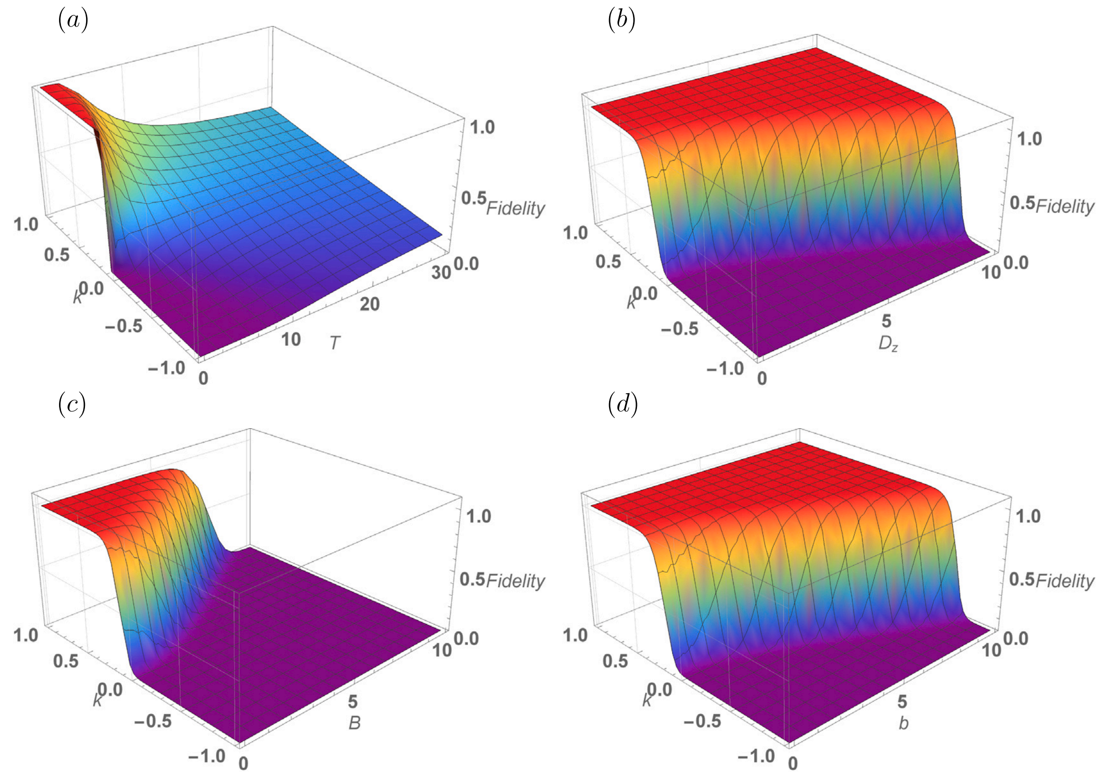

Fig. 6. The behavior of Fidelity against the different parameters. In (a) is plotted between T and k at $B = 1, b = 1, D_z = 1$, in (b)is plotted between D_z and k at $B = 1, b = 1, T = 1$, in (c) is plotted between B and k at $T = 1, D_z = 1, b = 1$, and in (d) is plotted between b and k at $B = 1, T = 1, D_z = 1$ with ($\gamma = 2, J = -1$).

4. Conclusions

This research study explores the effects of nonlinear coupling on different phenomena that occur in both homogeneous and inhomogeneous magnetic fields. The study reveals that nonlinear coupling can significantly impact entanglement. It is a fundamental concept in quantum mechanics that describes the correlations between two or more particles that exist even when a considerable distance separates them. The study further investigates the impact of nonlinear coupling on passivity and teleportation accuracy. Passivity measures how well a quantum system resists changes to its state. At the same time, teleportation is a process that allows the transfer of quantum information from one place to another without physically transporting the information carrier. The results of the study show that nonlinear coupling has a positive effect on both passivity and teleportation accuracy. Additionally, the study demonstrates that nonlinear coupling can positively affect the negativity and accuracy of teleportation. Negativity measures the degree of entanglement between two quantum systems, while accuracy measures how closely the teleported state matches the original state. These factors are crucial in quantum information processing, and the study suggests that nonlinear coupling could be a valuable tool in developing quantum technologies.

In summary, this research study provides insights into the effects of nonlinear coupling on various phenomena in quantum mechanics. The study's results suggest that nonlinear coupling can positively impact entanglement, passivity, and teleportation accuracy, which are essential factors in the development of quantum technologies.

CRediT authorship contribution statement

Nour Zidan: Investigation, Methodology, Supervision, Writing – review & editing. **Ahmed Redwan:** Methodology, Writing – original draft. **Tarek El-Shahat:** Methodology, Supervision, Writing – original draft. **Montasir Qasymeh:** Funding acquisition, Writing – review & editing. **Mahmoud Abdel-Aty:** Investigation, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] C. Bennett, D. DiVincenzo, Quantum information and computation, Nature 404 (2000) 247–255, http://dx.doi.org/10.1038/35005001.

[2] L. Zhou, H.S. Song, Y.Q. Guo, C. Li, Enhanced thermal entanglement in an anisotropic Heisenberg XYZ chain, Phys. Rev. A 68 (2) (2003) 024301, http://dx.doi.org/10.1103/PhysRevA.68.024301.

[3] G.-F. Zhang, S.-S. Li, Thermal entanglement in a two-qubit Heisenberg XXZ-spin chain under an inhomogeneous magnetic field, Phys. Rev. A 72 (2) (2005) 034302, http://dx.doi.org/10.1103/PhysRevA.72.034302.

[4] J.L. Guo, H.S. Song, Exact solutions of the generalized two-photon and two-qubit Rabi models, Eur. Phys. J. D 56 (2010) 265–274, http://dx.doi.org/10.1140/epjd/e2013-40183-5.

[5] T. Werlang, G. Rigolin, Thermal and magnetic quantum discord in Heisenberg models, Phys. Rev. A 81 (2010) 044101, http://dx.doi.org/10.1103/PhysRevA.81.044101.

[6] J.L. Guo, Y.J. Mi, J. Zhang, H.S. Song, Thermal quantum discord of spins in an inhomogeneous magnetic field, J. Phys. B: At. Mol. Opt. Phys. 44 (2011) 065504, http://dx.doi.org/10.1088/0953-4075/44/6/065504.

[7] I. Dzyaloshinskii, A thermodynamic theory of weak ferromagnetism of antiferromagnetics, J. Phys. Chem. Solids 4 (1958) 241, http://dx.doi.org/10.1016/0022-3697(58)90076-3.

[8] T. Moriya, Anisotropic superexchange interaction and weak ferromagnetism, Phys. Rev. Lett. 120 (1960) 91, http://dx.doi.org/10.1103/PhysRev.120.91.

[9] G.L. Kamta, A.F. Starace, Anisotropy and magnetic field effects on the entanglement of a two Qubit Heisenberg XY chain, Phys. Rev. Lett. 88 (2002) 107901, http://dx.doi.org/10.1103/PhysRevLett.88.107901.

[10] A.R. Mohammed, T.M. El-Shahatm, Study the entanglement dynamics of an anisotropic two-qubit Heisenberg XYZ system in a magnetic field, J. Quantum Inf. Sci. 7 (4) (2017) 160–171, http://dx.doi.org/10.4236/jqis.2017.74013.

[11] Y. Dakir, A. Slaoui, A.-B.A. Mohamed, R. Laamara, H. Eleuch, Quantum teleportation and dynamics of quantum coherence and metrological non-classical correlations for open two-qubit systems, Sci. Rep. 13 (2023) 20526, http://dx.doi.org/10.1038/s41598-023-46396-2.

[12] N. Zidan, A. ur Rahman, S. Haddadi, Quantum teleportation in a two-superconducting qubit system under dephasing noisy channel: role of Josephson and mutual coupling energies, Laser Phys. Lett. 20 (2023) 025204, http://dx.doi.org/10.1088/1612-202X/acb043.

[13] A. Redwan, A.-H. Abdel-Aty, N. Zidan, T. El-Shahat, Dynamics of the entanglement and teleportation of thermal state of a spin chain with multiple interactions, Chaos 29 (2019) 013138, http://dx.doi.org/10.1063/1.5085784.

[14] A. Redwan, A.-H. Abdel-Aty, N. Zidan, T. El-Shahat, Dynamics of classical and quantum information on spin-chains with multiple interactions, Inf. Sci. Lett. 7 (2) (2018) 29–33, http://dx.doi.org/10.12785/isl/070201.

[15] N. Zidan, Quantum teleportation via two-qubit Heisenberg XYZ chain, Can. J. Phys. 92 (5) (2014) 406–410, http://dx.doi.org/10.1139/cjp-2013-0404.

[16] N. Zidan, Dynamics of correlations in the presences of intrinsic decoherence, Internat. J. Theoret. Phys. 55 (2016) 1274–1284, http://dx.doi.org/10.1007/s10773-015-2768-y.

[17] H. Wei, T. Zeng, R. Fang, Quantum networks consist of semiconductor quantum dot and cavity, in: 2012 Symposium on Photonics and Optoelectronics, Shanghai, China, 2012, pp. 1–4, http://dx.doi.org/10.1109/SOPO.2012.6270434.

[18] W.B. Gao, P. Fallahi, E. Togan, J. Miguel-Sanchez, A. Imamoglu, Observation of entanglement between a quantum dot spin and a single photon, Nature 491 (2012) 426–430, http://dx.doi.org/10.1038/nature11573.

[19] D. Gunlycke, J.H. Jefferson, T. Rejec, A. Ramak, D.G. Pettifor, G.A.D. Briggs, Entanglement between static and ying qubits in a semiconducting carbon nanotube, J. Phys.: Condens. Matter. 18 (2006) S851–S866, http://dx.doi.org/10.1088/0953-8984/18/21/S11.

[20] B.B. Blinov, D.L. Moehring, L.-M. Duan, C. Monroe, Observation of entanglement between a single trapped atom and a single photon, Nature 428 (2004) 153–157, http://dx.doi.org/10.1038/nature02377.

[21] Ye. Yeo, Teleportation via thermal entangled states of a two-qubit Heisenberg XX chain, Phys. Rev. A 66 (2002) 2312, http://dx.doi.org/10.1103/PhysRevA.66.062312.

[22] R. Fortes, G. Rigolin, Probabilitic quantum teleportation via thermal entanglement, Phys. Rev. A 96 (2017) 022315, http://dx.doi.org/10.1103/PhysRevA.96.022315.

[23] L. Campos, C. Degli, M. Roncaglia, Qubit teleportation and transfer across antiferromagnetic spin chains, Phys. Rev. Lett. 99 (2007) 060401, http://dx.doi.org/10.1103/PhysRevLett.99.060401.

[24] S.M. Hassan, A.A. Altwaty, Optical solitons of the extended Gerdjikov-Ivanov equation in DWDM system by Extended Simplest equation method, Appl. Math. Inf. Sci. 14 (5) (2020) 901–907, http://dx.doi.org/10.18576/amis/140517.

[25] M. Mirzazadeh, M. Eslami, A.H. Bhrawy, B. Ahmed, Anjan Biswas, Solitons and other solutions to complex-valued Klein–Gordon equation in $\Phi-4$ field theory, Appl. Math. Inf. Sci. 09 (6) (2015) 2793–2801, http://dx.doi.org/10.12785/amis/090605.

[26] D. Vernardou, A. Kazas, M. Apostolopoulou, N. Katsarakis, E. Koudoumas, Hydrothermal growth of MnO2 at 95 oC as an anode material, Int. J. Thin. Film. Sci. Technol. 5 (2) (2016) 121–127, http://dx.doi.org/10.18576/ijtfst/050207.

[27] M.S. Al-Qrinawi, T.M. El-Agez, M.S. Abdel-Latif, S.A. Taya, Capacitance-voltage measurements of hetero-layer OLEDs treated by an electric field and thermal annealing, Int. J. Thin. Film. Sci. Technol. 10 (3) (2016) 217–226, http://dx.doi.org/10.18576/ijtfst/100311.

[28] A.A. Elhadary, A. El-Zein, m. Talaat, G. El-Aragi, A. El-Amawy, Studying the effect of the dielectric barrier discharge non- thermal plasma on colon cancer cell line, Int. J. Thin. Film. Sci. Technol. 10 (3) (2021) 161–168, http://dx.doi.org/10.18576/ijtfst/100305.

[29] M. Shapaan, D.C. Conductivity, Thermal stability and crystallization kinetics of the semiconducting 30P2O5 (50-x)V2O5 xB2O3 20Fe2O3 oxide glasses, Int. J. Thin. Film. Sci. Technol. 5 (3) (2016) 143–153, http://dx.doi.org/10.18576/ijtfst/050301.

[30] V. Sanap, B.H. Pawar, Influence of annealing temperature on properties of nanocrystalline CdO thin films synthesized via thermal oxidation process, Int. J. Thin. Film. Sci. Technol. 2 (2) (2013) 107–111, http://dx.doi.org/10.12785/ijtfst/020206.

[31] K.Y. Bliokh, H. Punzmann, Hua Xia, F. Nori, M. Shats, Field theory spin and momentum in water waves, Sci. Adv. 8 (2022) 1295.

[32] G. Brodin, M. Marklund, J. Zamanian, M. Stefan, Spin and magnetization effects in plasmas, Plasma Phys. Control Fusion 53 (2011) 074013, http://dx.doi.org/10.1088/0741-3335/53/7/074013.

[33] R. Wieser, Standing spin waves in finite quantum spin spiral chains, Ann. Phys. 459 (2023) 169502, http://dx.doi.org/10.1016/j.aop.2023.169502.

[34] G. Csaba, A. Papp, W. Porod, Spin-wave based realization of optical computing primitives, J. Appl. Phys. 115 (17) (2014) 741, http://dx.doi.org/10.1063/1.4868921.

[35] A. Mahmoud, F. Ciubotaru, F. Vanderveken, A.V. Chumak, S. Hamdioui, C. Adelmann, S. Cotofana, Introduction to spin wave computing, J. Appl. Phys. 128 (16) (2020) 161101, http://dx.doi.org/10.1063/5.0019328.

[36] S. Kumar Ranu, D.D. Stancil, Single-magnon excited states of a Heisenberg spin chain using a quantum computer, Phys. Rev. B 106 (2022) 184402, http://dx.doi.org/10.1103/PhysRevB.106.184402.

[37] S. Rakhmanova, D.L. Mills, Nonlinear spin excitations in finite Heisenberg chains, Phys. Rev. B 45 (1996) 9225, http://dx.doi.org/10.1103/PhysRevB.54.9225.

[38] K. Takano, Y. Kito, Y. Ono, K. Sano, Nonlinear sigma model method for the J_1-J_2 Heisenberg model: disordered ground state with plaquette symmetry, Phys. Rev. Lett. 91, 197202, http://dx.doi.org/10.1103/PhysRevLett.91.197202.

[39] M.M. Latha, C.C. Vasanthi, An integrable model of (2+1)-dimensional Heisenberg ferromagnetic spin chain and soliton excitations, Phys. Scr. 89 (2014) 065204, http://dx.doi.org/10.1088/0031-8949/89/6/065204.

[40] S. Varbev, I. Boradjiev, R. Kamburova, H. Chamati, Control of a qubit state by a soliton propagating through a Heisenberg spin chain, Phys. Rev. E 105 (2022) 034207, http://dx.doi.org/10.1103/PhysRevE.105.034207.

[41] H. Lane, C. Stock, S.-W. Cheong, F. Demmel, R.A. Ewings, F. Krüger, Nonlinear soliton confinement in weakly coupled antiferromagnetic spin chains, Phys. Rev. B 102 (2020) 024437, http://dx.doi.org/10.1103/PhysRevB.102.024437.

[42] A.A. Gippius, E.N. Morozova, A.S. Moskvin, A.V. Zalessky, A.A. Bush, M. Baenitz, H. Rosner, S.-L. Drechsler, NMR and local-density-approximation evidence for spiral magnetic order in the chain cuprate $LiCu_2O_2$, Phys. Rev. B 70 (2004) 020406 (R), http://dx.doi.org/10.1103/PhysRevB.70.020406.

[43] M. Enderle, C. Mukherjee, B. Fåk, R.K. Kremer, J.-M. Broto, H. Rosner, S.-L. Drechsler, J. Richter, J. Malek, A. Prokofiev, W. Assmus, S. Pujol, J.-L. Raggazzoni, H. Rakoto, M. Rheinstädter, H.M. Rønnow, Quantum helimagnetism of the frustrated spin-$\frac{1}{2}$ chain $LiCuVO_4$, Europhys. Lett. 70 (2005) 237, http://dx.doi.org/10.1209/epl/i2004-10484.

[44] N. Büttgen, H.-A. Krug von Nidda, L.E. Svistov, L.A. Prozorova, A. Prokofiev, W. Aßmus, Spin-modulated quasi-one-dimensional antiferromagnet $LiCuVO_4$, Phys. Rev. B 76 (2007) 014440, http://dx.doi.org/10.1103/PhysRevB.76.014440.

[45] L.A. Prozorova, S.S. Sosin, L.E. Svistov, N. Büttgen, J.B. Kemper, A.P. Reyes, S. Riggs, A. Prokofiev, O.A. Petrenko, Magnetic field driven 2D-3D crossover in theS=1/2 frustrated chain magnet $LiCuVO_4$, Phys. Rev. B 91 (2015) 174410, http://dx.doi.org/10.1103/PhysRevB.91.174410.

[46] S.-L. Drechsler, O. Volkova, A.N. Vasiliev, N. Tristan, J. Richter, M. Schmitt, H. Rosner, J. Málek, R. Klingeler, A.A. Zvyagin, B. Büchner, Frustrated cuprate route from antiferromagnetic to ferromagnetic spin-1/2 Heisenberg chains: Li_2ZrCuO_4 as a missing link near the quantum critical point, Phys. Rev. Lett. 98 (2007) 077202, http://dx.doi.org/10.1103/PhysRevLett.98.077202.

[47] M. Matsuda, K. Ohoyama, M. Ohashi, Magnetic ordering of the edge-sharing CuO_2 chains in $Ca_2Y_2Cu_5O_10$, J. Phys. Soc. Japan 68 (1999) 269, http://dx.doi.org/10.1143/JPSJ.68.269.

[48] S.-L. Drechsler, J. Richter, A.A. Gippius, A. Vasiliev, A.A. Bush, A.S. Moskvin, J. Málek, Yu. Prots, W. Schnelle, H. Rosner, Helical ground state and weak ferromagnetism in the edge-shared chain cuprate $NaCu_2O_2$, Europhys. Lett. 73 (2006) 83, http://dx.doi.org/10.1209/epl/i2005-10356-y.

[49] G. Vidal, R.F. Werner, Computable measure of entanglement, Phys. Rev. A 65 (2002) 032314, http://dx.doi.org/10.1103/PhysRevA.65.032314.

[50] C. Eltschka, J. Siewert, Negativity as an estimator of entanglement dimension, Phys. Rev. Lett. 111 (2013) 100503, http://dx.doi.org/10.1103/PhysRevLett.111.100503.

[51] S. Luo, Heisenberg uncertainty relation for mixed states, Phys. Rev. A 72 (2005) 042110, http://dx.doi.org/10.1103/PhysRevA.72.042110.

[52] Le. Song, G.-H. Yang, Entanglement and measurement-induced disturbance about two-qubit Heisenberg XYZ model, Internat. J. Theoret. Phys. 53 (2014) 1985–1992, http://dx.doi.org/10.1007/s10773-014-2005-0.

[53] G. Bowen, S. Bose, Teleportation as a depolarizing quantum channel, relative entropy, and classical capacity, Phys. Rev. Lett. 87 (2001) 267901, http://dx.doi.org/10.1103/PhysRevLett.87.267901.

[54] M. Horodecki, P. Horodecki, R. Horodecki, General teleportation channel, singlet fraction, and quasidistillation, Phys. Rev. A 60 (1999) 1888, http://dx.doi.org/10.1103/PhysRevA.60.1888.

[55] A. Peres, Separability criterion for density matrices, Phys. Rev. Lett. 77 (1996) 1413, http://dx.doi.org/10.1103/PhysRevLett.77.1413.