Dynamics of two-qubit quantum nonlocality in a Heisenberg chain model with the intrinsic decoherence

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Dynamics of two-qubit quantum nonlocality in a Heisenberg chain model with the intrinsic decoherence

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This paper investigates the dynamics of two-spin nonlocality generation in a Heisenberg XXX chain with Dzyaloshinskii-Moriya (DM) and Kaplan-Shekhtman-Entin-Wohlman-Aharony (KSEA) interactions. We analyze the two-spin nonlocality dynamics by using uncertainty-induced nonlocality, maximal Bell inequality, and log-negativity. We demonstrate that a separable two-spin Heisenberg XXX chain state, induced by two-spin antiferromagnetic interaction as well as x-component of DM and KSEA interactions, could evolve to maximal two-spin nonlocality state. The ability of preserving the maximal uncertainty-induced nonlocality can be enhanced by increasing the coupling strength of the spin-spin interaction coupling. The hierarchy principle is maintained for the two-spin Bell nonlocality and log-negativity entanglement. The two-spin log-negativity dynamics exhibits the phenomenon of sudden death and birth as a result of the intrinsic decoherence, which also causes a reduction in the two-spin nonlocalities. The two-spin uncertainty-induced nonlocality is more robust, against the intrinsic decoherence, than the other types of the nonlocality. The results indicate that by boosting the two-spin antiferromagnetic interaction, the produced nonlocalities (resulting from the DM and KSEA x-component interactions) can be shielded from the intrinsic decoherence effect.

Keywords: two-spin, Heisenberg chain, intrinsic decoherence, nonlocality

I. INTRODUCTION

The center of several quantum studies is the examination of the capacity of quantum qubit systems to produce two-qubit quantum information resources (including, coherence, nonlocality, and its variants: Bell nonlocality, entanglement, quantum discord, and other quantum correlations) [1–6]. Two-qubit quantum information resources have several potential applications in quantum information [7–10]. Bell nonlocality and entanglement are the first introduced types of the quantum nonlocality. They verify the hierarchy principle [11–13], according to which all quantum states that exhibit nonlocality identified by the violation of the maximum Bell inequality are necessary but not sufficiently entangled. After proving that some separable quantum states with quantum discord can be utilized to perform quantum information processing [14], several quantum information resources, including but not limited to entanglement, have been proposed to be implemented. These resources include quantum discord and skew information [15–17].

A significant amount of effort has been spent on studying and characterizing the coherence, and nonlocality [18, 19] using the Heisenberg spin chain models due to the prospective uses of two-qubit quantum information resources (HSCMs). The spin-spin interactions are described by the HSCM, which is also essential in explaining the magnetic and thermodynamic properties of many-particle systems [20–22]. In addition, numerous additional quantum systems, such as quantum dots [23] and nuclear spins [24], are also described by HSCM. Previously, the two-qubit quantum information resources have been investigated in several Heisenberg spin chain models [25, 26]. The effects of Dzyaloshinskii-Moriya (DM) and Kaplan-Shekhtman-Entin-Wohlman-Aharony (KSEA) interactions on the two-spin nonlocality in Heisenberg XYZ chain models were studied at thermal equilibrium [27, 28]. It is discovered through these research that adding the DM and KSEA z-component interactions to the Heisenberg spin chain can improve the generation of the two-qubit quantum information resources.

Understanding the impact of intrinsic decoherence [29] on dynamics and the capacity of quantum two-qubit real systems to produce two-qubit quantum information resources play a crucial role in controlling quantum features [30–33].

Only the Heisenberg spin chain models for two-qubit X-matrix states have been examined in previous studies of the two-qubit quantum information resources. For the two-spin Heisenberg chain with x-components of DM and KSEA interactions, in particular, the dynamics of two-qubit nonlocality, of uncertainty-induced nonlocality, maximal Bell function, as well as logarithmic negativity, with the non-X-state/general state, remain unexplored.

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Therefore, in the current study, using different quantifiers of information, namely; uncertainty-induced nonlocality, the maximal Bell function, and logarithmic negativity, we will examine the dynamics of variously produced two-spin nonlocalities in a Heisenberg chain with intrinsic decoherence and DM and KSEA x-component-interactions.

The manuscript is structured as follows: The physical model of the Heisenberg XXX chain with DM and KSEA x-component interactions in the presence of intrinsic decoherence is addressed in Section (2). Section (3) introduces the two-spin quantum information quantifiers. We present and analyze the outcomes of the numerical simulations in Section (4). We end up in section (5) by a conclusion.

II. THE INTRINSIC DECOHERENCE TWO-SPIN MODEL

Here, we investigate a two-spin Heisenberg XXX-chain under the influence of the x-component interactions with the DM and KSEA. The Milburn equation [29] describes the system's dynamics when there is intrinsic decoherence:

$$\frac{d}{dt}\hat{M}(t) = -i[\hat{H},\hat{M}] - \frac{\gamma}{2}[\hat{H},[\hat{H},\hat{M}]], \tag{1}$$

$\hat{M}(t)$ designs the time-dependent two-spin density matrix. γ represents the decoherence rate. The Hamiltonian $\hat{H}$ of the two $\frac{1}{2}$-spins isotropic Heisenberg XXX chain with DM and KSEA x-direction interactions, is [26, 27]:

$$\begin{aligned}\hat{H} &= \sum_{k=x,y,z} F\,\hat{\sigma}_A^k\hat{\sigma}_B^k + K(\hat{\sigma}_A^y\hat{\sigma}_B^z + \hat{\sigma}_A^z\hat{\sigma}_B^y)\\ &\quad + D(\hat{\sigma}_A^y\hat{\sigma}_B^z - \hat{\sigma}_A^z\hat{\sigma}_B^y),\end{aligned} \tag{2}$$

The first term designs the two-spin interaction with the exchange two-spin coupling F (the chain is antiferromagnetic for $F > 0$ and ferromagnetic for $F < 0$). K represents the KSEA interaction coupling. D designs DM interaction coupling strength.

In the two-spin basis: $\{|\varpi_1\rangle = |11\rangle, |\varpi_2\rangle = |10\rangle, |\varpi_3\rangle = |01\rangle, |\varpi_4\rangle = |00\rangle\}$, the two-spin Hamiltonian eigenstates $|E_i\rangle (i = 1, 2, 3, 4)$ and the eigenvalues E_i of the Eq. (2) are given by

$$\begin{aligned}|E_1\rangle &= \frac{1}{2}(|\varpi_1\rangle - i|\varpi_2\rangle - i|\varpi_3\rangle + |\varpi_4\rangle),\\ |E_2\rangle &= \frac{1}{2}(|\varpi_1\rangle + i|\varpi_2\rangle + i|\varpi_3\rangle + |\varpi_4\rangle),\\ |E_3\rangle &= \frac{-1}{\sqrt{2}}(H^-|\varpi_1\rangle + iR^-|\varpi_2\rangle - iR^-|\varpi_3\rangle - H^-|\varpi_4\rangle),\\ |E_4\rangle &= \frac{-1}{\sqrt{2}}(H^+|\varpi_1\rangle - iR^+|\varpi_2\rangle + iR^+|\varpi_3\rangle - H^+|\varpi_4\rangle),\end{aligned} \tag{3}$$

with

$$\begin{aligned}H^{\mp} &= \frac{D}{\sqrt{2D^2 + 2F^2 \mp 2F\sqrt{D^2 + F^2}}},\\ R^{\mp} &= \frac{\sqrt{K + F^2} - F}{\sqrt{2D^2 + 2F^2 \mp 2F\sqrt{D^2 + F^2}}}\end{aligned}$$

and the corresponding eigenvalues are,

$$\begin{aligned}E_1 &= F + 2K, \qquad E_2 = F - 2K\\ E_3 &= -F + 2\sqrt{D^2 + F^2},\\ E_4 &= F - 2\sqrt{D^2 + F^2}.\end{aligned} \tag{4}$$

By utilizing the eigenvalues E_k $(k = 1, 2, 3, 4)$ and the two-spin Hamiltonian eigenstates $|E_k\rangle$, the time evolution of the two-spin density matrix of Eq. (1) is [29],

$$\hat{M}(t) = \sum_{m,n=1}^{4} X_{mn}(t)\, Y_{mn}(t)\, \langle E_m|\hat{M}(0)|E_n\rangle\, |E_m\rangle\langle E_n|. \tag{5}$$

The terms of the interaction $X_{mn}(t)$ and the intrinsic decherence $Y_{mn}(t)$ can be written as

$$X_{mn}(t) = e^{-i(E_m-E_n)t}, \qquad Y_{mn}(t) = e^{-\frac{\gamma}{2}(E_m-E_n)^2 t}. \tag{6}$$

In the following, we use an initial pure state (that does not have quantum nonlocality) to explore the ability of the two-spin antiferromagnetic, Dzyaloshinskii-Moriya, and Kaplan-Shekhtman-Entin-Wohlman-Aharony interactions to generate quantum nonlocality via maximal Bell inequality, uncertainty-induced nonlocality, and logarithmic-negativity.

III. QUANTUM NONLOCALITY QUANTIFIERS

- **Uncertainty-induced nonlocality (UIN):** By using the correlation matrix $R = [r_{ij}]$,

$$r_{ij} = Tr\{\sqrt{\hat{M}(t)}(\sigma_i \otimes I)\sqrt{\hat{M}(t)}(\sigma_j \otimes I)\}, \tag{7}$$

$\sigma_i, i = 1, 2, 3$ are the Pauli matrices, the uncertainty-induced nonlocality [15, 17] expression is,

$$U(t) = \begin{cases} 1 - \lambda_{\min}(R), & \vec{b} = 0;\\ 1 - \frac{1}{|\vec{b}|^2}\vec{b}\, R\, \vec{b}^T, & \vec{b} \neq 0,\end{cases} \tag{8}$$

where $\lambda_{\min}(R)$ designs the smallest eigenvalue of the correlation matrix. $\vec{b}$ represents the Bloch vector. The UIN is used to estimate nonlocality beyond the entanglement [39].

- **Maximal Bell inequality (MBI):**
The two-spin Bell-nonlocality is determined by the fact that the maximal Bell inequality ($B_{\max}(t) < 2$) is violated [40, 41]. The two-spin state has Bell-nonlocality when the maximal Bell inequality is violated. In this work, we redefine the maximal Bell inequality as: $B(t) = B_{\max}(t) - 1$, i.e., the two-spin Bell-nonlocality exists when the maximal Bell function (MBF) $B(t)$ is grater than one. The analytical expression of the maximal Bell function $B(t)$ is given by

$$B(t) \;=\; 2\sqrt{\xi_1 + \xi_2} - 1, \tag{9}$$

$\xi_k (k = 1, 2)$ represent the two largest eigenvalues of the matrix $K = T^{\dagger}T$, which depend on the correlation matrix $T = [t_{ij}]$ [41], with $t_{ij} = \mathrm{Tr}\{\hat{M}(t)\sigma_i^{(1)}\sigma_j^{(2)}\}$, $i, j = 1, 2, 3$.

- **Logarithmic negativity**
The logarithmic-negativity [42] of the two-spin density matrix $M(t)$ can be used to determine the entanglement generated due to the two-spin anti-ferromagnetic, Dzyaloshinskii-Moriya, and Kaplan-Shekhtman-Entin-Wohlman-Aharony interactions. The closed form of the logarithmic negativity is given by

$$N(t) \;=\; \log_2[1 + 2\mu_T], \tag{10}$$

where μ_T [42] is the two-spin density matrix negativity.

IV. QUANTUM INFORMATION GENERATION DYNAMICS

Uncertainty-induced nonlocality, maximal Bell function, and logarithmic negativity are utilized to analyze the dynamics of different types of nonlocality for the two-spin system. When the considered system starts with an initial pure state as:

$$M(0) \;=\; |11\rangle\langle 11|, \tag{11}$$

the initial values of the considered quantifiers are: $U(0) = N(0) = 0$ and $B(0) = 1$.

From Fig.1a, we find that for the case where the Dzyaloshinskii-Moriya interaction coupling strength is very weak ($D = 0.05$) and the intrinsic decoherence is absent, the relatively weak spin-spin and Kaplan-Shekhtman-Entin-Wohlman-Aharony interactions, $F = K = 0.5$, have high ability to generate regular nonlocalities. The two-spin uncertainty-induced nonlocality, the maximal Bell function, and the logarithmic negativity grow and oscillate with π-period indicating that the two-spin nonlocality states can be produced by the two-spin and KSEA interactions. The generated two-spin nonlocalities have the same oscillatory dynamics.

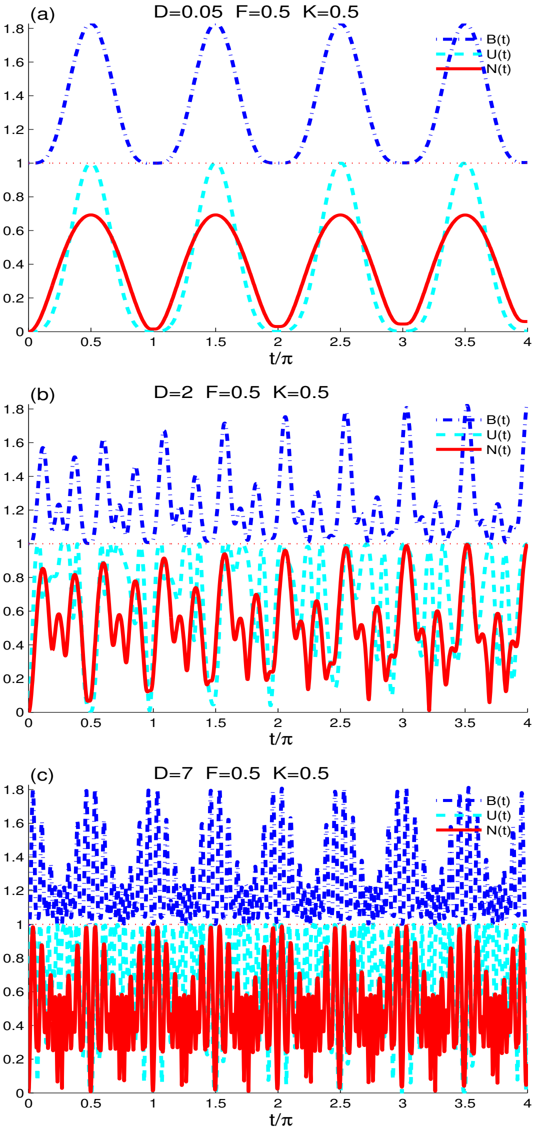

FIG. 1. Dynamics of the two-spin nonlocality of the Bell nonlocality $B(t)$, uncertainty-induced nonlocality $U(t)$ as well as the logarithmic negativity $N(t)$ for the spin-spin and KSEA interaction coupling which are relatively weak $F = K = 0.5$ in the absence of the decoherence. Different DM interaction coupling strengths are considered: $D = 0.05$ in (a), $D = 2$ in (b), and $D = 7$ in (c).

The two spins have maximally correlated states at $t = \frac{1}{2}(2n+1)\pi (n = 0, 1, 2, ...)$, where $U_{\max} = 1$, $N_{\max} = 1$ and $B_{\max} = 2\sqrt{2} - 1 \approx 1.8284$.

Fig.1b shows the effect of the increase of the Dzyaloshinskii-Moriya interaction coupling strength, $D = 2$. UIN, MBF, and logarithmic-negativity two-spin quantum information are increased, quickly, due to the increase of the Dzyaloshinskii-Moriya interaction coupling strength. The regular dynamics of the two-spin quantum information functions disappears. The amplitudes of the maximal Bell function, and the log-

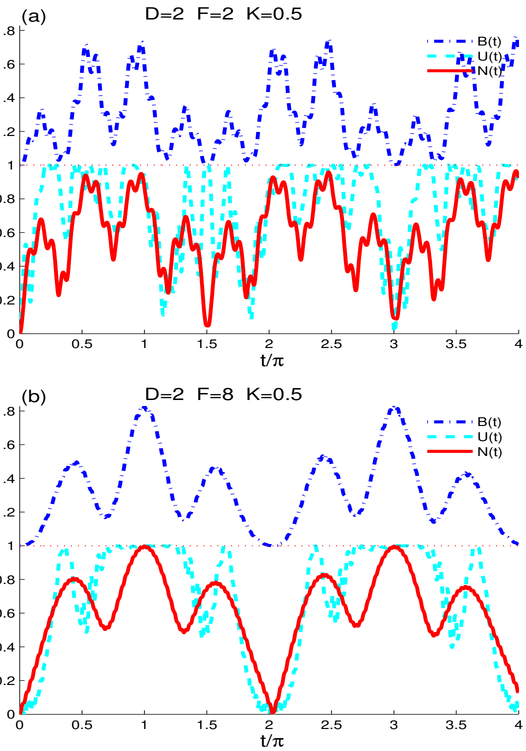

FIG. 2. Dynamics of the two-spin nonlocality generation of Fig.1b are plotted but for different values of the spin-spin interaction coupling strength: $F = 2$ in (a) and $F = 8$ in (b).

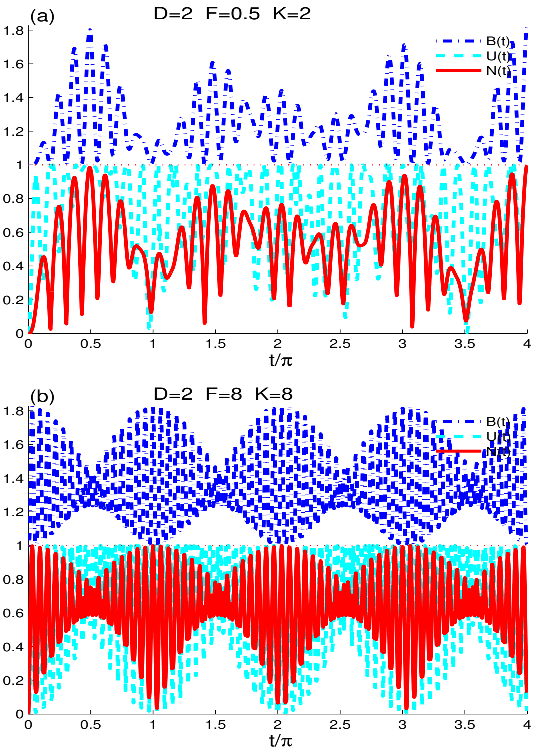

FIG. 3. Dynamics of the two-spin nonlocality of Fig.1b are plotted but for different cases of the two-spin antiferromagnetic, DM, and KSEA interaction couplings: $(D, F, K) = (2, 0.5, 2)$ in (a) and $(D, F, K) = (2, 8, 8)$ in (b).

negativity decrease while of uncertainty-induced nonlocality increases.

For high Dzyaloshinskii-Moriya interaction coupling strength $D = 7$, see Fig.1c, we note that the generated two-spin uncertainty-induced nonlocality, Bell-function, and the log-negativity effects are enhanced. The increase of the amplitudes and frequencies of the quantum quantifiers enhance the probability of maximally correlated states. Previous investigations reveal that the generated skew information and Bell function quantum correlation of a two-qubit Heisenberg XYZ chain are both greatly improved by the DM [39].

Fig.2 illustrates how the amount, regularity, and speed of the dynamics of the produced UIN, MBF, and log-negativity two-spin quantum information resources are affected by an increase in the two spin antiferromagnetic interaction coupling. By comparing the results of Figs.1b and 2a, we note that the increase of the spin-spin interaction coupling ($F = 2$) enhances the amplitudes and frequencies of the quantum quantifiers $U(t)$, $B(t)$ and $N(t)$. The generated two-spin Heisenberg XXX states tend to have maximal UIN, MBF, and log-negativity nonlocalities. UIN stabilizes to its maximal value $U_{\max} = 1$, i.e., the generated two-qubit Heisenberg XXX has high ability to preserve maximal UIN nonlocality. Fig.2b shows that the ability of preserving the maximal UIN nonlocality can be enhanced by increasing the coupling strength of the spin-spin interaction coupling ($F = 8$). In this case, the two-spin uncertainty-induced nonlocality is stabilized for a long time. For the relatively strong Dzyaloshinskii-Moriya interaction coupling, the increase of the two spin antiferromagnetic interaction coupling improves the generation of the maxmal two-spin uncertainty-induced nonlocality, Bell-function, and logarithmic-negativity nonlocalities. The results demonstrate that the two spin antiferromagnetic interaction has a great capacity to produce a two-qubit Heisenberg XXX chain state with maximal nonlocality as a result of the increase of the Dzyaloshinskii-Moriya coupling. The time windows for the Bell's inequality violation get larger for the two-spin Heisenberg XXX chain. UIN amplitudes increase and outpace logarithmic-negativity amplitudes.

Fig.3 shows the ability of strong two-spin antiferromagnetic as well as the DM and KSEA x-component interactions to generate maximal two-qubit Heisenberg XXX nonlocality. By comparing the results of Figs.1b and 3a, we find that the amplitudes and frequencies of the two-spin Bell-function, uncertainty-induced nonlocality, and the log-negativity quantifiers are enhanced, as the time evolves, due the increase of the KSEA interaction. This mean that after considering relatively strong Kaplan-Shekhtman-Entin-Wohlman-Aharony interaction ($K = 2$), the generated two-spin Heisenberg XXX chain states tend to have maximal UIN, MBF, and log-negativity nonlocalities.

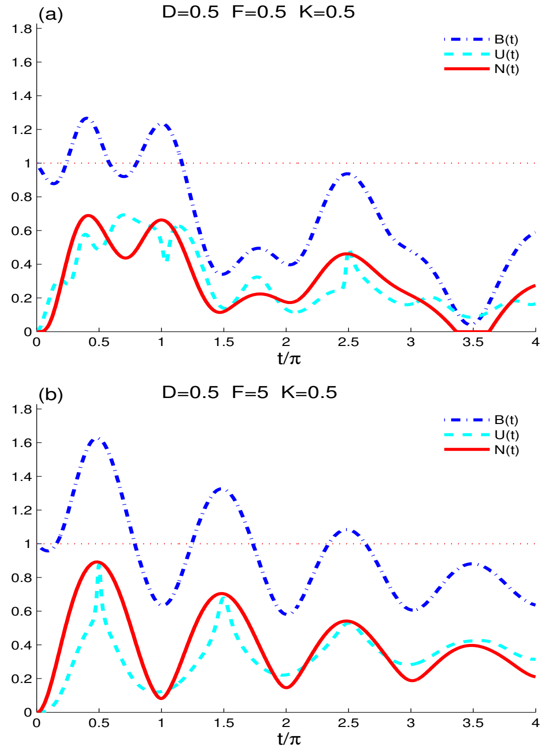

FIG. 4. Dynamics of the two-spin nonlocality under the effect of the intrinsic decoherence $\gamma = 0.05$ are investigated for DM and KSEA interaction couplings $D = K = 0.5$ and different values of the spin-spin couplings $F = 0.5$ on (a) and $F = 5$ in (b).

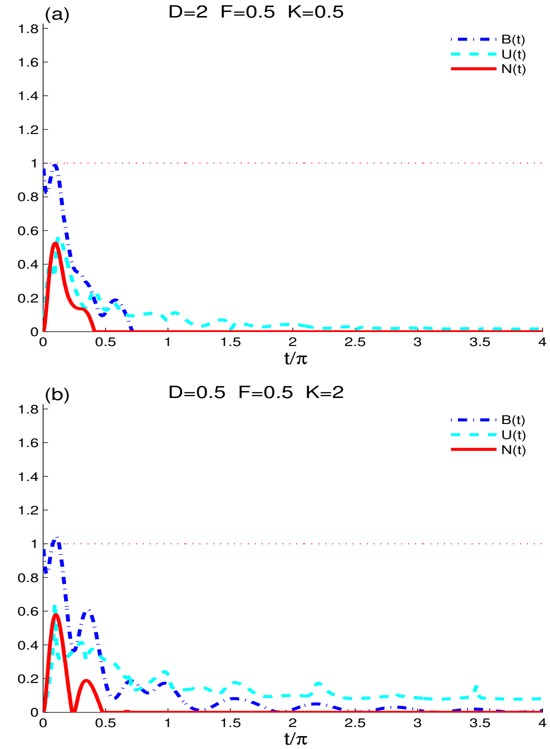

FIG. 5. The two-spin nonlocality generation dynamics of Fig.4a are investigated under strong DM and KSEA interaction couplings.

By comparing the results of Figs.2b and 3b, we observe that the strong two-spin antiferromagnetic interaction enhances the ability of the Kaplan-Shekhtman-Entin-Wohlman-Aharony x-component interaction to increase the two-spin nonlocalities. π-period oscillations of the two-spin nonlocalities have large amplitudes and high frequencies.

In Fig.4, the effect of the intrinsic decoherence for small rate ($\gamma = 0.05$) is investigated for DM and KSEA interaction couplings $D = K = 0.5$ and different values of the spin-spin couplings $F = 0.5$ in (a) and $F = 5$ in (b). Fig.4a illustrates that the amplitudes and the frequencies of the generated Bell-function, uncertainty-induced nonlocality, and log-negativity nonlocalities (decrease, as the time evolves, due the intrinsic decherence. For particular time windows, the entanglement log-negativity disappears suddenly and then arises suddenly, i.e., the phenomenon of the sudden death and birth in the two-spin log-negativity dynamics [43] occurs. The uncertainty-induced nonlocality is more robustness than the other measures of the nonlocality. In several time intervals, the violation of Bell's inequality does not occur and the two-spin state possesses partial entanglement log-negativity in the same time intervals. This is in line with the hierarchy principle [11–13]. The Bell nonlocality implies entanglement log-negativity nonlocality, which is necessary and is not sufficiently. In Fig.4b, we observe that the grow of the two-spin antiferromagnetic interaction can protect the generated nonlocalities (due to the DM and KSEA x-component interactions) against the intrinsic decoherence. The phenomenon of the entanglement sudden death and birth disappears.

Fig.5 shows the effect of the intrinsic decoherence for $\gamma = 0.05$ on the generated nonlocality, due to strong DM and KSEA x-component interactions (for x-component interactions $D = K = 0.5$). We notice that the effect of the intrinsic decoherence is accelerates as the DM and KSEA interaction coupling strengths rise. After a brief period of time, the created negativity entanglement and Bell nonlocality are completely destroyed. When the DM and KSEA x-component interactions are strengthened, uncertainty-induced nonlocality is accelerated and becomes more strong and stable. While the entanglement and Bell nonlocality are extremely vulnerable to the intrinsic decoherence.

V. CONCLUSION

We have examined the two-spin nonlocality dynamics resulting from the two-spin antiferromagnetic, Dzyaloshinskii-Moriya, and Kaplan-Shekhtman-Entin-Wohlman-Aharony x-component interactions in the Heisenberg XXX model. By adjusting the intensity of the interactions between the two-spin antiferromagnetic, Dzyaloshinskii-Moriya, Kaplan-Shekhtman-Entin-Wohlman-Aharony, as well as the intrinsic-decoherence rate, we can control the produced nonlocality of the two-spin Heisenberg XXX chain state. The capacity to create two-spin Heisenberg XXX chain states with partial and maximal nonlocality is highly enhanced by the DM and KSEA x-component interactions. The generated two-qubit Heisenberg XXX chain has a high capacity to maintain maximal UIN nonlocality. The two-spin antiferromagnetic interactions have the ability to enhance the amplitudes and frequencies of the two-spin nonlocalities. The hierarchy principle is shown to be respected for the produced two-spin Bell nonlocality and log-negativity entanglement. Due to interactions between the DM and KSEA x-components in the Heisenberg XXX chain model, an increase in intrinsic decherence causes a decrease in the Bell-function, uncertainty-induced nonlocality, and log-negativity. The phenomenon of sudden death and birth occurs in the dynamics of the log-negativity. Compared to the other function nonlocalities, the two-spin uncertainty-induced nonlocality is more resilient against the intrinsic decoherence. By strengthening the two-spin antiferromagnetic interaction, the generated nonlocalities may be shielded from the intrinsic decoherence. While a strengthening of the DM and KSEA x-component interaction coupling speeds up the intrinsic decoherence effect.

DECLARATIONS

Ethics approval and consent to participate

Not applicable

Consent for publication

Not applicable

Availability of data and materials

Correspondence and requests for materials should be addressed to A.-B. A. Mohamed.

Competing interests

The authors declare that they have no competing interests

Author Contributions

A.-B. A. Mohamed and F. M. Aldosari prepared all the figures and performed the mathematical calculations. A.-B. A. Mohamed and H. Eleuch wrote the original draft. H. Eleuch and F. M. Aldosari reviewed and edited the draft. All authors have read and agreed to the published version of the manuscript.

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