Improved Modeling Capabilites with Reduced-Order Integration

Osama A. Mohamed, P.E.[1]; Doyle E. Byrd[2]; and John O. Dow[3]

Abstract: Many rectangular isoparametric finite elements contain a modeling error known as parasitic shear. Parasitic shear can be removed by applying the correct reduced-order Gauss quardrature integration rule. When the 1×1 rule is blindly applied another error, spurious zero energy modes, can be introduced. An analysis procedure is presented that identifies the correct Gauss integration rule for eliminating parasitic shear without introducing spurious zero energy modes. Displacement polynomials expressed in transparent notation assists in the identification of error sources. The procedure is developed with the four-node membrane element and then applied to analyze and correct four- and nine-node Mindlin plate bending elements. Eigenvalue analyses and sample problems are presented to demonstrate the improvements produced by the procedure.

DOI: 10.1061/(ASCE)0733-9399(2005)131:1(2)

CE Database subject headings: Finite elements; Models; Errors; Displacement; Eigenvalues; Gaussian process.

Introduction

Many displacement-based elements are found to be overly stiff (Dohetry et al. 1969; Zinkiewicz et al. 1971; Kavanagh and Key 1972; Pawsey 1972; Prathap and Bhashyam 1982; Stolarski and Belytschko 1983; Hughes 1987; Dow and Mohamed 2002). This modeling deficiency is often caused by the existence of normal strain terms in the shear strain expressions (Dow and Byrd 1988). These improper terms which couple the normal strains to the shear strains at the kinematic level represent violations of the principles of continuum mechanics. This modeling deficiency is known as parasitic shear. Another mechanism can cause an element to be overly stiff when it has a high aspect ratio. This modeling error can be controlled and will not be discussed here.

The four-node membrane quadrilateral is known to be infected with parasitic shear (Cook 1972; 1975). The effects of parasitic shear can be eliminated in this element by using a 1×1 Gauss quadrature integration rule to evaluate the shear strain component of the strain energy expression. The strain energy terms associated with normal strains are evaluated with a 2×2 Gauss quadrature rule. The process of using a lower-order numerical integration scheme to eliminate the effects of parasitic shear is called reduced-order integration.

The successful use of 1×1 reduced-order Gauss quadrature integration to eliminate parasitic shear in the four-node membrane element has led to attempts to use this exact strategy to eliminate this modeling deficiency from other elements. However, attempts to correct parasitic shear in an element through the indiscriminate use of reduced-order integration can introduce other modeling errors in the element. The most notable is the introduction of spurious zero energy modes (Parisch 1979; Huang and Hinton 1984).

The objective of this presentation is to provide a clear explanation of the effects of reduced-order Gauss quadrature integration on the modeling characteristics of isoparametric finite elements. This analysis explains why the overstiffening is sometimes corrected and why spurious zero energy modes are sometimes introduced when reduced-order integration is used. With this knowledge, isoparametric elements can be formed in which parasitic shear is removed and other modeling errors are not introduced. It will be seen that the careful application of reduced-order Gauss quadrature integration produces these improved characteristics in isoparametric elements.

The analysis procedure for identifying the appropriate reduced-order Gauss quadrature rules for eliminating parasitic shear without introducing other modeling errors uses a physically based notation (Dow 1999). The physically based notation allows the equations developed during the element formulation procedure to be directly related to the equations of continuum mechanics that describe the physical system being analyzed. The ability to directly compare the discrete and continuous mathematical representations of the problem being solved allows modeling errors in individual elements to be identified during the formulation process. By identifying the terms that are the source of the modeling error, appropriate reduced-order integration rules can be specified that eliminate the erroneous terms and retain the correct terms. In this way, parasitic shear can be eliminated without introducing spurious zero energy modes or other modeling errors.

The procedure is developed for the familiar case of the four-node membrane quadrilateral. Then the four- and nine-node Mindlin plate bending elements are examined. Appropriate integration rules are identified that eliminate parasitic shear without introducing zero energy modes into these elements. An eigenvalue analysis of the four-node membrane element and the four-node Mindlin plate element are performed to show that the selec-

[1]Assistant Professor, Univ. of Hartford, West Hartford, CT 06117.

[2]Director of Production Operations, ABAQUS Inc., Pawtucket, RI 02860.

[3]Associate Professor, Univ. of Colorado, Boulder, CO 80309.

Note. Associate Editor: Henry K. Stolarski. Discussion open until June 1, 2005. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on April 29, 2003; approved on June 4, 2004. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 131, No. 1, January 1, 2005. ©ASCE, ISSN 0733-9399/2005/1-2–11/$25.00.

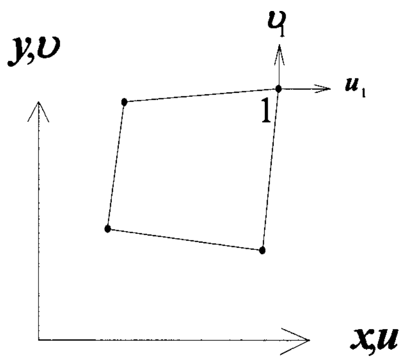

Fig. 1. Four-node membrane quadrilateral

tive use of reduced-order integration rules is successful. Then, corrected and uncorrected elements are used to analyze example plate bending problems to demonstrate the improvements in the modeling characteristics produced by the techniques presented here.

Four-Node Membrane Quadrilateral

The four-node membrane quadrilateral is shown in Fig. 1. This element has two in-plane displacements at each node. The displacement and strain expressions for this element will be derived using strain gradient notation. The use of this physically interpretable notation identifies the modeling characteristics of the element during the formulation stage (Dow and Byrd 1986; Dow et al. 1986). If parasitic shear is present, the appropriate integration rules can be identified that eliminates parasitic shear and does not generate any spurious zero-energy modes in the element.

The first step in this process is to express the displacement polynomials in terms of strain gradients. The displacement polynomials for the four-node quadrilateral consist of four terms for both the u and v displacements. The polynomial expansion for each displacement includes the l, x, y, and xy terms. These are linear polynomials with the additional xy term chosen for invariance. The strain gradient representations for the displacement representations in a four-node quadrilateral are expressed in strain-gradient notation as (Dow 1999)

$$u(x,y) = [u]_0 + [\varepsilon_x]_0 x + [\gamma_{xy}/2 - r]_0 y + [\varepsilon_{x,y}]_0 xy \tag{1}$$

$$v(x,y) = [v]_0 + [\gamma_{xy}/2 + r]_0 x + [\varepsilon_y]_0 y + [\varepsilon_{y,x}]_0 xy \tag{2}$$

From these displacement expressions, it is seen that the element is capable of modeling the following eight linearly independent strain states:
rigid body modes

$$(1)\ [u]_0 \quad (2)\ [v]_0 \quad (3)\ [r]_0$$

constant strain states

$$(4)\ [\varepsilon_x]_0 \quad (5)\ [\varepsilon_y]_0 \quad (6)\ [\gamma_{xy}]_0$$

Bilinear variations

$$(7)\ [\varepsilon_{x,y}]_0 \quad (8)\ [\varepsilon_{y,x}]_0$$

Each of these eight strain states produces a specific displacement pattern in the element. For example, if the element is representing a constant strain of unity in the x direction, then the coefficient $[\varepsilon_x]_o$ will have a specific value of unity and the u displacement will vary linearly with x. The v displacement for this case would be zero.

Table 1. Differentiation of Strains of the Four-Node Membrane Quadrilateral into Correct and Parasitic Polynomial Terms

Strain	Correct terms	Parasitic terms
ε_x	$1, y$ Constant in x, linear in y	None
ε_y	$1, x$ Linear in x, constant in y	None
γ_{xy}	1 Constant in x and y	x, y Linear in x and y

The associated strain expressions are produced when the displacements given in Eqs. (1) and (2) are substituted into the strain–displacement definitions. The strain expressions for the four-node membrane element are represented as

$$\varepsilon_x(x,y) = \frac{\partial u}{\partial x} = [\varepsilon_x]_0 + [\varepsilon_{x,y}]_0 y \tag{3}$$

$$\varepsilon_y(x,y) = \frac{\partial v}{\partial y} = [\varepsilon_y]_0 + [\varepsilon_{y,x}]_0 x \tag{4}$$

$$\gamma_{xy}(x,y) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = [\gamma_{xy}]_0 + [\varepsilon_{x,y}]_0\, x + [\varepsilon_{y,x}]_0 y \tag{5}$$

It is seen that the normal strain expressions are incomplete Taylor series expressions since they contain only the constant strain term plus one of the first two derivative terms. For example, the normal strain ε_x consists of a constant term plus a linear variation in the y direction.

However, the shear strain expression is contaminated with derivatives of the normal strain. This gives a coupling between the normal and shear terms at the kinematic level and causes parasitic shear. If the element is representing the purely flexural strain state of $[\varepsilon_{x,y}]_0$, then not only is strain energy produced from the normal strains, but strain energy is also produced in shear strains expressions. This parasitic shear strain energy is incorrect and should be eliminated from the element (Byrd 1988). It is noted that if compete polynomials are used in an element, then parasitic shear will not exist in the element. A six-node triangular element is an example of an element developed from complete second-order expansions for the displacements and is not covered in this paper.

The strains for the four-node membrane quadrilateral element are separated in Table 1 into the correct and incorrect parasitic shear terms. It is seen that the normal strains do not contain any incorrect terms while the shear strain has both correct and parasitic terms.

The parasitic shear terms can be removed from the four-node element by integrating the strain energy contribution of the shear strain with a 1×1 Gauss quadrature integration rule instead of with a 2×2 rule. As will be shown later, the indiscriminate use of a 1×1 rule is not generally successful. It should be noted that the normal strain contribution to the strain energy is integrated with a 2×2 rule. It is the use of a lower-order integration the process its name, reduced-order integration. The reason for the success of this approach with the four-node element will now be discussed.

The strain energy for this element will be developed using the strain gradient expressions given in Eqs. (3), (4), and (5) as follows:

J. Eng. Mech., 2005, 131(1): 2-11

$$\{\varepsilon\} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x & y \end{bmatrix} \{\varepsilon_{\text{sg}}\} = [\mathrm{T}_{\text{sg}}]\{\varepsilon_{\text{sg}}\} \tag{6}$$

where

$$\{\varepsilon\} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}^T$$

and

$$\{\varepsilon_{\text{sg}}\} = \{[u]_0 \quad [v]_0 \quad [r]_0 \quad [\varepsilon_x]_0 \quad [\varepsilon_y]_0 \quad [\gamma_{xy}]_0 \quad [\varepsilon_{x,y}]_0 \quad [\varepsilon_{y,x}]_0\}^{\mathrm{T}}$$

The coupling between the normal and shear is evident in the last row in the transformation matrix $[T_{\text{sg}}]$. The incorrect normal strain terms are introduced by the x and y terms in the last two columns. The internal strain energy for the element is computed as

$$U = \frac{1}{2}\int_{\text{Vol}} \{\varepsilon\}^{\mathrm{T}}\{\sigma\} dV \tag{7}$$

with the constitutive relationship given as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{8}$$

The strain energy is written as

$$U = \frac{1}{2}\int_{\text{Vol}} \{\varepsilon\}^{\mathrm{T}}[D]\{\varepsilon\} dV \tag{9}$$

This is expanded in component form to yield

$$U = \frac{1}{2}\int_{\text{Vol}} (D_1\varepsilon_x^2 + D_1\varepsilon_y^2 + D_2\varepsilon_x\varepsilon_y + D_3\gamma_{xy}^2) dV \tag{10}$$

Substitution of Eq. (6) into Eq. (9) yields

$$U = \tfrac{1}{2}\{\varepsilon_{\text{sg}}\}^{\mathrm{T}}[U]\{\varepsilon_{\text{sg}}\} \tag{11}$$

where

$$[U] = \int_{\text{Vol}} [\mathrm{T}_{\text{sg}}]^{\mathrm{T}}[D][\mathrm{T}_{\text{sg}}] dV$$

is a strain energy matrix in strain gradient coordinates. The strain energy matrix is separated into the contributions from the normal strains and the shear strain as

$$[U] = [U_\varepsilon] + [U_\gamma] \tag{12}$$

where $[U_\varepsilon]$ corresponds to the first two rows of the transformation matrix $[\mathrm{T}_{\text{sg}}]$ and $[U_\gamma]$ corresponds to the last row.

For the flexural strain energy matrix, the maximum integrand to be evaluated is second-order polynomials in the x and y directions. For isoparametric elements that are mapped onto a natural domain, the required order of Gauss quadrature is a two-point rule in the x direction and a two-point rule in the y direction.

If the flexural strain energy matrix is separated into the components due to ε_x, ε_y, and the coupling between the two due to Poisson's effect, then different rules apply to each. It is seen that the evaluation of the strain energy due to ε_x requires a 1×2 pattern in order to sufficiently integrate the required integrands. The 1×2 pattern refers to one point in the x direction and two point in the y direction. The evaluation of the strain energy due to ε_y would require a 2×1 pattern. The strain energy evaluation for the Poisson's coupling would require a 1×1 pattern since the maximum integrand to be evaluated is xy. However, a 2×2 rule is used for the combined flexural strain energy evaluation for convenience and time considerations.

Table 2. Eigenvalues for the Four-Node Membrane Quadrilateral

	Eigenvalues	
Mode number	Standard integration	Reduced integration
1	0.0	0.0
2	0.0	0.0
3	0.0	0.0
4	1.4835E4	1.0989E4
5	1.4835E4	1.0989E4
6	2.3077E4	2.3077E4
7	2.3077E4	2.3077E4
8	4.2857E4	4.2857E4
Sum	1.1868E5	1.1099E5

In order to evaluate the shear strain energy component, a 2×2 rule is required since the integrands are second order in both x and y. However, if the strain energy contributions due to the parasitic shear terms are to be eliminated, then the matrix $[U_\gamma]$ must be separated into the correct and incorrect terms. From Eqs. (5) and (6) along with Table 1, it is seen that the parasitic shear terms contribute to terms in $[U_\gamma]$. These terms contain integrands consisting of polynomial that are functions of x and y. The only term that is not associated with the parasitic shear consists of the integration of unity, which gives the volume of the element. In order to compute the required integrand of unity and not evaluate the integrands involving function of x and y, a 1×1 pattern for Gauss quadrature is required. The one-point rule correctly evaluates $2N-1=1$ or first-order polynomials. If the element domain is mapped onto a square domain with the origin at the center, thus, the one-point quadrature correctly evaluates the constant function, but gives zero for the other integrals involving functions of x and y. In other words, the use of a 1×1 pattern of Gauss quadrature for the evaluation of the shear strain component of the strain energy matrix eliminates the adverse effects of parasitic shear.

The use of the prescribed reduced integration does not eliminate any of the strain states that form the basis of the deformation modes for the element. Thus, no spurious zero-energy modes are developed with the use of the reduced integration.

Therefore, the appropriate rules for the Gauss quadrature for the four-node membrane quadrilateral are 2×2 for the flexural strain energy and 1×1 for the shear strain energy. With these orders for the Gauss quadrature for the evaluation of the strain energy, the influence of the parasitic shear terms is eliminated without the development of any spurious zero-energy modes.

These conclusions are confirmed from an eigenvalue analysis for a square element of unit thickness with the material properties of $E=30{,}000$ and $\upsilon=0.3$. The element is formed isoparametrically using both standard and reduced integration rules. The eigenvalues for the two elements are given in Table 2. Since the membrane elements are based on displacement representations, the element with the lowest eigenvalue spectrum will be the better element, according to Rayleigh–Ritz criteria. From Table 2, it is seen that the element formulated with the reduced integration for the shear strain energy is the better element. Also notice that only the energy content of Modes 4 and 5 are reduced. These modes correspond to the flexural modes that should be reduced due to the elimination of the parasitic shear terms. The eigenvalues of the last three modes, which are the shear and the extensional modes, are not reduced. This is expected since these strain states are not modified.

J. Eng. Mech., 2005, 131(1): 2-11

Table 3. Results of Cantilever Beam Analysis Using Four-Node Membrane Elements

Mesh used in beam model[a]	Normalized tip deflections for element with: Standard integration	Reduced integration
1×2	0.09	0.85
1×10	0.67	0.91
2×20	0.89	0.97
4×40	0.097	0.99
Exact deflection $=PL^3/3EI+PL/KGA$		

[a]Elements through depth by elements along length.

The improvement in the four-node membrane quadrilateral formed isoparametrically is illustrated with a cantilever beam problem. The beam has a length to depth ratio of ten and is subjected to an end shear load. The normalized tip displacements for various discretizations are given in Table 3. The element formulated with reduced integration gives much better results, especially for the coarse meshes. This beam problem has mainly flexural deformations, which means that the parasitic shear has heavy influence on the solution.

This illustrates that the removal of the parasitic shear provides an improvement in the element's modeling capabilities. The appropriate Gauss integration rules allow the parasitic shear to be easily removed from the element. These appropriate rules for the Gauss quadrature are 2×2 for the flexural contribution and 1×1 for the shear contribution. These rules are determined by examining the strain expressions for the element in physically interpretable strain gradients.

In the next section, the four-node Mindin plate-bending quadrilateral is examined for the proper integration rules that will eliminate the parasitic shear and not induce any spurious zero-energy modes of deformation.

Four-Node Mindlin Plate Bending Quadrilateral

The four-node plate-bending quadrilateral is shown in Fig. 2. The element has three degree of freedoms (DOFs) at each node consisting of a transverse displacement and two in-plane rotations. The Reissner–Mindlin plate-bending theory allows for the inclusion of through thickness shear strains. This means that the transverse displacements and the in-plane rotations are independent quantities and will be represented by separate polynomials.

The element is known to have a severe modeling deficiency. It is overstiff because of parasitic shear. An attempt to correct this modeling error through the use of reduced order integration of the shear strain contribution to the strain energy has been made (Hughes et al. 1977). The integration rules used in the reference were 2×2 for the bending terms and 1×1 for the shear terms. This approach introduced two spurious zero-energy modes into the element for a total of five zero-energy modes. It was noted that in most problems the elemental rank deficiency did not manifest itself into the global stiffness matrix and was of little consequence. However, for some problems, the global stiffness does indicate the existence of rank deficiency, which precludes the solution of the system of equations.

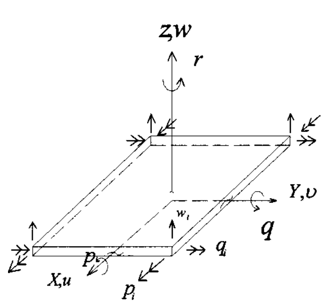

Fig. 2. Four-node plate bending quadrilateral

In the following, the procedure used to identify the parasitic shear terms in the four-node membrane element will be applied to the four-node Mindlin plate-bending element. With the parasitic shear terms identified, appropriate integration rules can be specified that eliminate the parasitic shear without causing the additional zero-energy modes in the element. During this examination, the source of the two additional zero-energy modes that occur with the use of 1×1 reduced integration for the shear strain energy is identified.

The procedure requires that the displacements and, consequently, the strains be expressed in terms of strain gradients so the terms that cause the modeling error can be immediately seen. The three governing displacement fields are associated with the w, u, and v displacements. Since each expression is independent, each would consist of a polynomial with four terms corresponding to the available DOF in the element. The w displacement would include the terms 1, x, y, and xy. Since plane sections are required to remain plane in first-order plate theory, the in-plane displacements have to be linear through the thickness. This means that the u and v displacements are linear in z. Thus the polynomials for the in-plane displacements u and v include the terms z, xz, yz, and xyz. The strain gradient representations for the transverse and in-plane displacements are (Byrd 1988)

$$w(x,y)=[w]_0+[\gamma_{xz}/2-q]_0x+[\gamma_{yz}/2+p]_0y+[(-\gamma_{xy,z}+\gamma_{yx,x}+\gamma_{xz,y})/2]_0xy \quad (13)$$

$$u(x,y,z)=[\gamma_{xz}/2+q]_0z+[\varepsilon_{x,z}]_0xz+[(\gamma_{xy,z}-\gamma_{yz,x}+\gamma_{xz,y})/2]_0yz+[(\varepsilon_{x,yz})]_0xyz \quad (14)$$

$$v(x,y,z)=[\gamma_{yz}/2-p]_0z+[(\gamma_{xy,z}+\gamma_{yz,x}-\gamma_{xz,y})/2]_0xz+[\varepsilon_{y,z}]_0yz+[\varepsilon_{y,xz}]_0xyz \quad (15)$$

The coefficients of Eqs. (13), (14), and (15) identify the 12 strain states or deformation patterns that the element is capable of representing. They are as follows:
rigid body modes

$$(1)\ [w]_0 \quad (2)\ [p]_0 \quad (3)\ [q]_0$$

flexural strain states

$$(4)\ [\varepsilon_{x,z}]_0 \quad (5)\ [\varepsilon_{x,yz}]_0 \quad (6)\ [\varepsilon_{y,z}]_0 \quad (7)\ [\varepsilon_{y,xz}]_0$$

shear strain states

$$(8)\ [\gamma_{xy,z}]_0 \quad (9)\ [\gamma_{xz}]_0 \quad (10)\ [\gamma_{xz,y}]_0 \quad (11)\ [\gamma_{yz}]_0 \quad (12)\ [\gamma_{yz,x}]_0$$

The rotations and strains found by substituting Eqs. (13)–(15) into the definition of rotations and strains from linear elasticity gives

$$p(x,y)=[p-\gamma_{yz}/2]_0+[(-\gamma_{xy,z}-\gamma_{yz,x}+\gamma_{xz,y})/2]_0x+[-\varepsilon_{y,z}]_0y+[-\varepsilon_{y,xz}]_0xy \quad (16)$$

J. Eng. Mech., 2005, 131(1): 2-11

Table 4. Differentiation of Strains of the Four-Node Mindlin Plate-Bending Quadrilateral into Correct and Parasitic Polynomial Terms

Strain	Correct terms	Parasitic terms
ε_x	z, yz Constant in x, linear in y	None
ε_y	z, xz Linear in x, constant in y	None
γ_{xy}	z Constant in x and y	xz, yz Linear in x and y
γ_{xz}	$1, y$ Constant in x, linear in y	x, xy Linear in x and y
γ_{yz}	$1, x$ Linear in x, constant in y	y, xy Linear in x and y

$$q(x,y) = [\gamma_{xz}/2 + q]_0 + [\varepsilon_{x,z}]_0 x + [(\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y})/2]_0 y + [\varepsilon_{x,yz}]_0 xy \tag{17}$$

$$\varepsilon_x = [\varepsilon_{x,z}]_0 z + [\varepsilon_{x,yz}]_0 yz \tag{18}$$

$$\varepsilon_y = [\varepsilon_{y,z}]_0 z + [\varepsilon_{y,xz}]_0 xz \tag{19}$$

$$\gamma_{xy} = [\gamma_{xy,z}]_0 z + [\varepsilon_{x,yz}]_0 xz + [\varepsilon_{y,xz}]_0 yz \tag{20}$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\varepsilon_{x,z}]_0 x + [\gamma_{xz,y}]_0 y + [\varepsilon_{x,yz}]_0 xy \tag{21}$$

$$\gamma_{yz} = [\gamma_{yz}]_0 + [\gamma_{yz,x}]_0 x + [\varepsilon_{y,z}]_0 y + [\varepsilon_{y,xz}]_0 xy \tag{22}$$

One of first things to note is that the in-plane shear γ_{xy} has parasitic shear terms. These are normal strain terms and must be eliminated to correct the element. Normally, reduced-order integration is only applied to integrands produced by the transverse shear strain and not on those produced by the in-plane shear strain. However from Eq. (20), it is apparent that the reduced integration should be applied to those produced by the in-plane shear strain in order to eliminate the parasitic shear. Table 4 shows the correct terms that should be retained and the parasitic terms that should be eliminated.

Since this expression is similar to the shear strain in the four-node membrane element, a reduced 1×1 pattern for the Gauss quadrature rule is required for the integration of the in-plane shear strain contribution to the strain energy. This would eliminate the contribution of the parasitic terms associated with x and y in the in-plane shear strain. Also, it does not eliminate any of the correct strain states that form part of the basis set for the element. Thus, this integration pattern would not generate any spurious zero-energy modes.

The parasitic shear terms existing in the transverse shear terms are obvious in Eqs. (21) and (22). For γ_{xz} the parasitic shear terms are associated with the x and xy terms and should be eliminated. The strain states associated with the 1.0 and y terms are not parasitic shear terms and should be retained. The integration rules that perform this operation are a 1×2 pattern. The one-point rule in the x direction will eliminate the terms that are a function of x, which are exactly those that are associated with the parasitic shear terms. The two-point rule in the y direction will evaluate the integrands associated with the good y terms up to the integrand of y^3 which is sufficient for the integrands existing in the strain energy expression for this transverse shear strain.

For γ_{yz}, the terms to be eliminated are associated with the y and xy polynomial terms and those to be kept are associated with the 1.0 and x polynomial terms. A Gauss quadrature pattern of 2×1 accomplishes this operation by eliminating the integrands that are a function of y and integrates the integrands that are polynomials up to x^3.

Table 5. Eigenvalues for Three Types of Four-Node Mindlin Plate Elements

Mode number	Eigenvalues: Normal integration	Appropriate reduced integration	Uniform reduced integration
1	0.0	0.0	0.0
2	0.0	0.0	0.0
3	0.0	0.0	0.0
4	0.11914E+04	0.91575E+03	0.0
5	0.23046E+04	0.91575E+03	0.0
6	0.23046E+04	0.11914E+04	0.91575E+03
7	0.32051E+04	0.19231E+04	0.91575E+03
8	0.51282E+04	0.32051E+04	0.19231E+04
9	0.67766E+04	0.35714E+04	0.19231E+04
10	0.10347E+05	0.10347E+05	0.35714E+04
11	0.19231E+05	0.19231E+05	0.19231E+05
12	0.19231E+05	0.19231E+05	0.19231E+05
Sum	0.69719E+05	0.60531E+05	0.47711E+05

Note: $E=30{,}000$; $\upsilon=0.3$; and $K=5/6$.

These integration patterns for the transverse shear strain components to the strain energy eliminate only the terms that are defined as parasitic shear terms. Other nonparasitic shear terms are not eliminated. These integration patterns do not produce any spurious zero-energy modes of deformation.

If a uniform 1×1 pattern is used for the reduced integration of the transverse shear contributions to the strain energy, then two spurious zero-energy modes of deformation are generated (Hughes et al. 1977). These are explained by examining the transverse shear strain expressions given in Eqs. (21) and (22). A 1×1 pattern of Gauss quadrature would eliminate all integrands that are functions of either x or y. This eliminates the strain state $[\gamma_{xz,y}]_0$ associated with the y term in γ_{xz} and eliminates the strain state $[\gamma_{yz,x}]_0$ associated with the x term in γ_{yz}. These are correct terms that are not to be eliminated. These are the only strain energy expressions in which these strain states exist and their elimination provides two modes of deformation, which contain no strain energy. The elimination of these two strain states produces the two spurious zero energy modes of deformation. Essentially, it eliminates two of the fundamental modes or strain states from the basis set for this element.

The appropriate patterns for the reduced integration of the four-node Mindlin plate bending element that eliminates the parasitic shear and does not produce any spurious zero-energy modes are: (1) a 1×2 pattern for ε_x; (2) a 2×1 pattern for ε_y; (3) a 1×1 pattern for Poisson's coupling between ε_x and ε_y; (4) a 1×1 pattern for γ_{xy}; (5) a 1×2 pattern for γ_{xz}; and (6) a 2×1 pattern for γ_{yz}. Thus the required strain energy integral evaluations can be accomplished with three integration patterns.

An eigenvalue analysis is performed to illustrate the differences between an isoparametric element generated with standard integration rules that do not exclude the parasitic shear, an isoparametric element generated using the appropriate integration pattern just discussed, and an isoparametric element generated with a uniform 1×1 reduced integration for the strain energy contributions for all the shear strains. The element to be analyzed is a square element with a length to thickness ratio of two. The eigenvalues for the three elements are given in Table 5. It is noted

J. Eng. Mech., 2005, 131(1): 2-11

Table 6. Results of Square Plate Analyses with Concentrated Central Load Using Four-Node Mindlin Plate Elements

Problem definition	Elements in quadrant	Deflections under point load		
		Element 1[a]	Element 2[b]	Element 3[c]
Square plate simply supported concentrated center load	1×1	4.278E-5	1.341E-2	2.405E-2
	2×2	1.343E-4	1.167E-2	1.262E-2
	4×4	4.846E-4	1.160E-2	1.181E-2
	8×8	1.690E-3	1.162E-2	1.170E-2
	12×12	3.203E-3	1.163E-2	1.170E-2
Exact deflection[d]			1.160E-2	
Square plate clamped edges concentrated center load				
	1×1	1.071E-5	1.071E-5	1.429E-5
	2×2	4.588E-5	4.948E-3	4.953E-3
	4×4	1.691E-4	5.454E-3	5.462E-3
	8×8	6.044E-4	5.586E-3	5.597E-3
	12×12	1.192E-3	5.613E-3	5.626E-3
Exact deflection[d]			5.600E-3	

[a]Element with normal Gauss numerical integration.

[b]Element with appropriate reduced Gauss numerical integration.

[c]Element with uniform reduced Gauss numerical integration.

[d]Exact solution for thin plate from Timoshenko and Woinowsky-Krieger (1959).

that six modes are different between the element with normal standard integration rules and the element with appropriate integration rules. These six modes correspond to the flexural modes that exist as parasitic shear in shear strains. The two additional zero-energy modes are evident in the element with uniform reduced integration for the shear strain energy.

It is seen that the element generated using uniform reduced integration is the softest element. The element generated with the appropriate integration rules is less stiff than the element generated with normal integration. Plate problems analyzed with finite elements do not exhibit inter-element compatibility so they do not obey the Rayleigh–Ritz criterion. As a result, they do not approach the correct result monotonically, so the relative magnitudes of the sum of the eigenvalues are not an indication of the quality of the element in an analysis. This is illustrated on square plate problems with simply supported and clamped edge conditions subjected to both concentrated center and uniformly varying loads.

The results for these analyses for the Mindlin plate bending elements with the integration schemes are given in Table 6 for a plate subjected to the concentrated center load. For the plate problem with a concentrated load, the element with appropriate reduced integration gives better results for the simply supported case while in the clamped edge case, the element with uniform reduced integration gives slightly better results. In both cases, the element with standard integration does not gives acceptable results, even in the refined discretizations.

In order to illustrate the deficiencies of elements that possess spurious zero-energy modes, the following example problem is analyzed. The problem consists of a square plate supported only at the corners resembling a table, which is a commonly used structure. The plate is subjected to both a concentrated load at the center and uniformly varying loads. The results of the analyses using the above Mindlin four-node elements are given in Table 7 for the problem with a uniform load.

An approximate value for the center deflection due to a uniformly varying load is 0.0265, computed using a series solution (Ballesteros and Lee 1960). The results for the various elements, excluding the Mindlin element generated with uniform integration, indicate that the elements are adequately modeling this problem. The results of these finite element analyses correspond to the accuracy reported when four-node Kirchhoff elements are used to model this problem (Zienkiewicz 1977).

For the isoparametric element with uniformly reduced integration, the results for the uniformly varying load case indicated the presence of singularities in the global stiffness matrix. The element with appropriate reduced integration gives smooth results for the deflections from the center to the edge. Since the element with appropriately reduced integration performed adequately, it is assumed that the oscillations are caused by the addition of the two zero-energy modes that exist in the element stiffness matrix for the isoparametric element generated with uniform reduced integration. The singularities and the oscillations in the transverse displacement for this type of problem create some suspicion concerning the use of the rank-deficient isoparametric elements with reduced integration (Cook 2002). This inclusion of the zero-energy deformation modes has been a topic of discussion in the literature (Hughes and Cohen 1978; Hughes and Tezduyar 1978; Pugh et al. 1978; Crisfield 1984) and is still an open question. However, with the clear identification of the source of these additional zero-energy modes previously presented, then these ele-

Table 7. Analysis of Square Plate with Uniform Load and Corner Supports Using Four-Node Plate Elements

Mesh[a]	Deflections at plate center		
	Element 1[b]	Element 2[c]	Element 3[d]
1×1	2.853E-5	1.203E-2	Singular
2×2	1.861E-4	2.190E-2	Singular
4×4	8.121E-4	2.461E-2	Singular
8×8	2.957E-3	2.531E-2	Singular
12×12	5.664E-3	2.545E-2	Singular

[a]Square elements in quarter plate, due to symmetry.

[b]Element with normal Gauss numerical integration.

[c]Element with appropriate reduced Gauss numerical integration.

[d]Element with uniform reduced Gauss numerical integration.

J. Eng. Mech., 2005, 131(1): 2-11

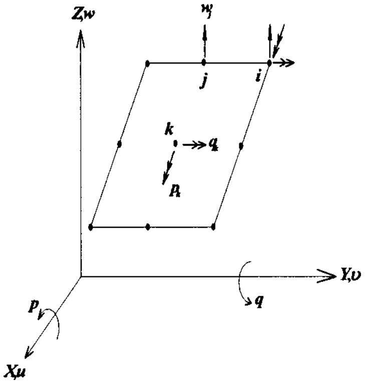

Fig. 3. Nine-node plate bending quadrilateral

ments need not be used anymore. The isoparametric elements with the appropriately reduced integration schemes could be used in the analyses.

This illustrates that the removal of the parasitic shear using an appropriate reduced integration technique provides an improvement in the element's modeling capabilities without introducing any spurious zero-energy modes. These appropriate rules for the Gauss quadrature can be applied in two fashions. The first is to use a 2×2 pattern for the flexural contribution, a 1×1 pattern for the in-plane shear contribution, a 1×2 pattern for the γ_{xz} contribution, and 2×1 for the γ_{yz} contribution. The second method of application, which may be slightly more efficient, is to use a 1×2 pattern for the contribution due to ε_x and γ_{xz}, a 2×1 pattern for the contribution due to ε_y and γ_{yz}, and a 1×1 pattern for the contribution due to the flexural coupling and γ_{xy}. These rules are determined by examining the strain expressions for the element that are derived in physically interpretable strain gradients.

In the next section, the nine-node Mindlin plate-bending quadrilateral is examined for the proper integration rules that will eliminate the parasitic shear and not induce any spurious zero-energy modes of deformation.

Nine-Node Mindlin Plate Bending Quadrilateral

The nine-node plate-bending quadrilateral is shown in Fig. 3. The element has three DOFs at each node consisting of a transverse displacement and two in-plane rotations. This element is also based on the Reissner–Mindlin plate-bending theory.

The element is known to have a severe modeling deficiency due to parasitic shear, which is alleviated through the use of reduced integration for the shear strain contribution to the strain energy (Zienkiewicz et al. 1971). The integration rules used in the references were 3×3 for the bending terms and 2×2 for the shear terms. It was discovered that this element had an additional zero-energy mode beyond those associated with the rigid body modes.

In the following, the previous procedure used in determining the parasitic shear terms will be applied to the nine-node Mindlin plate-bending element. With the parasitic shear identified, appropriate integration rules can be specified that eliminate the parasitic shear without causing the additional zero-energy mode that occur with the use of uniform 2×2 reduced integration for the shear strain energy.

The procedure requires that the displacements and strains be expressed in terms of strain gradients. The three governing displacement fields are associated with the u, v, and w displacements. Since each expression is independent, each would consist of a polynomial with nine terms corresponding to the available DOFs in the element. The w displacement would include the polynomial terms 1, x, y, x^2, xy, y^2, x^2y, xy^2, and x^2y^2. Since plane sections are required to remain plane in the first order plate theory, the in-plane displacements have to be linear through the thickness. This means that the u and v displacements are linear in z. Thus polynomials for the in-plane displacements u and v include the terms z, xz, yz, x^2z, xyz, y^2z, x^2yz, xy^2z, and x^2y^2z. The strain gradient expressions for the displacement are

$$\begin{aligned} w(x,y) = {} & [w]_0 + [\gamma_{xz}/2 - q]_0 x + [\gamma_{yz}/2 - p]_0 y \\ & + [(-\gamma_{xy,z} + \gamma_{yz,x} + \gamma_{xz,y})/2]_0 xy + [(\gamma_{xz,x} - \varepsilon_{x,z})/2]_0 x^2 \\ & + [(\gamma_{yz,y} - \varepsilon_{y,z})/2]_0 y^2 + [(\gamma_{xz,xy} - \varepsilon_{x,yz})/2]_0 x^2 y \\ & + [(\gamma_{yx,xy} - \varepsilon_{y,xz})/2]_0 xy^2 + [(\gamma_{xz,xyy} - \varepsilon_{x,yyz})/4]_0 x^2y^2 \end{aligned} \tag{23}$$

$$\begin{aligned} u(x,y,z) = {} & [\gamma_{xz}/2 + q]_0 z + [\varepsilon_{x,z}]_0 xz + [(\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y})/2]_0 yz \\ & + [(\varepsilon_{x,xz})/2]_0 x^2 z + [(\varepsilon_{x,yz})]_0 xyz + [(\gamma_{xy,yz} - \varepsilon_{y,xz})/2]_0 y^2 z \\ & + [(\varepsilon_{x,xyz})/2]_0 x^2yz + [(\varepsilon_{x,yyz})/2]_0 xy^2 z + [(\varepsilon_{x,xyyz})/4]_0 x^2y^2z \end{aligned} \tag{24}$$

$$\begin{aligned} v(x,y,z) = {} & [\gamma_{yz}/2 - p]_0 z + [(\gamma_{xy,z} + \gamma_{yz,x} - \gamma_{xz,y})/2]_0 xz + [\varepsilon_{y,z}]_0 yz \\ & + [(\varepsilon_{y,xz})]_0 xyz + [(\varepsilon_{y,yz})/2]_0 y^2 z + [(\gamma_{xy,xz} - \varepsilon_{x,yz})/2]_0 x^2 z \\ & + [(\varepsilon_{y,xxz})/2]_0 x^2yz + [(\varepsilon_{y,xyz})/2]_0 xy^2 z + [(\varepsilon_{y,xxyz})/4]_0 x^2y^2z \end{aligned} \tag{25}$$

The coefficients of Eqs. (23), (24), and (25) identify the 27 strain states or deformation patterns that constitute the basis set for the element. They are as follows:
rigid body modes

$$(1)\ [w]_0 \quad (2)\ [p]_0 \quad (3)\ [q]_0$$

flexural strain states

$$\begin{array}{lll} (4)\ [\varepsilon_{x,z}]_0 & (5)\ [\varepsilon_{x,xz}]_0 & (6)\ [\varepsilon_{x,yz}]_0 \\ (7)\ [\varepsilon_{x,xyz}]_0 & (8)\ [\varepsilon_{x,yyz}]_0 & (9)\ [\varepsilon_{x,xyyz}]_0 \\ (10)\ [\varepsilon_{y,z}]_0 & (11)\ [\varepsilon_{y,xz}]_0 & (12)\ [\varepsilon_{y,yz}]_0 \\ (13)\ [\varepsilon_{y,xxz}]_0 & (14)\ [\varepsilon_{y,xyz}]_0 & (15)\ [\varepsilon_{y,xxyz}]_0 \end{array}$$

shear strain states

$$\begin{array}{lll} (16)\ [\gamma_{xy,z}]_0 & (17)\ [\gamma_{xy,xz}]_0 & (18)\ [\gamma_{xy,yz}]_0 \\ (19)\ [\gamma_{xz}]_0 & (20)\ [\gamma_{xz,x}]_0 & (21)\ [\gamma_{xz,y}]_0 \\ (22)\ [\gamma_{xz,xy}]_0 & (23)\ [\gamma_{xz,xyy}]_0 & (24)\ [\gamma_{yz}]_0 \\ (25)\ [\gamma_{yz,x}]_0 & (26)\ [\gamma_{yz,y}]_0 & (27)\ [\gamma_{yz,xy}]_0 \end{array}$$

These 27 strain states define the set of deformation modes that form the basis for the element.

The strains associated with these displacements for the nine-node Mindlin plate element are

$$\begin{aligned} \varepsilon_x = {} & [\varepsilon_{x,z}]_0 z + [\varepsilon_{x,xz}]_0 xz + [\varepsilon_{x,yz}]_0 yz + [\varepsilon_{x,xyz}]_0 xyz + [\varepsilon_{x,yyz}]_0 y^2 z \\ & + [\varepsilon_{x,xyyz}]_0 xy^2 z \end{aligned} \tag{26}$$

$$\begin{aligned} \varepsilon_y = {} & [\varepsilon_{y,z}]_0 z + [\varepsilon_{y,xz}]_0 xz + [\varepsilon_{y,yz}]_0 yz + [\varepsilon_{y,xxz}]_0 x^2 z + [\varepsilon_{y,xyz}]_0 xyz \\ & + [\varepsilon_{y,xxyz}]_0 x^2yz \end{aligned} \tag{27}$$

J. Eng. Mech., 2005, 131(1): 2-11

Table 8. Differentiation of Strain of Nine-Node Mindlin Plate-Bending Quadrilateral into Correct and Parasitic Polynomial Terms

Strain	Correct terms	Parasitic terms
ε_x	$z, xz, yz, xyz, y^2z, xy^2z$ Linear in x, quadratic in y	None
ε_y	$z, xz, yz, xyz, x^2z, x^2yz$ Quadratic in x, linear in y	None
γ_{xy}	z, xz, yz, xyz Linear in x and y	x^2z, y^2z, x^2yz, xy^2z Quadratic in x and y
γ_{xz}	$1, x, y, xy, y^2, xy^2$ Linear in x, quadratic in y	x^2, x^2y, x^2y^2 Quadratic in x and y
γ_{yz}	$1, x, y, xy, x^2, x^2y$ Quadratic in x, linear in y	y^2, xy^2, x^2y^2 Quadratic in x and y

$$\gamma_{xy} = [\gamma_{xy,z}]_0 z + [\gamma_{xy,xz}]_0 xz + [\gamma_{xy,yz}]_0 yz + [\varepsilon_{x,xyz}]_0 x^2 z + [\varepsilon_{x,yyz} + \varepsilon_{y,xxz}]_0 xyz + [\varepsilon_{x,xyyz}/2]_0 x^2yz + [\varepsilon_{y,xyz}/2]_0 y^2 z + [\varepsilon_{y,xxyz}/2]_0 xy^2 z \tag{28}$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\gamma_{xz,x}]_0 x + [\gamma_{xz,y}]_0 y + [\gamma_{xz,xy}]_0 xy + [\varepsilon_{x,xz}/2]_0 x^2 + [(\gamma_{yz,xy} + \gamma_{xy,yz} - 2\varepsilon_{y,xz})/2]_0 y^2 + [\varepsilon_{x,xyz}/2]_0 x^2 y + [\varepsilon_{xx,xyy}/2]_0 xy^2 + [\varepsilon_{x,xyyz}/4]_0 x^2y^2 \tag{29}$$

$$\gamma_{yz} = [\gamma_{yz}]_0 + [\gamma_{yz,x}]_0 x + [\gamma_{yz,y}]_0 y + [(\gamma_{xz,xy} + \gamma_{xy,xz} - 2\varepsilon_{x,yz})/2]_0 x^2 + [\gamma_{yz,xy}]_0 xy + [\varepsilon_{y,yz}/2]_0 y^2 + [(\gamma_{xz,xyy} - \varepsilon_{x,yyz} + \varepsilon_{y,xxz})/2]_0 x^2 y + [\varepsilon_{y,xyz}/2]_0 xy^2 + [\varepsilon_{y,xxyz}/2]_0 x^2y^2 \tag{30}$$

These strains are separated into the correct and parasitic terms in Table 8. Also given in Table 8 is the maximum order of the polynomial terms that exist in the correct and parasitic terms.

The normal strains are biquadratic and do not contain any parasitic terms. From Table 8, the maximum integrand that would be produced by ε_x is $x^2y^4z^2$. A 3×3 integration pattern is required to appropriately integrate this polynomial. Similarly, the maximum integrand produced by ε_y is $x^4y^2z^2$ which requires a 3×2 integration pattern. The coupling between the two normal strains produces a maximum integrand of $x^3y^3z^2$. This requires a 2×2 pattern for appropriate integration.

The correct terms in the in-plane shear, γ_{xy}, are linear. The quadratic terms are parasitic. The correct terms produce a maximum integrand of $x^2y^2z^2$. The parasitic terms produce integrands of cubic or higher order. Remembering that the integration of an odd function on a symmetric domain is identically zero, then a 2×2 integration rule appropriately integrates the integrands associated with the good terms and eliminates the integrands associated with the parasitic terms.

The transverse shear γ_{xz} has correct terms that produce a maximum polynomial integrand of x^2y^4. The parasitic terms produce integrands that contain the term x^4. Thus, a 2×3 pattern of integration points will integrate the integrands associated with the good terms. The two points in the x direction will not integrate the integrands associated with the parasitic terms. Therefore, the 2×3 pattern of Gauss integration points will integrate all the integrands associated with the good terms and eliminate the integrands due to the parasitic terms.

Similarly, the appropriate integration pattern for γ_{yz} is 3×2 since the maximum integrand due to the good terms is x^4y^2 and all integrands due to the parasitic terms contain y^4. A two-point scheme in the y direction will eliminate the integrands due to the parasitic terms.

Therefore, the appropriate integration rules for the stiffness matrix for the nine-node Mindlin plate-bending element are 2×3 for the ε_x contribution, 3×2 for the ε_y contribution, 2×2 for the Poisson's coupling contribution, 2×2 for the γ_{xy} contribution, 2×3 for the γ_{xz} contribution, and 3×2 for the γ_{yz} contribution. These integration rules eliminate the influence of the parasitic shear terms without introducing any spurious zero-energy modes into the elemental stiffness matrix.

The conclusion that no zero-energy modes are introduced into the elemental stiffness matrix with the use of the appropriate integration rule is verified by examining the strain expressions that remain after the parasitic shear terms are eliminated. From the expressions in Eqs. (24)–(28) and the separation between the cor-

Table 9. Result of Square Plate Analyses with Concentrated Central Load Using Nine-Node Mindlin Plate Elements

Problem definition	Elements in quadrant	Deflections under point load		
		Element 1[a]	Element 2[b]	Element 3[c]
Square plate simply supported	1×1	8.041E-3	1.178E-2	1.277E-2
	2×2	1.037E-2	1.162E-2	1.188E-2
Concentrated center load	4×4	1.127E-2	1.163E-2	1.172E-2
	8×8	1.157E-2	1.165E-2	1.171E-2
Exact deflection[d]			1.160E-2	

[a]Element with normal Gauss numerical integration.

[b]Element with appropriate reduced Gauss numerical integration.

[c]Element with uniform reduced Gauss numerical integration.

[d]Exact solution for thin plate from Timosheno and Woinowsky-Krieger (1959).

J. Eng. Mech., 2005, 131(1): 2-11

rect and parasitic terms in Table 9, it is seen that all of the strain states that compose the basis set of deformation modes for this element still exist in the element. In other words, no strain state is totally eliminated from the strain expressions with the use of the appropriate integration rules. Thus, no additional zero-energy modes exist in the element when generated using the appropriate integration rule.

It was noted previously that when uniform reduced 2×2 integration for shear strain is utilized in the generation of the element that an additional zero-energy mode is developed. This is explained by examining the additional terms that are eliminated with the uniform 2×2 integration. Since the uniform 2×2 pattern has been specified as the appropriate rule for the in-plane shear, the additional zero-energy strains must be introduced in the integration of the strain energy of the transverse shear strains. The use of 2×2 integration pattern instead of the appropriate 2×3 pattern for the shear γ_{xz} eliminates these additional terms: $[(\gamma_{yz,xy}+\gamma_{xy,yz}-2\varepsilon_{y,xz})/2]_0 y^2$ and $[\gamma_{xz,xyy}/2]_0 xy^2$. The use of a 2×2 pattern instead of the appropriate 3×2 pattern for the shear γ_{yz} eliminates these additional terms: $[(\gamma_{xz,xy}+\gamma_{xy,xz}-2\varepsilon_{x,yz})/2]_0 x^2$ and $[(\gamma_{xz,xyy}-\varepsilon_{x,yyz}+\varepsilon_{y,xxz})/2]_0 x^2 y$. Examination of the strain states in these additionally eliminated terms indicate that all exist in other strain expressions except one. The one strain state that is totally eliminated is $\gamma_{xz,xyy}$. The total elimination of this strain state removes the strain energy associated with one of the deformation modes that form the basis set for the element. This produces the additional zero-energy mode that is associated with the use uniform 2×2 reduced integration for the shear in the nine-node Mindlin plate-bending element.

The square plate problem is analyzed using the nine-node elements with the various integration rules. The results for the concentrated load case are given in Table 9. It is seen that the element with the appropriate integration rules perform better for this problem than the element with normal or uniform reduced integration.

Thus, the use of appropriate reduced Gauss integration quadrature for the nine-node Mindlin plate bending element eliminates the influence of parasitic shear and does not induce any spurious zero-energy modes into elemental stiffness matrix.

Conclusion

The objective of defining the appropriate Gauss numerical integration rules that eliminate the artificial stiffening due to parasitic shear without inducing any spurious zero-energy modes into the element stiffness matrix has been accomplished. The objective is accomplished by expressing the polynomials that approximate the displacements and rotations using a physically interpretable notation. The use of this notation allows the equations that surface during the formulation of the finite element stiffness matrix to be compared to the equations of continuum mechanics. If the finite element model does not match the appropriate constitutive or strain–displacement relationships, the terms that cause modeling errors in the finite element are identified. That is to say, the precise terms that are to be retained and those to be eliminated are identified.

With the knowledge about which terms are to be retained and which are to be eliminated, appropriate integration patterns are selected to eliminate the parasitic shear terms while retaining the correct terms. The selection of the appropriate quadrature pattern is determined by the substitution of the strain expressions in terms of strain gradients into the definition of strain energy. Upon examination of the integrands that exist in the strain energy, the quadrature rule that will eliminate the unwanted terms is selected. In this paper, the appropriate Gauss quadrature rules are specified for the four-node membrane quadrilateral and two Mindlin plate-bending quadrilaterals, the four and nine node elements.

The source of spurious zero-energy modes that are developed in elements generated using uniform reduced numerical integration is identified through the use of this procedure based upon the strain gradient notation. The indiscriminate use of uniform reduced integration eliminates the effect of certain basis elements required in strain energy evaluation. This results in the creation of spurious zero energy modes. However, the use of appropriate reduced integration schemes does not create spurious zero-energy modes.

References

Ballesteros, P., and Lee, S. L. (1960). "Uniformly loaded rectangular plate supported at the corners." *Int. J. Mech. Sci.* 2, pp. 206–211.

Byrd, D. E. (1988). "Identification and elimination of errors in finite element analysis." PhD Dissertation, Univ. of Colorado, Boulder, Co.

Cook, R. D. (1972). "More on reduced integration and isoparametric elements." *Int. J. Numer. Methods Eng.* 5, pp 141–142.

Cook, R. D. (1975). "Avoidance of parasitic shear in plane element." *J. Struct. Div. ASCE* 101(6), 1239–1253.

Cook, R. D. (2002). *Concepts and applications of finite element analysis*, 4th Ed., Wiley, New York.

Crisfield, M. A. (1984). "A quadratic mindlin element using shear constraints." *Comput. Struct.* 18, pp 833–852.

Doherty, W. P., Wilson, E. L., and Taylor, R. L. (1969). "Stress analysis of axi-symmetric solids using higher-order quadrilateral finite elements." *Structural Engineering Laboratory Rep. No. SESM 69-3*, University of California at Berkeley, Berkeley, Calif.

Dow, J. O. (1999). *A unified approach to the finite element method and error analysis procedures*, Academic, New York.

Dow, J. O., and Byrd, D. E. (1986). "Exact energy plate finite element." *Proc., AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics, and Materials Conf.*, San Antonio, Tex.

Dow, J. O., and Byrd, D. E. (1988). "The identification and elimination of artificial stiffening errors in finite elements." *Int. J. Numer. Methods Eng.* 26, pp 743–762.

Dow, J. O., Cabiness, H. D., and Ho, T. H. (1986). "Linear strain element with curved edges." *J. Struct. Eng.* 112(4), 692–708.

Dow, J. O., and Mohamed, O. A. (2002). "Error identification in individual finite elements: a path to the integration of continuum and computational mechanics." *Proc., 43rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, Denver.

Huang, H. C., and Hinton, E. (1984). "A nine-node Lagrangian mindlin plate element with enhanced shear interpolation." *Eng. Comput.* 1, 369–379.

Hughes, T. J. R. (1987). *The finite element method: Linear static and dynamic finite element analysis*, Prentice-Hall, Englewood Cliffs, N.J.

Hughes, T. J. R., and Cohen, M. (1978). "The 'heterosis' finite element for plate bending." *Comput. Struct.* 9, 445–450.

Hughes, T. J. R., Taylor, R. L., and Kanoknukulchai, W. (1977). "A simple and efficient finite element for plate bending." *Int. J. Numer. Methods Eng.* 11, 1529–1543.

Hughes, T. J. R., and Tezduyar, T. E. (1978). "Finite elements based upon Mindlin plate theory, with particular reference to the four-node bilinear isoparametric element." *J. Appl. Mech.* 48, 587–596.

Kavanagh, K. T., and Key, S. W. (1972). "A note on selection and reduced integration techniques in the finite element method." *Int. J. Numer. Methods Eng.* 4(1), 148–151.

Parisch, H. (1979). "A critical survey of the 9-node degenerated shell element with special emphasis on thin shell application and reduced integration." *Comput. Methods Appl. Mech. Eng.* 20, 323–350.

J. Eng. Mech., 2005, 131(1): 2-11

Pawsey, S. F. (1972). “Discussion of papers by O. C. Zienkiewicz, R. L. Taylor, and J. M. Too and Pawsey and clough.” *Int. J. Numer. Methods Eng.* 4, 449–450.

Prathap, G., and Bhashyam, G. R. (1982). “Reduced integration and the shear flexible beam element.” *Int. J. Numer. Methods Eng.* 18, 195–210.

Pugh, E. D. L., Hinton, E., and Zienkiewicz, O. C. (1978). “A study of quadrilateral plate bending elements with ‘reduced’ integration.” *Int. J. Numer. Methods Eng.* 12, 1059–1079.

Stolarski, H., and Belytschko, T. (1983). “Shear and membrane locking in curved C^0 elements.” *Comput. Methods Appl. Mech. Eng.* 41, 279–296.

Timoshenko, S., and Woinowsky-Krieger, S. (1959). *Theory of plates and shells*, 2nd Ed., McGraw–Hill, New York.

Zienkiewicz, O. C. (1977). *The Finite Element Method*, 3rd Ed., McGraw-Hill, London.

Zienkiewicz, O. C., Taylor, R. L., and Too, J. M. (1971). “Reduced integration technique in general analysis of plates and shells.” *Int. J. Numer. Methods Eng.* 3, 275–290.

J. Eng. Mech., 2005, 131(1): 2-11