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Formulation and survey of ALE method in nonlinear solid mechanics

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Abstract

This paper investigates the applicability and accuracy of existing formulation methods in general purpose finite element programs to the finite strain deformation problems. The basic shortcomings in using such programs in these applications are then pointed out and the need for a different type of formulation is discussed. An arbitrary Lagrangian–Eulerian (ALE) method is proposed and a concise survey of ALE formulation is given. A consistent and complete ALE formulation is derived from the virtual work equation transformed to arbitrary computational reference configurations. Differences between the proposed formulations and similar ones in the literature are discussed. The proposed formulation presents a general approach to ALE method. It includes load correction terms and is suitable for rate-dependent and rate-independent material constitutive law. The proposed formulation reduces to both updated Lagrangian and Eulerian formulations as special cases.

Keywords: Arbitrary Lagrangian–Eulerian method (ALE); Finite deformation; Finite element formulation; Metal forming

1. Introduction

Although the application of the finite element method to many nonlinear problems has been successfully carried out, there are numerous areas of concern and investigation in that regard. One of such areas is the solution of large strain plasticity and metal-forming problems with pseudo-type boundary nonlinearities. A number of critical difficulties arise in the finite element analysis of such problems. Among these difficulties are: the proper formulation of the problem, mesh distortion during the deformation process, the proper modeling of the contact boundary conditions, incorporation of the plastic incompressibility condition and accounting for plastic anisotropy.

Foundations of large strain analysis of elastioplastic solids may be traced back to the early work of Hill [1]. It took some time, until Hibbit et al. [2] introduced, however, the first finite element

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formulation for large strain problems. In their approach they used a total Lagrangian formulation (TLF). Later on, McMeeking and Rice [3] pioneered the use of updated Lagrangian formulation (ULF) in the same area of applications. The two formulation methods have been widely used for both steady and non-steady static large plastic strain problems. On the other hand, Eulerian formulation (EF) has been initially introduced for finite element applications in the fluid mechanics area. Several trials aiming at adapting this formulation to large strain and metal-forming problems were attempted [4–7]. Owing to the difficulties in obtaining material time derivatives in spatial reference frame, no generally accepted Eulerian formulation is available for such problems. Some developments, e.g., Gadala et al. [6], give the proper material time derivatives in a spatial reference frame but original geometric parameters are included in the final equilibrium equation. Therefore, such approach may not be strictly considered as purely Eulerian. Also, since the mesh is spatially fixed in EF, it is not easy to simulate non-steady static or dynamic behavior. Trials have been made to relate the moving material points (in terms of FE Gauss points) to the fixed spatial mesh [6, 7, 8]. Much work is still required, however, to refine and establish this approach.

This paper investigate the applicability and accuracy of existing formulation methods in general purpose finite element programs to the finite strain deformation problems. The objective is to investigate some of the above aspects through the use of available FE commmercial programs; namely ANSYS [9] and NISA [10] provide an assessment for the existing formulation of such problems, then a new formulation is provided as well. The applications considered in this regard are flat punch indentation and plane strain extrusion problems.

2. Lagrangian-type of formulation

2.1. Sample examples

2.1.1. Flat punch identation

The problem is modeled as an axi-symmetric one with the punch and foundation being assumed rigid and frictionless. Axi-symmetric 8-node isoparametric elements together with 2-node (node-to-node) gap elements are used in the model. The mesh topologies before and after deformation is presented in Fig. 1. The radial displacement U_r, the second principle stress S_2, and the equivalent strain S_{eq} are used in the analysis of the results. The results obtained from two programs showed close agreement in displacement and strains but revealed noticeable discrepancies when comparing stress values. The results from both programs do not agree with ones given in other literature [11]. With respect to the incompressibility constraint, both programs predicted more than 1% volumetric plastic strain which is much greater than the $10^{-6 \sim -8}$ that is required for the constraint.

2.1.2. Metal extrusion

Th problem analyzed is the steady-state deformation process of plane-strain metal extrusion. Metal is forced into a symmetric die after sliding between smooth rigid plates. The die is approximated with a straight line and produce a 25% thickness reduction over a distance $1.2a$, where a is the half thickness of the original sheet. The material properties are Yong's modulus $E = 6.89 \times 10^4$ (MPa), Poisson's ratio $\nu = 0.3$, strain hardening $H = 1.14 \times 10^3$ (MPa) and initial yield stress $Y = 2.79 \times 10^2$ (MPa).

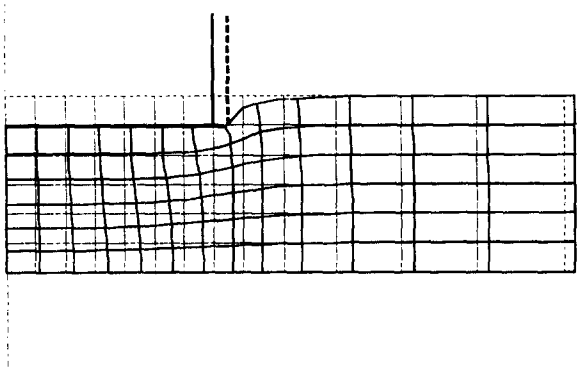

Fig. 1. The meshes before and after deformation.

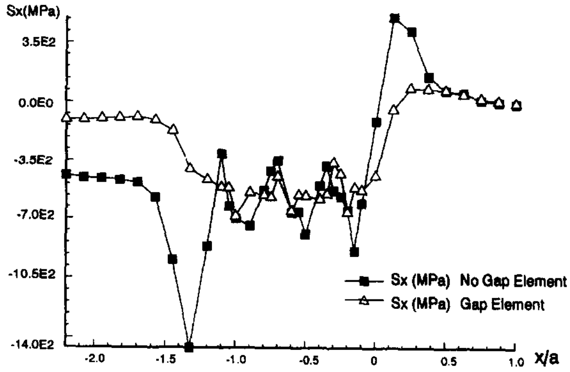

Fig. 2. Distribution of stress S_x on outer surface.

Table 1
Violation of incompressibility constrain in metal extrusion

	$U_{x0} = 0.5$		$U_{x0} = 3.0$	
	Gap element	No gap element	Material nonlinearity	Material & geometric nonlinearity
NISA	16.7%	34.0%	6.0%	24.0%
ANSYS	16.6%	12.6%	5.7%	5.1%

The metal-extrusion problem is modeled by plane strain 8-node isoparametric elements and 2-node gap elements. The distributions of displacement U_x and stress S_x, both of them in the extrusion direction, are examined. The distributions of U_x and S_x from NISA and ANSYS are very similar and the values of U_x are close to each other; the difference is less than 0.4%, but the values of S_x are quite different; the difference is almost 25%. Compared with the results given by other researchers [12], all values are quite different.

For frictionless die–workpiece interface, the results with and without gap elements, in both NISA and ANSYS, are significantly different. As a example, distribution of S_x on the outer surface of the workpiece is shown in Fig. 2. It is not expected that the introduction of node-to-node gap elements should influence the frictionless die case. In addition, the volume incompressibility constraint is also considerably violated as shown in Table 1. It is believed that certain deficiencies in the formulation may cause such problems.

Only the material nonlinearity and combination of material and geometric nonlinearity were used in the analysis. It is found that in NISA, combination of material and geometric nonlinearity makes the result violate the volume incompressibility constraint much more than using only the material nonlinearity option, as shown in Table 1. In ANSYS, on the other hand, the combination seems not to have significant effects and gives almost same results as only considering material nonlinearity.

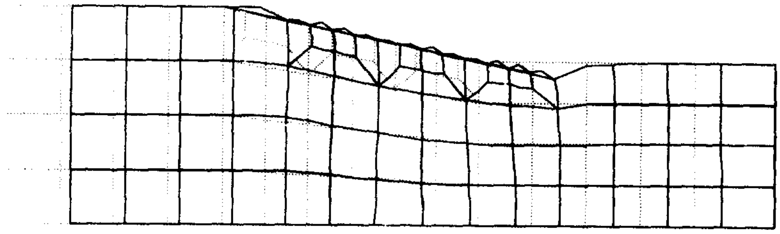

Fig. 3. Not updated boundary condition.

It is also important to report that both programs do not have a capability of automatically updating the boundary conditions. This is an important feature in handling most of metal-forming problems. Obviously, it is not possible to manually update boundary conditions after each incremental step. Because of that, the punch size as increased during the deformation, as shown in Fig. 1. For metal extrusion, the situation is worse. Fig. 3 shows a typical deformed mesh (superimposed on the original one) obtained from such programs. It is obvious that the reduction in thickness depicted by the program will not agree with the specified one and that the predicted deformation behavior is not practical. Another shortcoming is the fluctuations in loads predicted by Lagrangian methods. As an example, in the plane-strain metal extrusion, the extrusion pressure increases steadily up to the state when the billet fills the die. As the billet exits the die, the interface nodes that are constrainted to move on the die are released. This causes a drop in the strain energy of the billt which in turn causes a drop in the extrusion pressure. The studies performed by Lee [12] and Aravas [13] confirm the existence of these fluctuations.

2.2. Limitation of Lagrangian-type formulation

Most of the available commercial FE programs are based on Lagrangian type of formulation. Such formulations are efficient and quite suitable for handling nonlinear problems in which small strains prevail, boundary condition nonlinearities do not change with the course of deformation and where mesh distortion is not a critical factor in the analysis. Unfortunately, all of the above points are essential to any accurate metal-forming analysis.

The above examples reveal some of the main shortcomings in the application of Lagrangian type of formulations to metal-forming problems. It is shown that large plastic strains with friction and contact boundary conditions are not suitable to be analyzed by such formulations. Volume incompressibility constraint is usually violated and load fluctuations are evident with large strains, the problem of mesh distortions and element entanglement pose a serious drawback on the use of such formulation. It is not always feasible to update the mesh manually, and even if it is possible to do so, it will involve major interaction and time involvement in the user of such schemes.

On the other hand, some automatic mesh rezoning methods are developed for Lagrangian finite element analyses of metal-forming problems [14–16]. However, these methods are not so robust and efficient to remedy the mesh distortions. They are based on rediscretizing the deformed configuration after certain special deformation. Obviously, the determination of the special deformation is very important and can only be based on user's experience. The remesh has to be applied on the whole domain, which is actually not necessary. Furthermore, these methods cannot change the mesh topologies, so they may not improve the overall accuracy of the finite element calculations. The locations of nodes in the new mesh are calculated by algebraic interpolation method or by solving differential equations. The algebraic interpolation will introduce curve-fitting error, especially when the shape of boundary is complicated. Solving differential equation can generate higher quality finite element meshes, but it is time consuming to solve such kind of equations, and more important, some times, this may still produce poorly shaped meshes near boundaries.

The problems of contact and friction boundary conditions pose another serious concern in the solution of metal-forming problems. Node to node and node to surface contact elements have been recently introduced in commercial FE packages [9, 10]. Although these elements are quite useful in the analysis of such problems, many problems arise in convergence, specification of stiffness parameters for such elements, as well as accuracy of element formulation and assumptions.

Contact condition, which is defined from the geometrical compatibility on the contact surface or the impenetrability condition, has been introduced by the direct method, elements with special mechanical properties, the Lagrangia multiplier method and the penalty function method [17]. Since the contact area is a prior unknown, the boundary conditions of the contact problems are determined as part of the solution.

Although the contact surfaces obtained by the Lagrangian multiplier method satisfy the contact conditon in the integral sense, additional unknowns are required and the total number of unknowns in the system equations increases. In addition, the associated tangent matrix may be indefinite and has zero diagonal entries that pose some difficulties in the solution steps [18]. For the penalty function method, the solution results satisfy the contact conditions only approximately. The accuracy of the approximate solution depends strongly on the penalty parameter, so the correcte choice for this parameters is the essence of the algorithm. When the penalty parameter is chosen to be too large, it leads to numerical problems in the form of loss of accuracy in the solution. On the other hand, a too small choice, results in unacceptable penetration of one body into the other.

The node to node or node to surface, gap or contact elements have been recently introduced in commercial FE package [9, 10]. Such an approach is quite natural to the formulation of the FE method because the overall stiffness assembly process is unchanged; the contact elements can be treated as a seperate material property group such as plasticity, creep, etc. In spite of their various positives, contact or gap element exhibit several important shortcomings [19]. They undergo anomalous response behavior when employed in situations where large deformation kinematics are needed to generate closure, that is reflected in improper stiffness characterizations, i.e., poorly conditioned Jacobians. They are also awkward to apply in situations requiring friction effects and they require significant amount of equilibrium iteractions. Furthermore, parameters of the contact element have to be chosen by the user, and the convergence speed and the accuracy of the solution are quite dependent on the choice of these parameters. Sometimes, especially when friction effect is incorporated in, convergence may be difficult to achieve.

3. Eulerian formulation method

The main difference between the Eulerian formulation and other formulations is that the deformation of the material moving through a fixed region in space is determined as a function of the current position and the time t instead of determining the deformation of the material element by following its motion in space [6]. Because the independent variables in the Eulerian description are current position $\boldsymbol{x}$ of the body-point $\boldsymbol{X}$ and time t, for the material motion, $\boldsymbol{x}$ itself becomes dependent on the time t, which complicates material time derivatives and other relations when approached strictly by Eulerian, or unique spatial formulation. Although many authors have treated the Lagrangian and the updated Lagrangian formulation, there has been little effort concerning the development of a consistent Eulerian formulation. In the finite element analysis based on the Eulerian formulation, the fact that $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ is usually overlooked and gross approximations are introduced to overcome the difficulty of calculating of material time derivative [7]. Some developments, e.g. Gadala et al. [6], give the proper material time derivative in spatial description but original geometric parameters are included in the final equilibrium equation. Therefore, such approach may not be strictly considered as purely Eulerian formulation.

In the Eulerian method, the finite element mesh is fixed in space and does not move with material points. If displacements are taken as primary unknowns, it is necessary to determine the state of the material-associated properties (e.g. strain, stress) of the body points momentarily occupying the integration points in the beginning of each step. However, the method of updating the material-associated properties is not well developed and further investigations need to be considered. Abo-Elkhier [20] utilizes an imaginary finite element mesh to update the datum, but no discussion nor proper assessment of the accuracy of the method is given. Derbalian [8] discussed in detail the updating or interpolation of stress in Eulerian scheme that was proposed to simulate the steady static metal extrusion. Derbalian's method is based on an updated Lagrangian formulation, although coupled with spatially fixed finite element mesh. Therefore, strictly speaking, the method is not Eulerian one. A novel approach is presented, however, that discusses interpolating the known fixed mesh stresses (for example, integration stations) at time t to some particular points which are brought to the fixed mesh points at time $t + \Delta t$. The conjugate method proposed by Oden et al. [21] was used in [8] to interpolate stress. By constructing a set of shape functions conjugate (biorthogonal) to those used to represent the velocity field in the finite element approximation, a consistent approximation for the stress fieldd can be obtained which is continuous across interelement boundaries and involves less mean error. However, this method introduces a large system of equations which must be solved many times during each incremental step, which makes the calculation much more laborious. Also, the conjugate method is originally proposed for improving the accuracy of incremental stress fields, rather than the total stress.

Eulerian formulation, where the domain of interest is a fixed region in space and has to be known a prior, is suitable for the study of flow problems. Therefore, the analysis of steady-state metal-forming processes may be achieved by Eulerian formulation. Because it introduces other difficulties like appropriate representation of free body, it is less suited for domains whose boundaries or interfaces move substantially and it is not easy to simulate non-steady static or dynamic behavior within the frame work of this formulation.

4. Arbitrary Lagrangian–Eulerian method

From the above discussion, it is believed that a new formulation type that may combine the advantages of the Lagrangian and Eulerian ones is essential for the accurate simulation of metal forming processes. This new type of formulation is called arbitrary Lagrangian–Eulerian (ALE). Generally, ALE is a finite element formulation in which the reference system (computational mesh) is not a priori fixed in space or attached to the body, but an arbitrary computational reference system. In other words, in an ALE formulation, the finite element mesh need not adhere to the material or be fixed space but may be moved arbitrarily relative to the material. A proper ALE formulation should reduce to Lagrangian formulation if we choose to use the same motion for the computational and materials meshes. On the other hand, if we choose to fix the computational mesh, an ALE formulation should reduce to Eulerian formulation.

Combining the merits of both Lagrangian and Eulerian formulation, ALE is easy to handle mesh distortion and entanglement. More importantly, if the nodes on the current interface of tool–workpiece are specified as Eulerian points, it may eliminate load fluctuations, may describe precisely any contact boundary conditions and make boundary condition updating no longer necessary after each incremental step. Thus, it is evident that ALE method is ideally more suited for solving a variety of complex problems in solid mechanics, especially those dealing with finite strain deformation and fracture.

The concept of ALE was first proposed in the mid-sixties [22] under the name of "coupled Eulerian–Langrangian method"; a finite difference scheme for two-dimensional hydrodynamics problems with moving fluid boundaries. Later, the ALE method was introduced into the finite element method [23] in response to the need for nonlinear simulation techniques for nuclear safety analysis. Since then, ALE method has mainly been used in fluid and linear-path independent solids, where stress states are solely determined by the instantaneous displacement or velocity fields [24, 25]. Only recently the ALE method has been applied to finite strain deformation problems in solid mechanics.

Huetink [26, 27] introduced a finite element method to simulate metal-forming process uner the name of "combined Eulerian–Lagrangian formulation". In his formulation, the material rate of change of the equation of virtual power, with changing integration area, was calculated to derive the final equation. However, in the final discretized equilibrium equations, only material velocities were included. As in the updated Lagrangian method, the material velocities have to be calculated out from the final equations, then the new finite element mesh can be updated by arbitrarily moving the old finite element mesh in the material domain. On the boundaries, however, the mesh can only more in the tangential direction of the material domain. Following this is the updating of the material-associated properties. In the process, material motion and mesh motion can not happen at the same time and the discretized equilibrium equation is not a simultaneous equation including mesh velocities and material velocities as unknowns. In this regard, Huetink formulation is simply an ULF coupled with remeshing after each incremental step.

An improved version of ALE formulation was presented by Haber [28]. In Haber's work two displacement variable were considered to be primary unknowns; the Eulerian and Lagrangian type of displacements. This allowed some distinction between the material and the computational meshes. The new technique was applied to large-deformation frictional contact and fracture mechanics. It was demonstrated that the mixed Eulerian–Lagrangian description (i.e. ALE) can be used to vary the crack length in a continuous way and may easily handle contact boundary condition. The paper does not, however, clearly indicate which displacement stands for mesh motion and there is no discussion about how to move the computational finite element mesh. Furthermore, one of the most demanding aspects in ALE, e.g., the updating of material-associated properties was not addressed. It is also assumed that the load is independent of both Lagrangian and Eulerian displacement, so the final formulation may not be applicable to the deformation-dependent loads. Because of the assumption made in the derivation, the formulation is limited to linear elastic materials.

Similar to the above work, Yamada introduced two types of displacements called the material and spatial incremental displacements and derived an ALE formulation for plane deformation of hyperelastic material [29]. For such kind of materials, the stress can be uniquely determined throug the stain energy density from the deformation gradient tensor independent of strain histories. Thus, no state quantity of the particles needs to be introduced except the deformation gradient tensor. The application of formulation to only path-independent hyperelastic material models eliminated the need for rigorously addressing the problem of calculating and updating material-associated properties, and makes the formulation not applicable to the general solid mechanics deformation problems.

An ALE formulation specifically derived for solid mechanics was described by Schreurs [30, 31]. In [31], it is indicated clearly that in ALE, the difference of the CRS (computational reference system) derivative and MRS (Material reference system) derivative of a physical quantity has to be distinguished and a systematic basic description is introduced. Starting from the equilibrium equation in material domain, the weak form was set up by the principle of weighted residuals. The final discretized equilibrium equations were obtained by transforming integration over material domain to a reference domain. However, the physical meaning of the reference domain, i.e. the integral volume is not indicated. Neither the relationship of the reference domain with the physical domain; only fundamental ideas were discussed and no detailed or complete formulation was given.

Because the number of unknowns surpasses the number of equations, specifying the mesh motion is necessary. A mesh moving method was discussed. The grid point was considered as a material point of a linear isotropic fictitious body. Assuming one shape of a stress-free element being optimal, a linear algebraic equation was set up to have the deformed mesh element recover to the optimal shape. By solving the equations, the grid nodal displacements were determined. This method is time consuming and looks hard to guarantee the optimal mesh because the real material model is elastic-plastic linear strain hardening material.

Hughes [32] elaborated on some basic concepts related to ALE. An important one is the relationship between the material time derivative $f^{\cdot}$ and referential time derivative $f^{\wedge}$ of a physical quantity f. These should satisfy the following relation:

$$f^{\cdot} = f^{\wedge} + \frac{\partial f}{\partial x_i}(v_i - v_i^{\wedge}) \tag{1}$$

where x_i is material coordinate, v_i and $v_i^{\wedge}$ are material and grid point velocities individually.

Liu [33] applied Eq. (1) to the momentum equation and obtained an equation with respect to arbitrary reference volume. Petrov–Galerkin formulation was then utilized to set up the final discretized equilibrium equation. A stress updating procedure is developed to calculated the stress values at the quadrature and nodal points. The constitutive law is reformulated by introducing a stress-velocity product; the Petrov–Galerkin finite element method is applied to the reformulated equations to set up the equivalent weak form equations. These equations are solved simultaneously with the final equilibrium equations to get the final solution. With this scheme, some wave propagation problems and elastic-plastic dynamic deformation processes are simulated. In the same paper, Liu also transformed the integration extending over the material domain to referential domain and derived the referential time derivative of internal virtual work. In the derivation, he introduced a referential time rate of Cauchy stress which lacks grounds of physical meaning. In a later development [34], Liu considered the frictional interface of plane deformation problems and derived equilibrium equations for such case. The Laplace differential equation and fourth-order differential equation were proposed as mesh generator to manage mesh movement. The application of proposed ALE algorithm to metal rolling simulation is presented. It is shown that for small rollers, both Lagrangian and ALE meshes are feasible and the agreement between the two is quite good. For realistic roller size, a Lagrangian mesh failed to complete the simulation. However, the ALE mesh performs quite well.

Extended from fluid mechanics, an ALE formulation for solid mechanics was reported by Ghosh et al. [35–37]. Different from the methods discussed previously, the Reynolds transport theorem widely used in fluid mechanics was applied to an arbitrary moving control volume to obtain the field equations of mechanics with respect to an arbitrarily moving grid point. The weak forms of the differential equations were obtained by using appropriate weighting functions and integrating over the current grid volume.

In Ghosh's work, however, it was assumed that the motion of the material with respect to the grid is quasistatic, e.g. $(\partial v_i/\partial t)_\chi = 0$, which is the time rate of v_i with respect to grid point. In the opinion of the authors, this assumption is not warranted. In quasistatic problems, it is the material point acceleration that may be neglected instead of the motion of the material with respect to the grid points, because the $(\partial v_i/\partial t)_\chi$ contributes only partially to the material point acceleration. If assuming $(\partial v_i/\partial t)_\chi = 0$, the material acceleration can not be zero, so this assumption may not be correct from the kinematic point of view. On the other hand, the exclusive characteristics of the ALE that the grid points can move arbitrarily, is restricted so much by the assumption, so the assumption is not suitable for an ALE formulation. Using implicit time integration scheme, Ghosh integrated the weak from equation over the current control volume. This makes the calculation laborious because, unlike transient field problems, the integration volume here is changing with time. In [36] Ghosh described a way to update variables to nodal points of the arbitrary motion. Pseudo-material elements were constructed first and material-associated variables were evaluated by interpolation. Again because of the application of implicit time integration scheme, specifically the middle strategy, the updating of material-associated properties become much more complicated and tedious than explicit method. Local algebraic or elliptic mesh generator was implemented to perform mesh management in highly local deformation areas [37]. Although Ghosh mentioned that the specific cases of ALE formulation should be either an updated Lagrangian formulation or an Eulerian formulation, no verification is presented in the papers showing that the formulation may reduce to an updated Lagrangian formulation or Eulerian formulation.

A particular ALE implementation, ALE3D in metal-forming simulations, is recently introduced by Cough et al. [38]. The basic computational cycle consists of a Lagrangian step followed by an advection step. At the end of Lagrangian phase of the cycle the velocities and nodal positions are updated. At this point, the user has several options. If the user opts to run in a pure Lagrangian mode, no further action is taken and the code proceeds to the next time step. If a pure Eulerian calculation is desired, the nodes are placed back in their original positions. The user has options available to tailor the evaluation of the mesh in order to maximize either efficiency or accuracy. The mesh updating scheme implemented in ALE3D is a finite-element-based equipotential method. For the constitutive law, the Jauman rate derivative is used for the stress tensor and Von Mises yield condition is applied. In the paper [38], only program features are described and no formulation or rigorous algorithm is given.

An overall description of ALE is presented in [39] by Huerta and Casadei. It is clearly indicated that ALE is now fairly well established in the fluid mechanics field. Although important lines of research are still open, the most important challenge for the ALE techniques lies in its extension to solid and continuum mechanics in general, and, in particular, to non-linear solid mechanics where path-dependent material behavior is fairly common. It is pointed out that the best choice for the mesh motion or velocities and a low cost algorithm for updating the material-related properties constitute the major problems. However, no particular schemes are given or presented. Some primary governing issue, e.g., the conservation laws, constitutive equations and boundary conditions, are presented, but no complete formulation is given. The applications considered in the paper are some elastioplastic problems, with concentration on fast-transient solid dynamic problems and on showing the effectiveness of ALE to impact problems when the explicit integration scheme is used.

Although several forms of ALE schemes have been discussed in the literature, it is noted, however, that these formulations always concentrate on certain aspects of the problem which makes the outcome formulation incomplete and sutable only for specific application. Crucial points in the formulation, e.g., the final expression of equilibrium equations in ALE frame, the relation between computational and material points, evaluation of material related data, still need further investigation.

5. Formulation of ALE

Assuming a material reference system (MRS) point $P(x_1, x_2, x_3)$ at time t has velocity v_i $(i = 1, 2, 3)$, and the corresponding computational reference system (CRS) point $P(\chi_1, \chi_2, \chi_3)$ at time t has velocity $v_i^\wedge$ $(i = 1, 2, 3)$, as shown in Fig. 4; then for any quantity f, the CRS time derivative $f^\wedge$ (CRS held constant) and MRS time derivative $f^\cdot$ (MRS held constant) have the relationship expressed in Eq. (1) above [32].

The principle of virtual work at time t is

$$\int_V \sigma_{ij}\delta e_{ij}\,\mathrm{d}V = \int_V f_i^{\mathrm{B}}\delta u_i\,\mathrm{d}V + \int_S f_i^{\mathrm{S}}\delta u_i\,\mathrm{d}S, \tag{2}$$

where

$$\delta e_{ij} = \delta\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right),$$

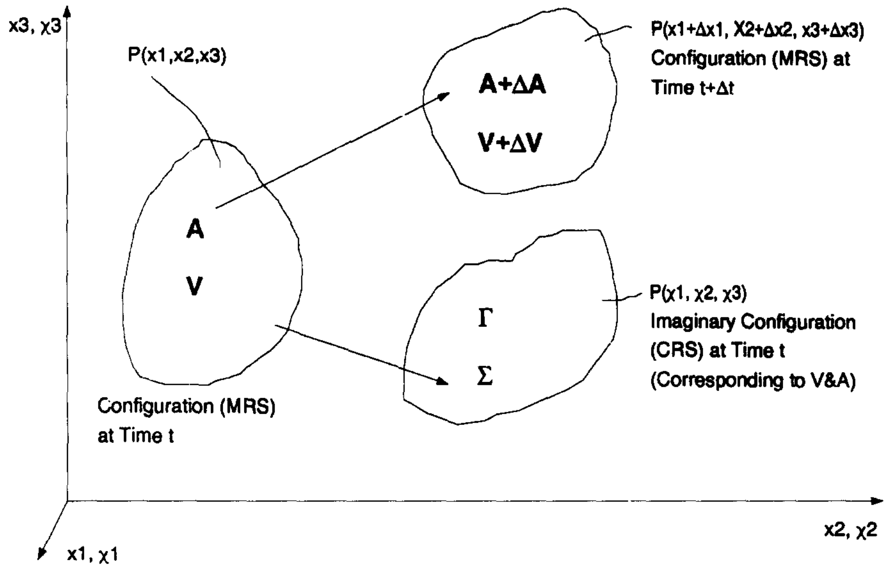

Fig. 4. MRS and CRS systems.

e_{ij} is the deformation tensor, f_i^{B} is components of body forces, f_i^S the components of the externally applied surface tractions, V the volume of deformed body, S the surface of deformed body, on which traction are prescribed, and σ_{ij} the component of Cauchy stress.

Utilizing the following relations

$$\int_V \sigma_{ij}\delta e_{ij}\,\mathrm{d}V = \int_V \sigma_{ij}\left(\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)\right)\mathrm{d}V = \int_V \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}\,\mathrm{d}V,$$

$$f_i^S = \sigma_{ij} n_j,$$

$$n_j\,\mathrm{d}S = \mathrm{d}A_j,$$

where $\mathrm{d}A_j$ is the projection of $\mathrm{d}S$ on coordinat plane having x_j as normal direction, then Eq. (2) may be written in the following form:

$$\int_V \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}\,\mathrm{d}V = \int_V f_i^{\mathrm{B}}\delta u_i\,\mathrm{d}V + \int_S \sigma_{ij} n_j \delta u_i\,\mathrm{d}S = \int_V f_i^{\mathrm{B}}\delta u_i\,\mathrm{d}V + \int_{A_j} \sigma_{ij}\delta u_i\,\mathrm{d}A_j. \tag{3}$$

At this point of the formulation, we require to ensure a one-to-one mapping between the MRS configuration at time $t(x_i)$ and imaginary CRS configuration at time $t(\chi_i)$, i.e.,

$$\chi_i = \chi_i(x_1, x_2, x_3) \text{ and } x_i = x_i(\chi_1, \chi_2, \chi_3) \qquad (i = 1, 2, 3)$$

which implies that the boundary of the CRS must always coincides with the boundary of the material body (MRS) and the Jacobian:

$$J = \det\left(\frac{\partial \chi_i}{\partial \chi_j}\right) \quad (i, j = 1, 2, 3)$$

is not vanishing. By the one-to-one mapping, MRS volume V, surface A_j and surface increment $\mathrm{d}A_j$ are transformed into Σ, B_j and $\mathrm{d}B_j$, respectively. Correspondingly, the integration of Eq. (3) over the MRS configuration at time t is transformed to the imaginary CRS configuration, so

$$\int_{\Sigma} \frac{\partial \delta u_i}{\partial \chi_k} \frac{\partial \chi_k}{\partial x_j} \sigma_{ij} J \,\mathrm{d}\Sigma = \int_{\Sigma} f_i^{\mathrm{B}} \delta u_i J \,\mathrm{d}\Sigma + \int_{B_i} \sigma_{ij} \delta u_i \bar{J} \,\mathrm{d}B_j, \tag{4}$$

where

$$\mathrm{d}V = J\,\mathrm{d}\Sigma$$

$$\mathrm{d}A_j = \bar{J}\,\mathrm{d}B_j \quad \text{and} \quad \bar{J} = \det \begin{pmatrix} \dfrac{\partial x_i}{\partial \chi_i} & \dfrac{\partial x_i}{\partial \chi_k} \\ \dfrac{\partial x_k}{\partial \chi_i} & \dfrac{\partial x_k}{\partial \chi_k} \end{pmatrix}_{x_j = \chi_j = 0}.$$

Consider the rate change of Eq. (4) with CRS configuration held constant, and denote the rate by superscript cap "^". The left-hand side will be

$$\mathrm{LHS} = \int_{\Sigma} \frac{\partial \delta u_i}{\partial \chi_k} \left[\left(\frac{\partial \chi_k}{\partial x_j} \right)^{\wedge} \sigma_{ij} J + \left(\frac{\partial \chi_k}{\partial x_j} \right) \sigma_{ij}^{\wedge} J + \left(\frac{\partial \chi_k}{\partial x_j} \right) \sigma_{ij} J^{\wedge} \right] \mathrm{d}\Sigma \tag{5}$$

since

$$\left(\frac{\partial \chi_k}{\partial x_j} \right)^{\wedge} \left(\frac{\partial x_l}{\partial \chi_k} \right) = -\left(\frac{\partial \chi_m}{\partial x_j} \right) \left(\frac{\partial x_l}{\partial \chi_m} \right)^{\wedge} = -\left(\frac{\partial \chi_m}{\partial x_j} \right) \left(\frac{\partial x_l^{\wedge}}{\partial \chi_m} \right)$$

from Eq. (1)

$$x_l^{\wedge} = x_i + (v_l^{\wedge} - v_l) = v_l^{\wedge}$$

so that

$$\left(\frac{\partial \chi_k}{\partial x_j} \right)^{\wedge} = -\left(\frac{\partial \chi_m}{\partial x_j} \right) \left(\frac{\partial v_l^{\wedge}}{\partial \chi_m} \right) \left(\frac{\partial \chi_k}{\partial x_l} \right) = -\left(\frac{\partial \chi_k}{\partial x_l} \right) \left(\frac{\partial v_l^{\wedge}}{\partial x_j} \right), \tag{6}$$

we know from [40]:

$$J^{\wedge} = = J \frac{\partial v_p^{\wedge}}{\partial x_p} \tag{7}$$

and from Eq. (1):

$$\sigma_{ij}^{\wedge} = \dot{\sigma_{ij}} + (v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p}, \tag{8}$$

substitute (6), (7) and (8) into (5),

$$\mathrm{LHS} = \int_{\Sigma} \frac{\partial \delta u_i}{\partial \chi_k} \frac{\partial \chi_k}{\partial x_j} \left[\left(-\frac{\partial v_j^{\wedge}}{\partial x_l} \right) \sigma_{il} + \dot{\sigma_{ij}} + (v_i^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij} \frac{\partial v_p^{\wedge}}{\partial x_p} \right] J \,\mathrm{d}\Sigma$$

transform back to the MRS configuration at time t:

$$\mathrm{LHS} = \int_V \frac{\partial \delta u_i}{\partial x_j}\left[\left(-\frac{\partial v_j^{\wedge}}{\partial x_l}\right)\sigma_{il} + \dot{\sigma_{ij}} + (v_i^{\wedge} - v_p)\frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij}\frac{\partial v_p^{\wedge}}{\partial x_p}\right]\mathrm{d}V, \tag{9}$$

using Jaumann stress rate tensor σ_{ij}^*, which is frame indifferent,

$$\sigma_{ij}^* = \dot{\sigma_{ij}} - \sigma_{ip}\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x_p} - \frac{\partial v_p}{\partial x_j}\right)\right] - \sigma_{jp}\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_p} - \frac{\partial v_p}{\partial x_i}\right)\right]$$

therefore,

$$\dot{\sigma_{ij}} = \sigma_{ij}^* + \sigma_{ip}\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x_p} - \frac{\partial v_p}{\partial x_j}\right)\right] + \sigma_{jp}\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_p} - \frac{\partial v_p}{\partial x_i}\right)\right],$$

substituting into Eq. (9):

$$\begin{aligned}\mathrm{LHS} = &\int_V \frac{\partial \delta u_i}{\partial x_j}\left[\left(-\frac{\partial v_j^{\wedge}}{\partial x_p}\right)\sigma_{ip} + \sigma_{ij}^* + \frac{\sigma_{ip}}{2}\left(\frac{\partial v_j}{\partial x_p} - \frac{\partial v_p}{\partial x_j}\right)\right]\mathrm{d}V \\ &+ \int_V \frac{\partial \delta u_i}{\partial x_j}\left[\frac{\sigma_{jp}}{2}\left(\frac{\partial v_i}{\partial x_p} - \frac{\partial v_p}{\partial x_i}\right) + (v_p^{\wedge} - v_p)\frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij}\frac{\partial v_p^{\wedge}}{\partial x_p}\right]\mathrm{d}V.\end{aligned} \tag{10}$$

Similarly, the rate change of the right-hand side of Eq. (4) will be

$$\mathrm{RHS} = \int_\Sigma \delta u_i (f_i^{\mathrm{B}\wedge} J + f_i^{\mathrm{B}} J^{\wedge})\,\mathrm{d}\Sigma + \int_{B_j} \delta u_i (\sigma_{ij}^{\wedge} \bar{J} + \sigma_{ij} \bar{J}^{\wedge})\,\mathrm{d}B_j \tag{11}$$

from Eq. (1):

$$f_i^{\mathrm{B}\wedge} = f_i^{\mathrm{B}\cdot} + (v_p^{\wedge} - v_p)\frac{\partial f_i^{\mathrm{B}}}{\partial x_p} \tag{12}$$

and similar to (7):

$$\bar{J}^{\wedge} = \bar{J}\left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p}\right) \tag{13}$$

substuting (12), (13) and (7), (8) in (11), then

$$\begin{aligned}\mathrm{RHS} = &\int_\Sigma \delta u_i \left(f_i^{\mathrm{B}\cdot} + (v_p^{\wedge} - v_p)\frac{\partial f_i^{\mathrm{B}}}{\partial x_p} + f_i^{\mathrm{B}}\frac{\partial v_p^{\wedge}}{\partial x_p}\right) J\,\mathrm{d}\Sigma \\ &+ \int_{B_j} \delta u_i \left(\dot{\sigma_{ij}} + (v_p^{\wedge} - v_p)\frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij}\left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p}\right)\right)\bar{J}\,\mathrm{d}B_j\end{aligned}$$

transforming back to the MRS configuration at time t,

$$\begin{aligned}\mathrm{RHS} = &\int_V \delta u_i \left(f_i^{\mathrm{B}\cdot} + (v_p^{\wedge} - v_p)\frac{\partial f_i^{\mathrm{B}}}{\partial x_p} + f_i^{\mathrm{B}}\frac{\partial v_p^{\wedge}}{\partial x_p}\right)\mathrm{d}V \\ &+ \int_{A_j} \delta u_i \left(\dot{\sigma_{ij}} + (v_p^{\wedge} - v_p)\frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij}\left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p}\right)\right)\mathrm{d}A_j\end{aligned}$$

where $\mathrm{d}A_j = n_j \mathrm{d}S$,

So

$$\mathrm{RHS} = \int_V \delta u_i \left(f_i^{\mathrm{B}\cdot} + (v_p^{\wedge} - v_p) \frac{\partial f_i^{\mathrm{B}}}{\partial x_p} + f_i^{\mathrm{B}} \frac{\partial v_p^{\wedge}}{\partial x_p} \right) \mathrm{d}V$$
$$+ \int_S \delta u_i \left(\sigma_{ij}^{\cdot} n_j + (v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} n_j + \sigma_{ij} \left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p} \right) n_j \right) \mathrm{d}S$$

but $\sigma_{ij}^{\cdot} n_j = f_i^{\mathrm{S}\cdot}$ is the traction (or load) rate change with the displacements and gradients held fixed (corresponding to the $\Delta_{\mathrm{load}} T_0$ in Ref. [3]),

$$\int_S \delta u_i \left(\sigma_{ij}^{\cdot} n_j + (v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} n_j + \sigma_{ij} \left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p} \right) n_j \right) \mathrm{d}S$$
$$= \int_S \delta u_i \left(f_i^{\mathrm{S}\cdot} + (v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} n_j + \sigma_{ij} \left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p} \right) n_j \right) \mathrm{d}S$$

and,

$$\mathrm{RHS} = \int_V \delta u_i \left(f_{ij}^{\mathrm{B}\cdot} + (v_p^{\wedge} - v_p) \frac{\partial f_i^{\mathrm{B}}}{\partial x_p} + f_i^{\mathrm{B}} \frac{\partial v_p^{\wedge}}{\partial x_p} \right) \mathrm{d}V$$
$$+ \int_S \delta u_i f_i^{\mathrm{S}\cdot} \, \mathrm{d}S + \int_S \delta u_i \left((v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij} \left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p} \right) \right) n_j \, \mathrm{d}S. \tag{14}$$

Finally, using the LHS (Eq. (9)) and RHS (Eq. (14)), and arranging terms,

$$\int_V \frac{\partial \delta u_i}{\partial x_j} \left[\left(-\frac{\partial v_p^{\wedge}}{\partial x_p} \right) \sigma_{ip} + \sigma_{ij}^{*} + \frac{\sigma_{ip}}{2} \left(\frac{\partial v_j}{\partial x_p} - \frac{\partial v_p}{\partial x_j} \right) + \frac{\sigma_{jp}}{2} \left(\frac{\partial v_i}{\partial x_p} - \frac{\partial v_p}{\partial x_i} \right) \right] \mathrm{d}V$$
$$+ \int_V \frac{\partial \delta u_i}{\partial x_j} \left[(v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij} \frac{\partial v_p^{\wedge}}{\partial x_p} \right] \mathrm{d}V - \int_V \delta u_i \left((v_p^{\wedge} - v_p) \frac{\partial f_i^{\mathrm{B}}}{\partial x_p} + f_i^{\mathrm{B}} \frac{\partial v_p^{\wedge}}{\partial x_p} \right) \mathrm{d}V \tag{15}$$
$$- \int_S \delta u_i \left((v_p^{\wedge} - v_p) \frac{\partial \sigma_{ij}}{\partial x_p} + \sigma_{ij} \left(\frac{\partial v_p^{\wedge}(x_j = 0)}{\partial x_p} \right) \right) n_j \, \mathrm{d}S = \int_V \delta u_i f_i^{\mathrm{B}\cdot} \, \mathrm{d}V + \int_S \delta u_i f_i^{\mathrm{S}\cdot} \, \mathrm{d}S.$$

The above formulation is a complete and general one in the sense that it may be applied to various types of finite deformation problems with generalized types of loadings and boundary conditions. The formulation is also suitable for implementing various types of rate dependant and rate-independent material constitutive relations. Another unique feature in the formulation is the existence of generalized load correction terms that facilitates handling of deformation-dependent loads, e.g., follower loads.

The proposed formulation differs from similar ones in the literature in the following aspects:

- No specific assumptions were introduced afte the principle of virtual work. Also, velocities, instead of displacements, are used as primary variables. This makes it more straightforward to implement rate-dependent material laws.
- Consistent load correction term to handle deformation-dependent loads.

- The integral volume and surface have clear physical meanings which are the material domain after last incremental step. This is arrived at by choosing the arbitrary reference domain coincident with the material domain after each incremental step.
- The formulation represents a strict ALE method because the mesh motion and deformation can occur independently at the same time, i.e., both material velocity and mesh velocity are included in Eq. (15).
- The formulation reduces to updated Lagrangian formulation when the material velocity is equal to mesh velocity, i.e. when $v_i = v_i^{\wedge}$ anywhere in V and on S Eq. (15) becomes

$$\int_V \frac{\partial \delta u_i}{\partial x_j}\left[\sigma_{ij}^{*} - \frac{\sigma_{ip}}{2}\left(\frac{\partial v_j}{\partial x_p} + \frac{\partial v_p}{\partial x_j}\right) + \frac{\sigma_{jp}}{2}\left(\frac{\partial v_i}{\partial x_p} - \frac{\partial v_p}{\partial x_i}\right) + \sigma_{ij}\frac{\partial v_p}{\partial x_p}\right]\mathrm{d}V$$
$$- \int_V \delta u_i\left(f_i^{\mathrm{B}}\frac{\partial v_p}{\partial x_p}\right)\mathrm{d}V - \int_S \delta u_i\left(\sigma_{ij}\left(\frac{dv_p(x_j=0)}{\partial x_p}\right)\right)n_j\,\mathrm{d}S$$
$$= \int_V \delta u_i f_i^{\mathrm{B}\cdot}\,\mathrm{d}V + \int_S \delta u_i f_i^{\mathrm{S}\cdot}\,\mathrm{d}S \tag{16}$$

which is the same as the updated Lagrangian formulation given by McMeeking [3], except for the two load correction matrices:

$$\int_V \left(f_i^{\mathrm{B}}\frac{\partial v_p}{\partial x_p}\right)\mathrm{d}V \quad \text{and} \quad \int_S\left(\sigma_{ij}\left(\frac{\partial v_p(x_j=0)}{\partial x_p}\right)\right)n_j\,\mathrm{d}S.$$

- On the other hand, when mesh velocity equal to zero i.e., $v_p^{\wedge} = 0$, the formulation reduces to a general Eulerian formulation, as shown in the following equation:

$$\int_V \frac{\partial \delta u_i}{\partial x_j}\left(\dot{\sigma_{ij}} - v_p\frac{\partial \sigma_{ij}}{\partial x_p}\right) = \int_V \delta u_i\left(f_i^{\mathrm{B}\cdot} - v_p\frac{\partial f_i^{\mathrm{B}}}{\partial x_p}\right)\mathrm{d}V + \int_S \delta u_i\left(f_i^{\mathrm{S}\cdot} - v_p n_j\frac{\partial \sigma_{ij}}{\partial x_p}\right)\mathrm{d}S \tag{17}$$

7. Application of arbitrary Lagrangian–Eulerian method

The problem of mesh distortion is easier to handle in ALE method. The ALE method can not only handle the mesh distortion but also improve the calculation accuracy. Because the motion of the mesh is arbitrary, the new mesh is always chosen to give an optimal results. Instead of using gap element or multi-degree constraints, ALE can deal with contact boundary simply.

On the interface of tool and workpiece, the nodes will be kept as Lagrangian points in the direction of tool movement, i.e., nodes moving with tool at same speed, and be kept as Eulerian points in the direction perpendicular to the direction of tool movement, i.e., nodes being fixed in that direction. Therefore, contact boundary condition can be accurately described in all of the deformation process, even the tool with sharp edges and corners. The updating of boundary condition is no longer necessary. The "punch increment", and load fluctuations as mentioned earlier, should not occur. This may greatly increase the efficiency and accuracy of simulation.

8. Conclusions

It is indicated that when the conventional Lagrangian methods are applied to simulate finite strain deformation processes, many problems arise: extensive mesh distortion or entanglement, load fluctuations, incorrect description of contact boundary condition with sharp edge or corners and deficiencies in satisfying the incompressibility constraint. Although the existing rezoning technique overcomes mesh distortion and entanglement, other problems are difficult to be eliminated. ALE method has the potential to eliminate or alleviate the problems above, and provides a promising method in the application to finite strain deformation problems. However, until now, in solid mechanics, ALE method has not been well developed. A general consistent ALE formulation is proposed in this paper. It is applicable to general loading and various material models.

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