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NUMERICAL METHODS FOR SOLUTION OF IMPULSIVE DIFFERENTIAL EQUATIONS AND STABILITY ANALYSIS

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Abstract

The qualitative study of impulsive differential equations began in 1960 with the work of Mil'man and Myshkis [9, 10]. In the recent years, there have been intensive studies on the qualitative theory and behavior of the solutions of impulsive differential equations. Impulsive differential equations provide a natural framework for mathematical

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modeling in many real life problems, such as population dynamics, ecology, optimal control, industrial robotics, radio technology, pharmacokinetics, microorganism reproduction, economics, production theory, and so on. However, many impulsive differential equations cannot be solved analytically or the solution process is more complicated. Therefore, introducing a suitable numerical method is a good attempt. To the best of our knowledge, in the literature, there are limited numbers of studies and research on this domain. Because of its technical nature, the presence of impulses complicates the stability analysis of the system. We study the numerical methods and the stability of the solutions of the impulsive differential equations by introducing several methods.

1. Introduction to the Numerical Solutions of Impulsive Differential Equations

Because of the physical nature and functional purposes, the wide practical applications of impulsive differential equations explain the growing interest in technical problems. Impulsive differential equations have appeared in mathematical models of many processes in biology, medicine and technology. The study of dynamic systems with impulse effects has assumed importance in view of the fact that many evolutionary processes experience an abrupt change of state at certain moments of time.

Let us denote the set of real and integer numbers with $\mathbb{R}$, $\mathbb{Z}$, respectively, and let $X \in \mathbb{R}$ and $T = \{\tau_k \mid k \in \mathbb{Z}\} \subset \mathbb{R}$, where $\tau_k < \tau_{k+1}$ for all $k \in \mathbb{Z}$, $\tau_k \to +\infty$ when $k \to +\infty$ and $\tau_k \to -\infty$ when $k \to -\infty$ and $\tau_k^+ = \tau_k + 0$, $\tau_k^- = t_k - 0$.

Assume that $\Omega \subset \mathbb{R}$ is any real interval and $\mathbf{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ is a vector of the state function, $\mathbf{f}(t, \mathbf{x}) : \Omega \times X \to X$ and

$$\begin{aligned} &\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \quad t \neq \tau_k, \\ &\Delta\mathbf{x}|_{t=\tau_k} = \mathbf{x}(\tau_k^+) - \mathbf{x}(\tau_k^-) = I_k(\mathbf{x}(\tau_k)), \end{aligned} \tag{1}$$

where $I_k : X \to X$ $(k \in \mathbb{Z})$ are continuous operators.

Let $\mathbf{x}(t)$ be a solution of the system (1) which satisfies the initial condition $\mathbf{x}(\tau_0^+) = \mathbf{x}_0$. Then the solution can be represented in the form

$$\mathbf{x}(t) = \begin{cases} \mathbf{x}_0 + \int_{\tau_0}^{t} \mathbf{f}(s, \mathbf{x}(s))ds + \sum\limits_{\tau_0 < \tau_k < t} I_k(\mathbf{x}(\tau_k)), & t \in \Omega^+, \\ \mathbf{x}_0 + \int_{\tau_0}^{t} \mathbf{f}(s, \mathbf{x}(s))ds - \sum\limits_{t < \tau_k \le \tau_0} I_k(\mathbf{x}(\tau_k)), & t \in \Omega^-, \end{cases} \tag{2}$$

where Ω^+ and Ω^- are the maximal intervals on which the solutions can be continued to the right and to the left of the point τ_0, respectively. Let $t_q > \tau_0$, i.e., $t_q \in \Omega^+$ be a fixed value of *t*. Then an approximate solution $[x_1(t_q), x_2(t_q), ..., x_n(t_q)]^T$ of the system (2) can be obtained using the following steps of the algorithm for solving impulsive differential equations [7]:

1. Use the initial value $\mathbf{x} := \mathbf{x}_0$ at the moment $t = \tau_0$ and initialize $k = 0$.

2. Use a numerical method (NM) on the half-segment $(\tau_k, \tau_{k+1}]$, i.e., $\mathbf{x}^{[j+1]} := NM(\mathbf{x}^{[j]})$ and increase the counter $k := k + 1$.

More precisely, we divide $(\tau_k, \tau_{k+1}]$ into *m* disjunctive subsegments:

$$(\tau_k, \tau_{k+1}] = \bigcup_{j=0}^{m-1} (t_k^{[j]}, t_k^{[j+1]}],$$

where $t_k^{[0]} = \tau_k$, $t_k^{[m]} = \tau_{k+1}$, and apply a numerical method with knots $t_k^{[1]}, t_k^{[2]}, ..., t_k^{[m-1]}$.

3. At the jump point $t = \tau_k$, the impulsive operator I_k contributes rapidly to the changes on the state function $\mathbf{x}$ with the amount

$$\mathbf{J}(t_k) := I_k\left(\mathbf{x}(\tau_k) + \sum_{\tau_0 < \tau_m < \tau_k} \mathbf{J}(\tau_m)\right).$$

4. Repeat steps 2 and 3 for $\tau_{k+1} < t_q$.

5. Use the numerical method (NM) on the half-segment $(\tau_k, t_q]$, i.e., $\mathbf{x}^{[j+1]} := NM(\mathbf{x}^{[j]})$.

6. Add the sum of all jumps to the state function $\mathbf{x}$:

$$\mathbf{x} := \mathbf{x} + \sum_{\tau_0 < \tau_k < t_q} \mathbf{J}(\tau_k).$$

A similar procedure can be applied for the case $t_q < \tau_0$, i.e., $t_q \in \Omega^-$. The only condition is that the impulsive operators I_k for $k \leq 0$ are invertible operators, because when the value of t decreases at the moment $t = \tau_k$, the impulsive operator acts as its inverse mapping I_k^{-1}.

2. The θ Method

Consider an impulsive differential equation of the form [6]:

$$\dot{x}(t) = \alpha x, \quad t > 0, \quad t \neq k,$$

$$\Delta x = \beta x, \quad t = k, \quad x(0^+) = x_0, \tag{3}$$

where $\alpha \neq 0$, β, $x_0 \in \mathbb{R}$, $1 + \beta \neq 0$, $k \in \mathbb{N}$. System (3) has a unique solution on the interval $0 < t < \infty$:

$$x(t) = x_0 e^{\alpha t}(1+\beta)^{[t]}, \tag{4}$$

where $[t]$ denotes the greatest integer function of t. The trivial solution $x \equiv 0$ of (3) is asymptotically stable if and only if $|1 + \beta| e^{\alpha} < 1$.

The algorithm of the method:

$$x_{km,l} = x_{km,l-1} + h(\alpha(1-\theta)x_{km,l-1} + \alpha\theta x_{km,l}), \quad l = 1, 2, \ldots, m,$$

$$x_{km,0} = (1+\beta)x_{(k-1)m,m},$$

$$x_{0,0} = x_0, \tag{5}$$

where $x_{km,l}$ $(1 \le l \le m)$ and $x_{km,0}$ are approximations to the values of the solutions $x(t_{km,l})$ and $x(k+0)$, respectively, $t_{km,l} = k + lh$, $0 \le l \le m$, $k, m \in \mathbb{N}$, $h = 1/m$. System (5) can be rearranged as

$$x_{km,l} = \left(1 + \frac{\alpha h}{1 - \theta\alpha h}\right) x_{km,l-1}, \quad l = 1, 2, \ldots, m,$$

$$x_{km,0} = (1+\beta) x_{(k-1)m,m}, \quad x_{0,0} = x_0. \tag{6}$$

The numerical scheme is asymptotically stable at (α, β) if and only if

$$\left| (1+\beta)\left(1 + \frac{\alpha h}{1 - \theta\alpha h}\right)^m \right| < 1 \text{ for all } m > M,$$

where $M = |\alpha|$.

3. Runge-Kutta Method

For the approximate (numerical) solution of the system (3) applying the classical Runge-Kutta method [6], we get

$$x_{km,l} = x_{km,l-1} + \alpha h \sum_{i=1}^{s} b_i Y_i, \quad l = 1, 2, \ldots, m,$$

$$Y_i = x_{km,l-1} + \alpha h \sum_{j=1}^{s} a_{ij} Y_j, \quad i = 1, 2, \ldots, s,$$

$$x_{km,0} = (1+\beta) x_{(k-1)m,m}, \quad x_{0,0} = x_0. \tag{7}$$

The algorithm scheme can be rearranged in a simple form

$$x_{km,l} = (1 + \bar{h} b^T (I - \bar{h}A)^{-1} \mathbf{e}) x_{km,l-1}, \quad l = 1, 2, \ldots, m,$$

$$x_{km,0} = (1+\beta) x_{(k-1)m,m},$$

$$x_{0,0} = x_0, \tag{8}$$

where $A = (a_{ij})_{i,\, j=1}^{s}$, $b = (b_1, b_2, ..., b_s)^T$, $\mathbf{e} = (1, 1, ..., 1)^T$, h is a small step size such that the matrix $I - \alpha hA$ is invertible, and $\overline{h} = \alpha h$. For the pth order Runge-Kutta method, we denote $R(\overline{h}) = 1 + \overline{h} b^T (I - \overline{h}A)^{-1} e$ called the *stability function* of the Runge-Kutta method. Then there exists some constant $C > 0$ such that $| e^{\overline{h}} - R(\overline{h}) | \le Ch^{p+1}$. The scheme (7) is said to be *asymptotically stable* at (α, β) if and only if

$$|(1 + \beta) R^m(\overline{h})| < 1 \text{ for all } m > M.$$

The introduced Runge-Kutta scheme (7) for system (3) is said to be *asymptotically stable* at (α, β) if and only if $\exists M > 0$ such that for all $m > M$, $h = 1/m$, the matrix $I - \alpha hA$ is invertible, and for any $x_{km,\, l}$ $(0 \le l \le m)$ given by (8), we have $\lim_{k\to\infty} X_k = 0$, where $X_k = (x_{km,\,0}, x_{km,\,1}, ..., x_{km,\,m})$.

The well-known classical fourth order Runge-Kutta method is also applicable for the numerical solutions of the system (3):

$$\mathbf{x}^{[j+1]} - \mathbf{x}^{[j]} = \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4),$$

$$K_1 = \mathbf{f}(t^{[j]}, \mathbf{x}^{[j]}),$$

$$K_2 = \mathbf{f}\left(t^{[j]} + \frac{h}{2}, \mathbf{x}^{[j]} + \frac{h}{2}K_1\right),$$

$$K_3 = \mathbf{f}\left(t^{[j]} + \frac{h}{2}, \mathbf{x}^{[j]} + \frac{h}{2}K_2\right),$$

$$K_4 = \mathbf{f}(t^{[j]} + h, \mathbf{x}^{[j]} + hK_3). \tag{9}$$

4. Linear Multistep Methods

For the numerical solutions of the impulsive differential equations, one of the applicable and interesting methods is the linear multistep method [8]. Consider an impulsive equation in the form:

$$\mathbf{x}'(t) = f(t, \mathbf{x}), \quad t > t_0, \quad t \neq \tau_k,$$

$$\Delta \mathbf{x} = I_k(\mathbf{x}), \quad t = \tau_k, \quad k \in \mathbb{N},$$

$$\mathbf{x}(t_0^+) = \mathbf{x}_0, \tag{10}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\lim_{k\to+\infty} \tau_k \to +\infty$.

In particular, consider the following initial value problem:

$$x'(t) = ax, \quad t > 0, \quad t \neq k,$$

$$\Delta x = bx, \quad t = k,$$

$$x(0^+) = x_0, \tag{11}$$

where $a, b, x_0 \in \mathbb{R}$ and $b \neq -1$, $k \in \mathbb{N}$. The state function $x(t)$ is said to be a *solution* of the system (11) if:

1. $\lim_{t\to 0^+} x(t) = x_0$.

2. $x(t)$ is differentiable and $x'(t) = ax(t)$ for $t \in (0, +\infty), t \neq k, k \in \mathbb{N}$.

3. $x(t)$ is left continuous in $(0, +\infty)$ and $x(k^+) = (1 + b)x(k)$, $k \in \mathbb{N}$.

System (11) has a unique solution on the interval $(0, +\infty)$:

$$x(t) = \begin{cases} x_0 e^{at}(1+b)^{[t]}, & t > 0, t \neq k, \\ x_0 e^{ak}(1+b)^{k-1}, & t = k \in \mathbb{N}. \end{cases}$$

The standard form of the linear multistep method is

$$\sum_{i=0}^{k} \alpha_i x_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \tag{12}$$

where α_i and β_i are constants subject to the conditions:

$$\alpha_k = 1, \quad |\alpha_0| + |\beta_0| \neq 0$$

and $f_{n+i} := f(t_{n+i}, x_{n+i})$, $i = 0, 1, 2, ..., k$.

The algorithm for the linear multistep method for the impulsive differential equations is the following:

$$x_{0,0} = x_0,$$

$$x_{sm,l} = -\sum_{i=0}^{k-1} \frac{\alpha_i - ha\beta_i}{\alpha_k - ha\beta_k} x_{sm,l+i-k}, \quad s = 0, 1, 2, \dots \quad \text{and} \quad l = k, \dots, m,$$

$$x_{(s+1)m,0} = (1+b)x_{sm,m},$$

$$x_{(s+1)m,l} = -\sum_{i=0}^{k-l-1} \frac{\alpha_i - ha\beta_i}{\alpha_k - ha\beta_k} x_{sm,m+l+i-k}$$

$$-\sum_{i=k-l}^{k-1} \frac{\alpha_i - ha\beta_i}{\alpha_k - ha\beta_k} x_{(s+1)m,l+i-k}, \tag{13}$$

where $l = 1, 2, \dots, k-1$, and $h = 1/m$ is the step size. $x_{sm,l}$ and $x_{sm,0}$ denote the approximations to $x(t_{sm+l})$ and $x(s^+)$, respectively.

5. Using Spline Functions

One of the important trends in the investigation of impulsive differential equations is related to the periodic solutions of these equations. We will use the ideas of the numerical-analytical method to study the periodic impulsive equation [1]:

$$\frac{dx}{dt} = f(t, x), \quad t \neq \tau_k,$$

$$\Delta x = I_k(x), \quad t = \tau_k, \tag{14}$$

where $t \in \mathbb{R}$, $k \in \mathbb{Z}$, $x \in \Omega \subset \mathbb{R}^n$ and $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$, where $\{\tau_i\}_{i\in\mathbb{N}}$ is a strictly increasing sequence of impulsive moments such that $0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ and $\tau_k \to \infty$ as $k \to \infty$, and let us denote $J = [0, T]$ and $J' = J \backslash \{\tau_i\}_{i\in\mathbb{N}}$.

As usual, in the theory of impulsive differential equations [3, 5], at the points of discontinuity τ_i of the solution $x(t)$, we assume that $x(\tau_i) \equiv x(\tau_i - 0)$. It is clear that, in general, the derivatives $x'(\tau_i)$ do not exist. On the other hand, the limits $x'(\tau_i \mp 0)$ do exist. According to this convention, we assume $x'(\tau_i) \equiv x'(\tau_i - 0)$.

We assume that the system (14) is T-periodic in t, that is, there exists a $p \in \mathbb{N}$ such that

$$\tau_{k+p} = \tau_k + T, \quad f(t+T, x) = f(t, x), \quad I_{k+p}(x) = I_k(x)$$

for $t \in \mathbb{R}$, $k \in \mathbb{Z}$ and $x \in \Omega$. The function $f(t, x)$ is continuous (piecewise continuous with first kind discontinuities at $t = \tau_i$), this means: $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is continuous in the sets $(\tau_{k-1}, \tau_k] \times \Omega$ $(k \in \mathbb{Z})$ and for each $x \in \Omega$ and $k \in \mathbb{Z}$, there exists the finite limit of $f(t, y)$ as $(t, y) \to (\tau_k, x)$, $t > \tau_k$. The functions $I_k : \Omega \to \mathbb{R}^n$ $(k \in \mathbb{Z})$ are continuous in Ω.

If the function $x = \varphi(t)$ is a T-periodic solution of (14) for which $\varphi(0) = \varphi(T) = x_0$ and $\varphi(t) \in \Omega$, $t \in [0, T]$, then $\varphi(t) \equiv x(t, x_0)$ $(t \in [0, T])$ satisfies the relations

$$\varphi(t) = x(t, x_0) = x_0 + \int_0^t f(s, \varphi(s))ds + \sum_{0 \le \tau_k < t} I_k(\varphi(\tau_k)) \quad (t \in [0, T])$$

and

$$\int_0^T f(s, \varphi(s))ds + \sum_{0 \le \tau_k < T} I_k(\varphi(\tau_k)) = 0.$$

Lemma 1. *The following relations are valid for* $t \in [0, T]$:

$$\alpha(t) := \left(1 - \frac{t}{T}\right)\int_0^t ds + \frac{t}{T}\int_t^T ds = 2t\left(1 - \frac{t}{T}\right),$$

$$\left(1 - \frac{t}{T}\right)\int_0^t \alpha(s)ds + \frac{t}{T}\int_t^T \alpha(s)ds = \frac{1}{3}\alpha^2(t) + \frac{1}{6}T\alpha(t) \le \frac{1}{3}T\alpha(t).$$

Lemma 2. *If* $x_0 \in D$ *is fixed for t in the interval* $[0, T]$, *then we may recurrently define the sequence of functions* $\{x_m(t, x_0)\}$:

$$\begin{cases} x_0(t, x_0) \equiv x_0, \\ x_{m+1}(t, x_0) = x_0 + \left(1 + \frac{t}{T}\right)\int_0^t f(s, x_m(s, x_0))ds \\ \qquad - \frac{t}{T}\int_t^T f(s, x_m(s, x_0))ds \\ \qquad + \left(1 - \frac{t}{T}\right)\sum_{0 \le \tau_k < t} I_k(x_m(\tau_k, x_0)) \\ \qquad - \frac{t}{T}\sum_{t \le \tau_k < T} I_k(x_m(\tau_k, x_0)), \quad m = 0, 1, 2, ..., \end{cases}$$

where sequence tends uniformly to the T-periodic solution $\varphi(t)$ *of* (14) *for which* $\varphi(0) = x_0$.

Lemma 3. *The functions* $x_m(t, x_0)$ *satisfy the relations* $x_m(0, x_0) = x_0(T, x_0) = x_0$ *and* $x_m(t, x_0) \in \Omega$ $(t \in [0, T], m = 0, 1, 2, ...)$. *The sequence* $x_m(t, x_0)$ *is uniformly convergent in the interval* $[0, T]$ *and its limit* $x(t, x_0)$ *satisfies the relations* $x(0, x_0) = x(T, x_0) = x_0$. *If* $\varphi(t)$ *is a T-periodic solution of* (14), *then it follows that* $\varphi(t)$ *and* $x(t, x_0)$ *satisfy the equation*

$$x(t) = x_0 + \left(1 - \frac{t}{T}\right)\int_0^t f(s, x(s))ds - \frac{t}{T}\int_t^T f(s, x(s))ds$$

$$+ \left(1 - \frac{t}{T}\right)\sum_{0 \le \tau_k < t} I_k(x(\tau_k)) - \frac{t}{T}\sum_{t \le \tau_k < T} I_k(x(\tau_k))$$

for which $\varphi(0) = \varphi(T) = x_0$ *and* $\varphi(t) \in \Omega$, $\varphi(t) \equiv x(t, x_0)$, $t \in [0, T]$.

Spline functions approximations for solving the periodic impulsive differential equation (14)

Let Λ be the subpartition on each of the consecutive intervals of continuity $(\tau_k, \tau_{k+1}](k = 0, 1, 2, ...)$:

$$\Lambda : 0 < \tau_k = t_0^k < t_1^k < t_2^k < \cdots < t_j < \cdots < t_{N_k}^k = \tau_{k+1}$$

$$(j = 0, 1, 2, ..., N_k),$$

where N_k is a sufficiently large positive integer and the set $\{\tau_k\}$ is a strictly increasing sequence of impulsive moments. Let us denote $t_{j+1} - t_j = h < 1$ $(j = 0, 1, 2, ..., N_k - 1)$. Then we define the spline functions approximating the solution $x(t)$ of (14) by S_Λ on the interval $[\tau_k, \tau_{k+1})(k = 0, 1, 2, ...)$:

$$S_\Lambda \equiv S_k(t) = S_{k-1}(t_k) + \sum_{j=1}^{r} \{f^{(j)}[t_k, S_{k-1}(t_k)]\} \frac{(t - t_k)^j}{j!}$$

$$+ U_{k-1}(\tau_k) + \sum_{j=1}^{r} U_{k-1}^{(j)}(\tau_k, x_0) \frac{(t - \tau_k)^j}{j!}, \tag{15}$$

where $S_{-1}(t) = \varphi_0 = x_0$ and $U_{-1}(\tau_k, x_0) = I_{k-1}(\tau_k)$.

There exists a non-empty compact set $D \subset \Omega$ together with its $\left(\frac{1}{2} ML + MQ\right)$-neighborhood, where

$$L = \sup\{| \tau_{k+1} - \tau_k |, k = 0, 1, 2, ...\},$$

$$Q = \sup\left\{\left(1 - \frac{t_k}{L}\right) i[\tau_k, t_k) + \frac{t_k}{L} i[t_k, \tau_{k+1}) : t_k \in (\tau_k, \tau_{k+1})\right\}.$$

Also, on each of the intervals $[\tau_k, \tau_{k+1})$, the following inequality holds:

$$-1 + \frac{K_1 L}{3} + K_2 Q < K_1 K_2 \left(\frac{LQ}{3} - N\right) < 1,$$

where

$$N = \sup\left\{\left(1 - \frac{t_j}{L}\right) \sum_{0 \le \tau_k < t_j} \alpha(\tau_k) + \left(\frac{t_j}{L}\right) \sum_{t_j \le \tau_k < \tau_{k+1}} \alpha(\tau_k) : t_j \in (\tau_k, \tau_{k+1})\right\},$$

$$\alpha(t_j) = 2t_j\left(1 - \frac{t_j}{L}\right) \quad (j = 0, 1, 2, \dots).$$

The numbers Q and N defined above depend on the points τ_k of the interval $[\tau_k, \tau_{k+1})$ and satisfy the estimates

$$Q \le q, \quad N \le \sum_{0 \le \tau_k < \tau_{k+1}} \alpha(\tau_k) \le \frac{1}{2} qL, \quad \alpha(t_j) \le \frac{1}{2} L.$$

6. Description of the Spline Approximation Scheme

We assume that the functions $f^{(q)}(t, x)$ and $I_k^{(q)}(x)$ satisfy the Lipschitz condition in $x \in \Omega$ uniformly with respect to $t \in \mathbb{R}$, $k \in \mathbb{Z}$ with Lipschitz constants K_1 and $K_2 > 0$:

$$| f^{(q)}(t, x) - f^{(q)}(t, y) | \le K_1 | x - y |,$$

$$| I_k^{(q)}(x) - I_k^{(q)}(y) | \le K_2 | x - y | \text{ and}$$

$$| f(t, x) | \le M, \quad | I_k(x) | \le M \tag{16}$$

for $x, y \in \Omega$, $t \in \mathbb{R}$, $k \in \mathbb{Z}$ and $M > 0$ and for all (t, x), (t, y) in the domain of definition of the function $f^{(q)}$, where $q = 0, 1, 2, \dots$.

Let the exact solution of (14) be written in the following form:

$$x(t) = \sum_{j=0}^{r} \frac{x_k^{(j)}}{j!}(t - t_k)^j + x^{(r+1)}(\xi_k)\frac{(t - t_k)^{r+1}}{(r+1)!}$$

$$+ \sum_{j=0}^{r} \frac{I^{(j)}(x_k)}{j!}(t - \tau_k)^j + I^{(r+1)}(x_k(\eta_k))\frac{(t - \tau_k)^{r+1}}{(r+1)!}, \tag{17}$$

where $\xi_k, \eta_k \in (t_k, \tau_{k+1})$ and $k = 0, 1, 2, \dots$.

Before going to discuss the convergence of the spline approximations, let us introduce the following notations:

$$e(t) = |x(t) - S_{\Lambda}(t)|, \quad e_k = |x_k - S_{\Lambda}(x_k)|,$$

$$f_k^{(j)} = f^{(j)}[t_k, S_{k-1}(t_k)], \quad \widetilde{f}_k^{(j)} = f^{(j)}(t_k, x_k), \quad j = 0, 1, 2, ...,$$

$$I_k^{(j)} = I^{(j)}(x(\tau_k)), U_k^{(j)} = U^{(j)}(x(\tau_k)), \quad j = 0, 1, 2, ...$$

The estimate of $|x(t) - S_k(t)|$ can be obtained by using (15), (17) and the Lipschitz condition (16):

$$e(t) = |x_{k-1} - S_{k-1}(t_k)| + \sum_{j=1}^{r-1} |x_k^{(j)} - f^{(j)}| \frac{(t-t_k)^j}{j!}$$

$$+ |I_{k-1} - U_{k-1}| + \sum_{j=1}^{r-1} |I^{(j)} - U^{(j)}| \frac{(t-\tau_k)^j}{j!}.$$

If we let $B = |x_k^{(j)} - f^{(j)}|$, $\widetilde{B} = |I^{(j)} - U^{(j)}|$, $h = \max\{(t - t_k), (t - \tau_k)\}$ and noting that

$$\sum_{j=1}^{r-1} \frac{h}{j!} < e^h - 1 < e, \quad P = \max\{B, \widetilde{B}\},$$

then we get

$$e(t) \le |x_{k-1} - S_{k-1}(t_k)| + |I_{k-1} - U_{k-1}| + 2eP \le \frac{2^k P^k}{k!} h^k \omega(h),$$

where

$$\omega(h) = \max \sup\{|x_{k-1} - S_{k-1}(t_k)|, |I_{k-1} - U_{k-1}|\} = O(h).$$

Theorem 1. *The implicit nonlinear system* (14) *is uniquely solvable*, *provided that h is sufficiently small.*

Integrating system (14) and using the solution (17) on the interval (τ_k, τ_{k+1}), then the system becomes

$$\bar{\mathbf{x}}_{k+1} = \bar{\mathbf{x}}_k + \int_{t_k}^{t_{k+1}} f\Bigg(s,\ \mathbf{x}_k + \sum_{j=0}^{r} \frac{x_k^{(j)}}{j!}(s - t_k)^j + f^{(r)}(\xi_k)\frac{(s - t_k)^{r+1}}{(r+1)!}$$

$$+ \sum_{j=0}^{r} \frac{I^{(j)}(\mathbf{x}_k)}{j!}(s - \tau_k)^j + I^{(r+1)}(x_k(\eta_k))\frac{(s - \tau_k)^{r+1}}{(r+1)!}\Bigg) ds.$$

Defining the mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{A}(\mathbf{u}) = \bar{\mathbf{x}}_k + \int_{t_k}^{t_{k+1}} f\Bigg(s,\ \mathbf{u} + \sum_{j=0}^{r} \frac{f^{(j-1)}(t_{k+1}, \mathbf{u})}{j!}(s - t_k)^j$$

$$+ f^{(r)}(\xi_k)\frac{(s - t_k)^{r+1}}{(r+1)!}$$

$$+ \sum_{j=0}^{r} \frac{I^{(j)}(\mathbf{u})}{j!}(s - \tau_k)^j + I^{(r+1)}(x_k(\eta_k))\frac{(s - \tau_k)^{r+1}}{(r+1)!}\Bigg) ds,$$

then the system becomes $\mathbf{u} = \mathcal{A}\mathbf{u}$.

Consider $\mathbf{u}, \mathbf{v} \in \mathbb{R}$. Then

$$\| \mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v}) \|$$

$$\leq \int_{t_k}^{t_{k+1}} \Bigg\| f\Bigg(s,\ \mathbf{u} + \sum_{j=0}^{r} \frac{f^{(j-1)}(t_{k+1}, \mathbf{u})}{j!}(s - t_k)^j + f^{(r)}(\xi_k)\frac{(s - t_k)^{r+1}}{(r+1)!}$$

$$+ \sum_{j=0}^{r} \frac{I^{(j)}(\mathbf{u})}{j!}(s - \tau_k)^j + I^{(r+1)}(\mathbf{u}(\eta_k))\frac{(s - \tau_k)^{r+1}}{(r+1)!}\Bigg) ds$$

$$- f\Bigg(s,\ \mathbf{v} + \sum_{j=0}^{r} \frac{f^{(j-1)}(t_{k+1}, \mathbf{v})}{j!}(s - t_k)^j + f^{(r)}(\zeta_k)\frac{(s - t_k)^{r+1}}{(r+1)!}$$

$$+ \sum_{j=0}^{r} \frac{I^{(j)}(\mathbf{v})}{j!}(s - \tau_k)^j + I^{(r+1)}(x_k(\varrho_k))\frac{(s - \tau_k)^{r+1}}{(r+1)!}\Bigg) ds \Bigg\|$$

$$\leq \int_{t_k}^{t_{k+1}} L \Bigg\| \mathbf{u} + \sum_{j=0}^{r} \frac{f^{(j-1)}(t_{k+1}, \mathbf{u})}{j!}(s - t_k)^j + f^{(r)}(\xi_k)\frac{(s - t_k)^{r+1}}{(r+1)!}$$

$$+ \sum_{j=0}^{r} \frac{I^{(j)}(\mathbf{u})}{j!}(s - \tau_k)^j + I^{(r+1)}(\mathbf{u}(\eta_k))\frac{(s - \tau_k)^{r+1}}{(r+1)!}$$

$$- \mathbf{v} - \sum_{j=0}^{r} \frac{f^{(j-1)}(t_{k+1}, \mathbf{v})}{j!}(s - t_k)^j - f^{(r)}(\zeta_k)\frac{(s - t_k)^{r+1}}{(r+1)!}$$

$$- \sum_{j=0}^{r} \frac{I^{(j)}(\mathbf{v})}{j!}(s - \tau_k)^j - I^{(r+1)}(x_k(\varrho_k))\frac{(s - \tau_k)^{r+1}}{(r+1)!} \Bigg\| ds$$

$$\leq Lh\| \mathbf{u} - \mathbf{v} \| + L^2 \sum_{j=0}^{r} \frac{h^j}{j!} \| \mathbf{u} - \mathbf{v} \|.$$

Denote the Lipschitz constant of the mapping $\mathcal{A}$ by

$$L_{\mathcal{A}} = Lh\left(1 + L\sum_{j=0}^{r} \frac{h^j}{j!}\right) < 1$$

for h sufficiently small. This proves that $\mathcal{A}$ is a contraction and following the Picard-Banach fixed point theorem, the equation is uniquely solvable.

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