Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 3653851, 9 pages
https://doi.org/10.1155/2018/3653851

Research Article

Loss, Gain, and Singular Points in Open Quantum Systems

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Received 27 June 2017; Accepted 19 April 2018; Published 2 September 2018

Academic Editor: Soheil Salahshour

Non-Hermitian quantum physics is used successfully for the description of different puzzling experimental results, which are observed in open quantum systems. Mostly, the influence of exceptional points on the dynamical properties of the system is studied. At these points, two complex eigenvalues $\mathcal{E}_i \equiv E_i + i\Gamma_i/2$ of the non-Hermitian Hamiltonian $\mathcal{H}$ coalesce (where E_i is the energy and Γ_i is the inverse lifetime of the state i). We show that also the eigenfunctions Φ_i of the two states play an important role, sometimes even the dominant one. Besides exceptional points, other critical points exist in non-Hermitian quantum physics. At these points $a = a^{\mathrm{cr}}$ in the parameter space, the biorthogonal eigenfunctions of $\mathcal{H}$ become orthogonal. For illustration, we show characteristic numerical results.

1. Introduction

The properties of open quantum systems are described usually by averaging over the spectroscopic features of the individual states of the system. Open quantum systems are characterized by, e.g., their Markovian and non-Markovian behavior or by equilibrium and nonequilibrium properties. The present-day high-resolution experimental studies provide however more information, namely, concrete information on the spectroscopic properties of the individual states. It is a challenge for the theory to describe these results.

In many current theoretical studies, a non-Hermitian formalism is used for the description of individual states of an open quantum system. In these papers, the part of the system, which is localized in a certain finite space region, is considered to be embedded into an infinitely extended environment. Mathematically, the total function space consists of two parts: the localized part of the system and the extended environment. Mostly, the environment is assumed to be the continuum of scattering wavefunctions[1].

This formalism is, from the point of view of mathematics, much more complicated than the familiar Hermitian formalism. For example, the eigenfunctions of a non-Hermitian operator $\mathcal{H}$ are biorthogonal with nontrivial consequences for their normalization. Most important is however the existence of singularities.

In mathematics, the so-called exceptional points (EPs) are known for a long time; see the book [1] on the perturbation theory for linear operators. At the EPs, two eigenvalues of $\mathcal{H}$ coalesce. This coalescence is, on principle, different from the well-known degeneration of two eigenstates of a Hermitian operator H. In the last case, the eigenfunctions of the two states are different and orthogonal while those of two coalescing eigenstates of $\mathcal{H}$ are biorthogonal and differ only by a phase from one another at the EP. The related interesting topological features are studied also experimentally in detail [2–4].

Due to the important role of EPs in non-Hermitian systems, in most theoretical studies of non-Hermitian quantum physics only the eigenvalues of the non-Hermitian Hamiltonian $\mathcal{H}$ are considered in detail. The eigenfunctions are often assumed to be of minor interest. Examples are the method of complex scaling [5] and the review [6] on superradiance. The meaning of the eigenvalues as well as of the eigenfunctions of $\mathcal{H}$ is discussed theoretically in the review [7] and compared to experimental results in the review [8]. The eigenfunctions around EPs play an important role for the explanation of some experimental results that are puzzling from the point of view of Hermitian quantum physics. Examples are the so-called phase lapses [9] observed experimentally in mesoscopic systems and the high efficiency of the photosynthesis [10]. The role of EPs in quantum physics is discussed in many other papers [11–16].

The meaning of the eigenvalues and that of the eigenfunctions of $\mathcal{H}$ are less different from one another than it seems to be. On the one hand, the eigenfunctions of $\mathcal{H}$ play a role also in those cases in which they are usually not considered [7, 8]. Among others, they are responsible for the fact that the influence of the EPs occurs in a larger parameter range around the EP than expected from the behavior of the eigenvalues. Moreover, higher-order EPs are shielded from observation in physical systems [17] because of the mixing of the eigenfunctions of $\mathcal{H}$ via the environment. On the other hand, the eigenvalues of $\mathcal{H}$ determine the position of the EPs and influence therefore also the behavior of the eigenfunctions of $\mathcal{H}$.

Further studies in non-Hermitian quantum physics have shown the following nontrivial results. The coupling between system and environment may occur via an exchange of particles or of information (mostly in terms of excitons). We have the following in detail.

(i) Every state i of the system is related to different decay channels c according to the values of the so-called partial widths of the state i relative to the channels c. The coupling of the whole system to the environment is characterized by the sum of partial widths. This case is realized in many-body systems. In standard Hermitian quantum physics, the exchange of particles between system and environment is described as tunneling of particles.

(ii) Every state of the system is coupled exclusively to only one channel with, respectively, some gain from the environment and some loss to the environment. This case is realized, e.g., in the photosynthesis in which visible light is captured in the light-harvesting complex [10]. The exciton transfer occurs without excitation of any eigenstates of $\mathcal{H}$. It cannot be described in standard theory.

Another problem of non-Hermitian quantum physics is the following. In mathematics, the EPs are defined in relation to one environment (mostly called channel). Physical systems are related however generally to more than one channel. For example, transmission of particles through a system needs at least two channels, entrance and exit channel. In this case, the different channels are parts of the total environment and are orthogonal to one another. Another example is the above-mentioned processes with gain and loss each of which occurs relative to another channel. The role played by EPs in physical systems is therefore not at all clear.

Consideration of the eigenfunctions of the non-Hermitian Hamiltonian $\mathcal{H}$ in detail has shown further nontrivial properties of non-Hermitian quantum physics. For example, the EPs are not the only singular points in non-Hermitian quantum systems. Other singularities occur at maximum width bifurcation or level repulsion where the eigenfunctions of $\mathcal{H}$ are orthogonal (instead of biorthogonal). They are related to obviously nonlinear irreversible processes [17].

In the present paper, we will illustrate the relation between the eigenvalues $\mathcal{E}_i \equiv E_i + i/2\Gamma_i$ [2] and eigenfunctions Φ_i of the non-Hermitian Hamiltonian $\mathcal{H}$, on the one hand, as well as the appearance of gain and loss and different types of singularities in non-Hermitian quantum physics, on the other hand.

For this aim, we start from a system with two states $i = 1, 2$ the complex energies of which are $\varepsilon_i \equiv e_i + i/2\gamma_i$ [2]. We then consider the coupling of the system to two channels under the assumption that each state is coupled, above all, to only one channel: $\gamma_1^{(1)} \gg \gamma_1^{(2)}; \gamma_2^{(2)} \gg \gamma_2^{(1)}$. We vary parametrically the energies e_i of the states and determine the eigenvalues $\mathcal{E}_i \equiv E_i + i/2\Gamma_i$ of the non-Hermitian Hamiltonian $\mathcal{H}$. In such a case, width bifurcation causes automatically a transition from loss (determined by $\gamma_1^{(2)} < 0$ and $\gamma_2^{(1)} < 0$) to gain (determined by $\gamma_1^{(2)} > 0$ and $\gamma_2^{(1)} > 0$).

The paper is organized in the following manner. In Section 2, we sketch the formalism which is used in performing the calculations. Some numerical results are shown in Section 3. First we show a few eigenvalue trajectories together with the corresponding eigenfunction trajectories in Section 3.1. The resonance structure of the transmission and the transmission probability are given in Section 3.2. In the last Section 4 we summarize and discuss the results and draw some conclusions on non-Hermitian quantum physics of open quantum systems.

2. Non-Hermitian Hamiltonian $\mathcal{H}$ of an Open Quantum System

2.1. Eigenvalues and Eigenfunctions of $\mathcal{H}^{(2,1)}$. To begin with, we sketch the features typical for an open quantum system embedded in one common continuum. Details can be found in [7] and, above all, in [17]. They can be discussed by means of the genuine 2×2 non-Hermitian matrix:

$$\mathcal{H}^{(2,1)} = \begin{pmatrix} \varepsilon_1^{(1)} \equiv e_1^{(1)} + \frac{i}{2}\gamma_1^{(1)} & \omega^{(1)} \\ \omega^{(1)} & \varepsilon_2^{(1)} \equiv e_2^{(1)} + \frac{i}{2}\gamma_2^{(1)} \end{pmatrix}. \qquad (1)$$

Here, $\varepsilon_i^{(1)}$ are the complex energies of the two states $i = 1, 2$ which are coupled to the environment 1 (called also channel 1)[1]. $e_i^{(1)}$ are the energies of the states i and $\gamma_i^{(1)}$ are their widths. $\omega^{(1)}$ stand for the coupling matrix elements of the two states via the common environment 1. They are complex where $\mathrm{Re}(\omega^{(1)})$ arises from the principal value integral and $\mathrm{Im}(\omega^{(1)})$ from the residuum [7]. The complex eigenvalues $\mathcal{E}_i^{(1)} \equiv E_i^{(1)} + (1/2)\Gamma_i^{(1)}$ of $\mathcal{H}^{(2,1)}$ give energies $E_i^{(1)}$ and widths $\Gamma_i^{(1)}$ of the states of the localized part of the system[1].

In (1), a singular point may appear, the so-called EP, at which two eigenvalues of $\mathcal{H}^{(2,1)}$ coalesce [1]. At these points, the two corresponding eigenfunctions are not orthogonal. Instead

$$\begin{aligned} \Phi_1^{\mathrm{cr}} &\longrightarrow \pm i \Phi_2^{\mathrm{cr}}; \\ \Phi_2^{\mathrm{cr}} &\longrightarrow \mp i \Phi_1^{\mathrm{cr}} \end{aligned} \qquad (2)$$

according to analytical and numerical results [18–20]. An EP is, according to its definition, related to the common environment in which the system is embedded. In other words, it is well defined under the condition that the system is embedded in only one continuum.

The eigenfunctions of a non-Hermitian Hamilton operator are biorthogonal

$$\begin{gathered}\mathcal{H}\left|\Phi_i\right\rangle = \mathcal{E}_i\left|\Phi_i\right\rangle\\ \left\langle\Psi_i\right|\mathcal{H} = \mathcal{E}_i\left\langle\Psi_i\right|.\end{gathered}\tag{3}$$

In the case of the symmetric 2×2 Hamiltonian (1), it is

$$\Psi_i = \Phi_i^*\tag{4}$$

and the eigenfunctions should be normalized according to

$$\left\langle\Phi_i^* \mid \Phi_j\right\rangle = \delta_{ij}\tag{5}$$

in order to smoothly describe the transition from a closed system with discrete states to a weakly open one with narrow resonance states. As a consequence of (5), the values of the standard expressions are changed:

$$\begin{gathered}\left\langle\Phi_i \mid \Phi_i\right\rangle = \operatorname{Re}\left(\left\langle\Phi_i \mid \Phi_i\right\rangle\right);\\ A_i \equiv \left\langle\Phi_i \mid \Phi_i\right\rangle \geq 1\end{gathered}\tag{6}$$

$$\left\langle\Phi_i \mid \Phi_{j\neq i}\right\rangle = i\operatorname{Im}\left(\left\langle\Phi_i \mid \Phi_{j\neq i}\right\rangle\right) = -\left\langle\Phi_{j\neq i} \mid \Phi_i\right\rangle\tag{7}$$

$$\left|B_i^j\right| \equiv \left|\left\langle\Phi_i \mid \Phi_{j\neq i}\right\rangle\right| \geq 0.\tag{8}$$

Furthermore, the phase rigidity, which is a quantitative measure for the biorthogonality of the eigenfunctions,

$$r_k \equiv \frac{\left\langle\Phi_k^* \mid \Phi_k\right\rangle}{\left\langle\Phi_k \mid \Phi_k\right\rangle} = A_k^{-1},\tag{9}$$

is smaller than 1. Far from an EP, $r_k \approx 1$ while it approaches the value $r_k = 0$ when an EP is approached.

Additionally to the Hamiltonian (1), we will consider the non-Hermitian matrix

$$\mathcal{H}_0^{(2,1)} = \begin{pmatrix}\varepsilon_1^{(1)} \equiv e_1^{(1)} + \dfrac{i}{2}\gamma_1^{(1)} & 0\\ 0 & \varepsilon_2^{(1)} \equiv e_2^{(1)} + \dfrac{i}{2}\gamma_2^{(1)}\end{pmatrix}\tag{10}$$

which describes the system without any mixing of its states via the environment. In other words, $\omega = 0$ corresponds to vanishing EM of the eigenstates. The eigenfunctions Φ_i of $\mathcal{H}^{(2,1)}$ can be represented in the set of eigenfunctions $\{\Phi_i^0\}$ of $\mathcal{H}_0^{(2,1)}$,

$$\begin{gathered}\Phi_i = \sum b_{ij}\Phi_j^0;\\ b_{ij} = \left\langle\Phi_j^{0*} \mid \Phi_i\right\rangle\end{gathered}\tag{11}$$

under the condition that b_{ij} are normalized by $\sum_j (b_{ij})^2 = 1$. The coefficients $|b_{ij}|^2$ differ from the $(b_{ij})^2$. They contain the information on the strength of EM.

The main features characteristic of open quantum systems are described well by the eigenvalues and eigenfunctions of (1). Typical values related to the eigenfunctions are the phase rigidity (9) and the contribution of EM (11) to their purity. All these values contain the influence of the environment. They are proven experimentally; for details see [7, 8, 17].

2.2. Schrödinger Equation with $\mathcal{H}^{(2,1)}$. The Schrödinger equation $(\mathcal{H}^{(2,1)} - \mathcal{E}_i^{(1)}|\Phi_i^{(1)}\rangle = 0$ may be rewritten into a Schrödinger equation with source term [7, 17],

$$\left(\mathcal{H}_0^{(2,1)} - \mathcal{E}_i^{(1)}\right)\left|\Phi_i^{(1)}\right\rangle = -\begin{pmatrix}0 & \omega\\ \omega & 0\end{pmatrix}\left|\Phi_i^{(1)}\right\rangle.\tag{12}$$

In this representation, the coupling ω of the states i and $j \neq i$ of the localized system via the common environment of scattering wavefunctions (EM) is contained in the source term of the Schrödinger equation; for details see [7].

Far from EPs, the coupling of the localized system to the environment influences the spectroscopic properties of the system only marginally [7, 17]. The influence of the environment is however nonvanishing also in this case; see, e.g., the experimental results [21]. That means, the eigenstates i of $\mathcal{H}$ will never be the same as those of a Hermitian operator.

In the neighborhood of EPs, the coupling between system and environment cause, according to mathematical studies, nonlinear effects in the Schrödinger equation (12). Among others, these effects lead to a conservation of the resonance structure of the cross section in the one-channel case which is finally unaffected by EM and by the existence of EPs [17]. Thus, the one-channel case cannot be used in order to test the results of the non-Hermitian formalism.

2.3. Two-Channel Case with $\mathcal{H}^{(2,2)}$. Let us consider now the genuine 4×4 non-Hermitian matrix

$$\mathcal{H}^{(2,2)} = \begin{pmatrix}\varepsilon_1^{(1)} & \omega^{(1)} & 0 & 0\\ \omega^{(1)} & \varepsilon_2^{(1)} & 0 & 0\\ 0 & 0 & \varepsilon_1^{(2)} & \omega^{(2)}\\ 0 & 0 & \omega^{(2)} & \varepsilon_2^{(2)}\end{pmatrix}.\tag{13}$$

Here, $\varepsilon_i^{(1)} \equiv e_i^{(1)} + (i/2)\gamma_i^{(1)}$ and $\varepsilon_i^{(2)} \equiv e_i^{(2)} + (i/2)\gamma_i^{(2)}$ are the complex energies of the two states $i = 1, 2$ relative to the two different channels $c = 1, 2$ [1]. Usually $\varepsilon_i^{(1)} \neq \varepsilon_i^{(2)}$. $\omega^{(1)}$ and $\omega^{(2)}$ stand for the coupling matrix elements of the two states via environments 1 and 2, respectively. The Hamiltonian (13) includes the fact that the complex energy $\varepsilon_i^{(c)}$ of the localized state i may be different relative to the different channels c.

Also the eigenvalues $\mathcal{E}_i^{(c)} \equiv E_i^{(c)} + (i/2)\Gamma_i^{(c)}$ and eigenfunctions $\Phi_i^{(c)}$ of (13) are characterized by two numbers: the number i of the state ($i = 1, 2$) of the localized part of the system and the number c of the channel ($c = 1, 2$), called environment, in which the system is embedded. Usually, $E_i^{(1)} \neq E_i^{(2)}$ and $\Gamma_i^{(1)} \neq \Gamma_i^{(2)}$. Further, the wave functions $\Phi_i^{(1)}$ and $\Phi_i^{(2)}$ differ from one another due to the EM of the eigenstates via environments 1 and 2, respectively. That means the system has four states, from a mathematical point of view.

The two environments are different from and orthogonal to one another. Further, the two eigenstates with equal i and different c arise from the same state i of the localized part of the system. The zeros in the matrix (13) express the corresponding fact that the two eigenstates i relative to the two channels 1 and 2 of $\mathscr{H}^{(2,2)}$ cannot interact with one another.

We mention here that the Hamiltonian (13) is formally the same as the Hamiltonian (24) or rather (1) in [10]. There is however a fundamental difference: in (1) in [10], the transition of excitons is considered while (13) describes primarily the transition of particles. Furthermore, in (1) in [10] the whole system is fully embedded into both environments ($c = 1$) and ($c = 2$) which are both of completely different nature and exist independently of one another. In contrast to this, the Hamiltonian (13) describes a system, the states of which are primarily embedded partially in each of the two different environments. These two different environments are also independent of one another. They are, however, nothing but parts of the total environment.

Without an EP in the considered parameter range in relation to both channels, we have $E_i^{(1)} \approx E_i^{(2)}$, $\Gamma_i^{(1)} \approx \Gamma_i^{(2)}$, and $\Phi_i^{(1)} \approx \Phi_i^{(2)}$. Under the influence of an EP relative to $c = 1$ and/or relative to $c = 2$, the eigenvalues and eigenfunctions will be, however, different from one another; $E_i^{(1)} \neq E_i^{(2)}$, $\Gamma_i^{(1)} \neq \Gamma_i^{(2)}$, and $\Phi_i^{(1)} \neq \Phi_i^{(2)}$ in the corresponding parameter range.

In analogy to (10), we will consider also the non-Hermitian Hamiltonian:

$$\mathscr{H}_0^{(2,2)} = \begin{pmatrix} \varepsilon_1^{(1)} & 0 & 0 & 0 \\ 0 & \varepsilon_2^{(1)} & 0 & 0 \\ 0 & 0 & \varepsilon_1^{(2)} & 0 \\ 0 & 0 & 0 & \varepsilon_2^{(2)} \end{pmatrix} \tag{14}$$

which describes the system without any mixing of its states via any environment. In other words, $\omega^{(1)} = \omega^{(2)} = 0$ corresponds to vanishing EM of the eigenstates via an environment. The mixing of the eigenstates of (13) can be represented in a set of eigenfunctions of (14) in complete analogy to the relation (11) for two states coupled to one common environment.

2.4. Schrödinger Equation with $\mathscr{H}^{(2,2)}$.

2.4. Schrödinger Equation with $\mathscr{H}^{(2,2)}$. Using (14), we can write down the Schrödinger equation with source term for the two-channel case in analogy to (12) for the one-channel case. The corresponding equation reads

$$\begin{aligned} &\left(\mathscr{H}_0^{(2,2)} - \mathscr{E}_i^{(c)}\right)\left|\Phi_i^{(c)}\right\rangle \\ &= -\begin{pmatrix} 0 & \omega^{(1)} & 0 & 0 \\ \omega^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^{(2)} \\ 0 & 0 & \omega^{(2)} & 0 \end{pmatrix}\left|\Phi_i^{(c)}\right\rangle. \end{aligned} \tag{15}$$

The source depends on the coupling of the system to both channels, i.e., on $\omega^{(1)}$ and on $\omega^{(2)}$. It does not depend on the energies $\varepsilon_i^{(c)}$.

We repeat here that, according to their definition [1], EPs occur only in the one-channel case, i.e., only in the two submatrices related to channel 1 and channel 2, respectively. They are not defined in the 4×4 matrix (13). However, the EPs of the two submatrices in (13) may influence the dynamics of the open two-channel system.

2.5. Non-Hermitian Hamiltonian and Resonance Structure of the S Matrix. The resonance structure of the S matrix containing two resonance states can be obtained from the expression

$$S = \frac{\left(E - E_1 - (i/2)\,\Gamma_1\right)\left(E - E_2 - (i/2)\,\Gamma_2\right)}{\left(E - E_1 + (i/2)\,\Gamma_1\right)\left(E - E_2 + (i/2)\,\Gamma_2\right)} \tag{16}$$

which is derived in [22] by rewriting the Breit-Wigner expression for one or more isolated resonances that are coupled to a common continuum of scattering wavefunctions; see also [7, 17]. The expression (16) is unitary. According to (16), the resonance structure of the cross section is determined exclusively by the spectroscopic values of the localized part of the system, i.e., by the eigenvalues $\mathscr{E}_i = E_i + i/2\Gamma_i$ of the non-Hermitian Hamiltonian $\mathscr{H}$. The expression (16) allows us therefore to receive reliable results also in the two-channel case and, moreover, when the phase rigidity of the eigenfunctions of $\mathscr{H}$ is reduced, $r_k < 1$ and when the eigenfunctions of $\mathscr{H}$ contain EM, i.e., when they are mixed in the set of eigenfunctions $\{\Phi_i^0\}$ of $\mathscr{H}_0$ according to (11).

According to the results obtained in [17], the evolution of the system near to an EP is driven exclusively by the nonlinear source term of the Schrödinger equation (12) which describes the coupling of the localized part of the system to the common environment and is characteristic of the open quantum system embedded in one environment. The calculations in [17] are performed for fixed ω; i.e., ω can not be responsible for the width bifurcation occurring in these calculations under the influence of an EP. The only conclusion is therefore that the nonlinear source term is the driving force. It is able, in the one-channel case, to largely conserve the resonance structure of the cross section.

The one-channel case does, therefore, not allow us to prove the existence of the nonlinear effects and of EM, since the resonance structure of the cross section calculated with and without EM is the same in this case [17]. This result agrees, on the one hand, with the experience received from many different numerical studies in realistic cases which are performed without taking into account EM. On the other hand, it is not in contradiction with the conclusions received from the study of non-Hermitian physics of open quantum systems.

The conservation of the resonance structure of the cross section, which is possible in the one-channel case, is expected to be impossible, generally, in the two-channel (or more-channel) case.

3. Numerical Results

3.1. Eigenvalues and Eigenfunctions of $\mathscr{H}^{(2,2)}$. The aim of our numerical studies is first to show the influence of a singularity onto the eigenvalues and eigenfunctions of the non-Hermitian Hamilton operator $\mathscr{H}^{(2,2)}$. Knowing these results, we study then, in the following Section 3.2, the influence of the singularity onto observable values such as resonance structure and contour plot of the transmission probability.

In Figure 1, we provide numerical results for the parameter dependence of eigenvalues $\mathscr{E}_i^{(1,2)} \equiv E_i^{(1,2)} + (i/2)\Gamma_i^{(1,2)}$ and eigenfunctions $\Phi_i^{(1,2)}$ of the Hamiltonian $\mathscr{H}^{(2,2)}$. We consider the case that one of the states is coupled to a special channel much stronger than the other one, $\gamma_i^{(c)} \ll \gamma_{j\neq i}^{(c)}$. Without loss of generality, we choose the extreme case $\gamma_1^{(1)} \neq 0$; $\gamma_2^{(2)} \neq 0$; $\gamma_1^{(2)} = \gamma_2^{(1)} = 0$.

We are interested, above all, in the appearance of a critical parameter value $a = a^{\text{cr}}$ at which the spectroscopic values of the localized part of the system change. All our calculations show that a^{cr} is well defined: it is the value of maximum width bifurcation. At a^{cr}, the phase rigidity approaches the value 1, meaning that the two eigenstates of the non-Hermitian Hamiltonian $\mathscr{H}$ become orthogonal at this parameter value. However, the EM of the states via the continuum cannot be neglected at a^{cr}.

In Figure 1 we show a few typical numerical results. The results of the left column are obtained with $\omega^{(1)} = \omega^{(2)}$ and those of the right column with $\omega^{(1)} \neq \omega^{(2)}$. The other parameters are given in the figure caption.

The results are the following. The energies E_i cross at $a = a^{\text{cr}}$ (Figures 1(a) and 1(f)), while the widths Γ_i bifurcate (Figures 1(b) and 1(g)). Notably, the state with vanishing width gets a positive width due to the width bifurcation; i.e., it does not decay around $a = a^{\text{cr}}$.

Around the critical parameter value, the phase rigidity r_i decreases slightly (Figures 1(c), 1(d), 1(h), and 1(i)). It approaches the value 1 when $a \longrightarrow a^{\text{cr}}$. That means, the eigenfunctions of the non-Hermitian operator $\mathscr{H}$ become orthogonal when $a \longrightarrow a^{\text{cr}}$. They are mixed via the environment (Figures 1(e) and 1(j)).

When we start from parameter independent energies and parameter dependent widths in contrast to the case considered in Figure 1, then the biorthogonal eigenfunctions of $\mathscr{H}$ become orthogonal at maximum level repulsion. In any case, there is a critical parameter value at which the biorthogonal eigenfunctions of $\mathscr{H}$ become orthogonal.

We remark that not only do eigenstates with positive widths appear in the present study on systems embedded in two environments but also they are well known from different studies on one-channel systems.

3.2. Resonance Structure and Contour Plot of the Transmission Probability. Observable information on the spectroscopic properties of the localized part of the system is contained in the resonance structure of the cross section. Using (16), we have calculated the resonance structure of the cross section at the critical parameter value $a = a^{\text{cr}}$ and at the two values $a_1 = -3$ and $a_2 = 3$ of a far from a^{cr}.

In all cases we see the double-hump structure of the transmission which is characteristic of the resonance structure of a two-level system coupled to one channel [7, 17, 23]. An example is shown in Figure 2. That means the cross section in the two-channel case is very much like that in the one-channel case when the system is coupled with the same strength to both channels and the EM of the states is the same. The results obtained for other values of the coupling strength (e.g., for $\omega = 0.01i$; $0.1i$ or $0.5i$) are very similar.

Additionally, we have performed some calculations with different values of the original widths ($\gamma_i^{(1)} \neq \gamma_i^{(2)}$) under the condition $\omega^{(1)} = \omega^{(2)} \equiv \omega$. As a result, differences in the values of the original widths $\gamma_i^{(c)}$ almost do not influence the resonance structure of the cross section in the critical parameter range. This corresponds to the fact that $\gamma_i^{(c)}$ are not involved in the source term of the Schrödinger equation (15) and therefore do not contribute to the EM of the states via the environment.

When the EM of the eigenstates via the continuum of scattering wavefunctions is different for the two channels and different from zero, $\omega^{(1)} \neq \omega^{(2)}$, then the minimum of the cross section at a^{cr} will fill up. An example is shown in Figure 3. Similar results (not shown) are received with $\omega^{(1)} = 0.4i$ and $\gamma_1/2 = 0.4$. In both cases, several different values are used for $|\omega^{(2)}| < |\omega^{(1)}|$. The filling-up of the minimum causes a broad maximum of the cross section (after averaging) around $a = a^{\text{cr}}$ instead of the typical resonance peak. This can be seen well when $|\omega^{(1)}| \approx \gamma_1/2$.

Although the double-hump structure of the cross section, appearing under the condition $\omega^{(1)} = \omega^{(2)} \equiv \omega \neq 0$, is well pronounced, it differs, even for small ω, from that obtained for vanishing coupling strength $\omega = 0$ between system and environment (see Figure 4). This result differs from those obtained in the one-channel case; see [17]. That means, EM can generally not be neglected in the two-channel case.

Further information is contained in the contour plots of the cross section some of which are shown in the lower parts of Figures 2, 3, and 4. Most interesting is the padding of the minimum of the double-hump structure of the cross section in the critical parameter range around a^{cr} when the EM of the two states via the environment is different from one another. It causes an enhancement of the cross section around the critical parameter value a^{cr}.

In all cases, the two eigenstates of the non-Hermitian operator $\mathscr{H}^{(2,2)}$ become orthogonal in approaching the critical value $a = a^{\text{cr}}$. Here the eigenfunctions of $\mathscr{H}^{(2,2)}$ are mixed in the set of eigenfunctions of (14).

4. Discussion of the Results and Conclusions

The results of our calculations show very clearly that not only the eigenvalues $\mathscr{E}_i = E_i + i\Gamma_i/2$ of $\mathscr{H}$ play an important role in the non-Hermitian quantum physics of open quantum systems but also the eigenfunctions of $\mathscr{H}$ are of similar importance. It depends on the physical problem that is considered, whether the eigenvalues are more significant than the eigenfunctions or vice versa. In any case, the eigenvalues *and* the eigenfunctions must be considered in describing and explaining a physical problem.

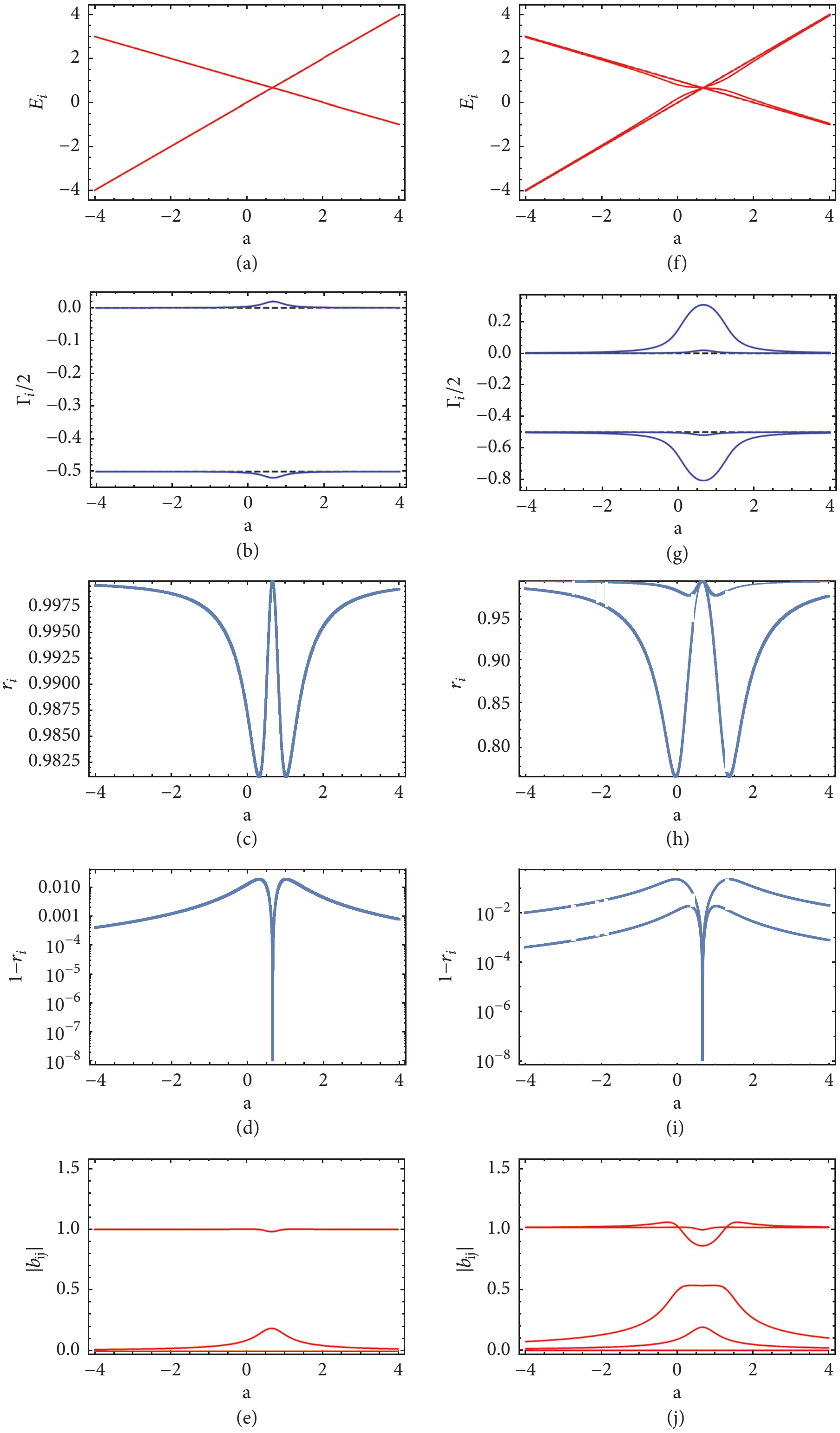

FIGURE 1: Eigenvalues $\mathcal{E}_i^{(1,2)} \equiv E_i^{(1,2)} + (i/2)\Gamma_i^{(1,2)}$ and eigenfunctions $\Phi_i^{(1,2)}$ of the Hamiltonian $\mathcal{H}^{(2,2)}$ as a function of a. $\omega^{(1)} = \omega^{(2)} = 0.1i$ (left), $\omega^{(1)} = 0.5i$; $\omega^{(2)} = 0.1i$ (right). Parameters: $e_1 = 1 - a/2$; $e_2 = a$; $\gamma_1^{(1)}/2 = \gamma_2^{(2)}/2 = -0.5$; $\gamma_1^{(2)}/2 = \gamma_2^{(1)}/2 = 0$ (dashed lines in a, b, f, and g). At the critical parameter value $a = a^{\mathrm{cr}} = 0.6494$, the phase rigidity r_i approaches value 1.

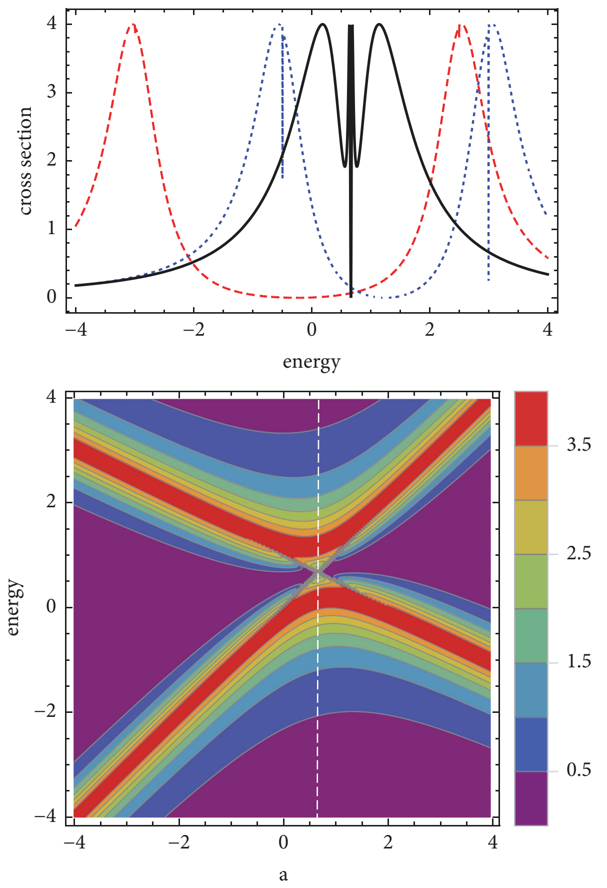

Figure 2: Resonance structure (above) and contour plot (below) of the transmission with $\omega^{(1)} = \omega^{(2)} \neq 0$. The parameters are the same as in Figure 1, left. Above: black: $a = a^{\mathrm{cr}}$; red: $a_1 = -3$; blue: $a_2 = 3$. Below: the white line marks the value $a = a^{\mathrm{cr}}$.

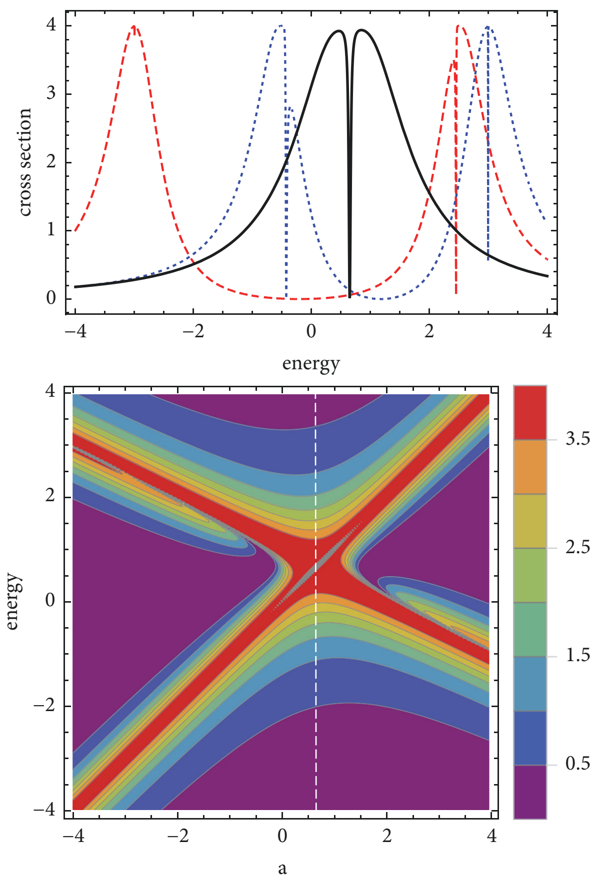

Figure 3: The same as Figure 2 but with $\omega^{(1)} \neq \omega^{(2)}$. The parameters are the same as in Figure 1, right.

While the influence of the eigenvalues is restricted, above all, to a small parameter range around some critical points, the eigenfunctions influence a much larger parameter range around these points. An example is the EM (which is a second-order effect). Its influence cannot be neglected over a comparably large parameter range. This behavior is known from the one-channel case [17] and holds true also in the two-channel case as can be seen from the results in Figure 1.

Instead of an EP, we see in Figure 1 the critical parameter value a^{cr} at which the energy trajectories cross and the width bifurcation is maximum. Here, the phase rigidity approaches the value 1, meaning that the eigenfunctions are orthogonal. They are strongly mixed via the continuum. Similar results are very well known also from one-channel studies [17]. The meaning of this result is not fully understood up to now.

Due to the width bifurcation, the width of the state with originally vanishing coupling strength to one of the channels becomes automatically positive. That means that this state gains something from the environment. Thus some gain from the environment in an open quantum system is not at all an exotic process. The different characteristic features of non-Hermitian quantum physics, which are considered in the Introduction, are really nothing but different sides of non-Hermitian quantum physics.

In conclusion we state the following. According to long-time experience, open quantum systems are described usually well by standard methods. There are however some exceptions which occur in a parameter range around singularities. These may be the well-known exceptional points (EPs). Others are related to the fact that the biorthogonal eigenfunctions of $\mathscr{H}$ become orthogonal when width bifurcation (or level repulsion) becomes maximum.

In any case, the non-Hermitian formalism is a very powerful method and is able to explain different phenomena that are puzzling in standard Hermitian quantum physics. There remain however many open questions to which an answer has to be found in future. One of these questions is the mathematical and physical meaning of the critical point a^{cr} at which the eigenfunctions of the non-Hermitian Hamilton operator $\mathscr{H}$ become orthogonal. Other questions arise from the meaning of EPs in many-channel systems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

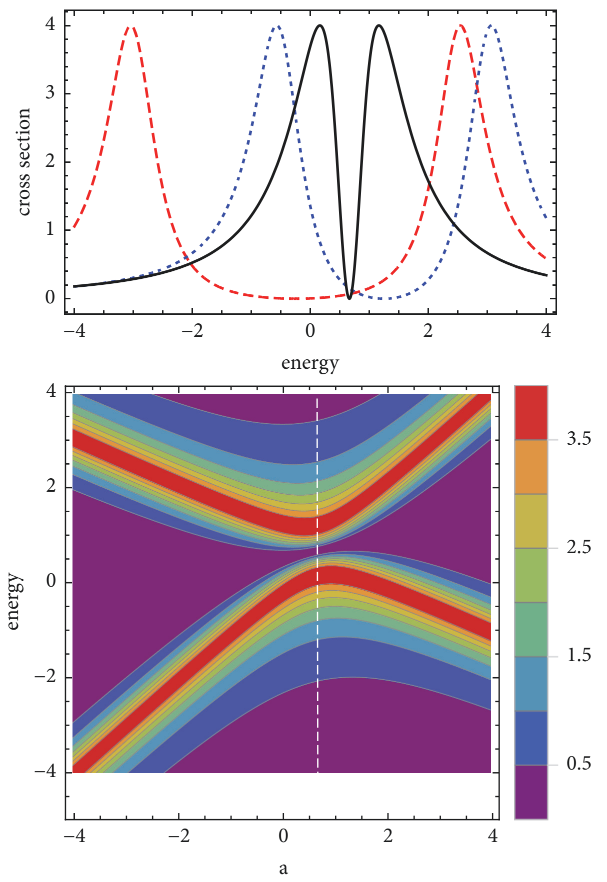

Figure 4: The same as Figure 2 but with vanishing external mixing, $\omega^{(1)} = \omega^{(2)} = 0$.

Acknowledgments

The authors thank Jon Bird for valuable discussions.

Endnotes

1. We consider open quantum systems described by a non-Hermitian Hamilton operator. This should not be confused with the consideration of PT-symmetric systems which are neither open nor closed, but nonisolated according to the definition in, e.g., C.M. Bender, Journal of Physics: Conference Series 631, 012002 (2015).
2. In contrast to the definition that is used in, for example, nuclear physics, we define the complex energies before and after diagonalization of $\mathscr{H}$ by $\varepsilon_k = e_k + (i/2)\gamma_k$ and $\mathscr{E}_k = E_k + (i/2)\Gamma_k$, respectively, with $\gamma_k \le 0$ and $\Gamma_k \le 0$ for decaying states. This definition will be useful when discussing systems with gain (positive widths) and loss (negative widths).

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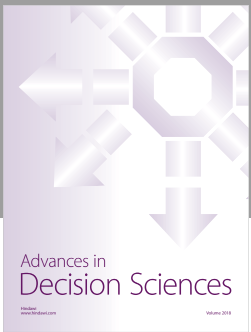
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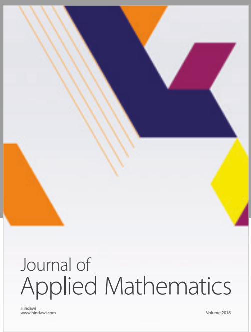
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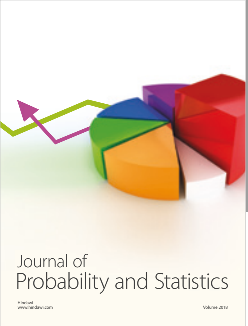
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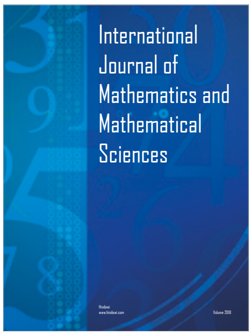
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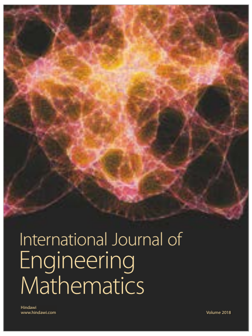
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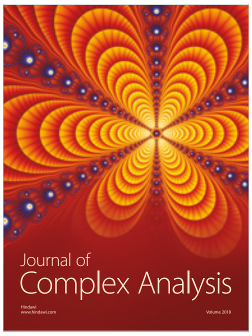
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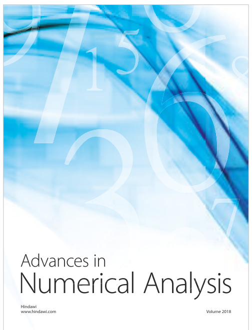
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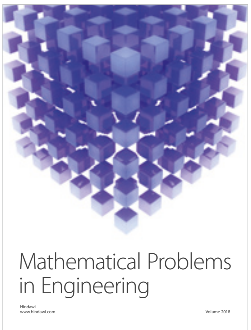
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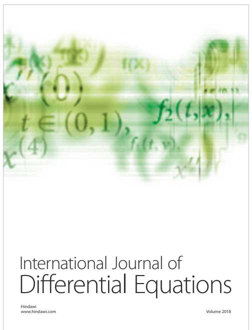
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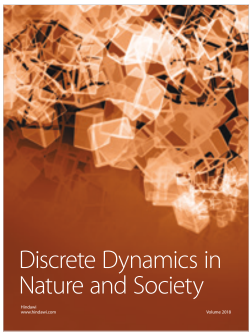
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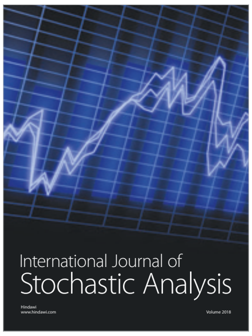
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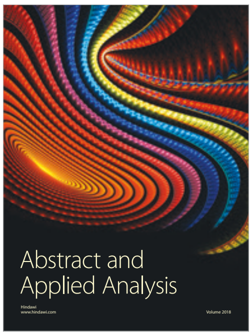
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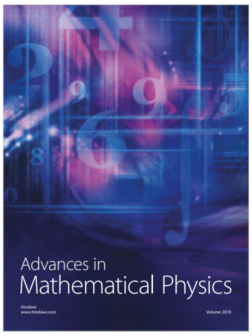
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