Image Processing and 'Noise Removal Algorithms'—The Pdes and Their Invariance Properties & Conservation Laws

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Image Processing and 'Noise Removal Algorithms'—The Pdes and Their Invariance Properties & Conservation Laws

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Abstract We analyse the symmetry, invariance properties and conservation laws of the partial differential equations (pdes) and minimization problems (variational functionals) that arise in the analyses of some noise removal algorithms.

Keywords Noise removal algorithms · Invariance · Conservation laws

Mathematics Subject Classification 58D19 · 58J70 · 70H33

1 Introduction

Amongst others, Rudin et al [1] and Esedoglu & Osher [2] have discussed and modelled, in considerable detail, the notion of *denoising* that arises in images wherein the goal is 'to remove noise from a corrupted digital image without blurring object boundaries'. Other detailed works in this regard by Osher are [3–5]. Essentially, they propose to denoise images by minimizing the total variation norm of the estimated solution. To this end, they 'derive a constrained minimization algorithm in which the constraints are determined by the noise statistics'. In this paper, we study a number of models that are developed and discussed in [1]. We study the symmetry, invariance properties and conservation laws of the differential equations and/or minimization problems (variational functionals) that are presented therein.

A number of other works and alternative approaches in the area of 'denoising' are available in [6–9], inter alia.

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We present some preliminaries that will be used in the analyses that follow.

Consider an rth-order system of partial differential equations of n independent variables $\underline{s} = (s_1, s_2, \dots, s_n)$ and m dependent variables $u = (u_1, u_2, \dots, u_m)$ viz.,

$$E^{\mu}(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0, \quad \mu = 1, \dots, \tilde{m}, \tag{1}$$

where a locally analytic function $f(\underline{s}, u, u_1, \dots, u_k)$ of a finite number of dependent variables $u, u_1, \dots, u_k$ denote the collections of all first , second ,..., kth-order partial derivatives and s is a multivariable, that is

$$u_i^{\alpha} = D_i\big(u^{\alpha}\big), \qquad u_{ij}^{\alpha} = D_j D_i\big(u^{\alpha}\big), \dots \tag{2}$$

respectively, with the total differentiation operator with respect to s^i given by,

$$D_i = \frac{\partial}{\partial s^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + \cdots, \quad i = 1, \dots, n. \tag{3}$$

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence. Firstly, if $(T^{s1}, T^{s1}, \dots)$ is a conserved vector corresponding to a conservation law, then

$$D_{s1}T^{s1} + D_{s1}T^{s1} + \cdots = 0 \tag{4}$$

along the solutions of the differential equation $E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0$.

Moreover, if there exists nontrivial differential functions Q^{μ}, called a 'multiplier', such that

$$Q^{\mu}(\underline{s}, u, u_{(1)}, \dots)E^{\mu}(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = D_{s1}T^{s1} + D_{s1}T^{s1} + \cdots, \tag{5}$$

for some (conserved) vector $(T^{s1}, T^{s1}, \dots)$, then

$$\frac{\delta}{\delta u}Q(\underline{s}, u, u_{(1)}, \dots)E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{6}$$

where $\frac{\delta}{\delta u}$ is the Euler operator. Hence, one may determine the multipliers, using (6) and then construct the corresponding conserved vectors; several approaches for this exists of which the better known one is the 'homotopy' approach.

If the system of differential equations is derived from a variational principle, then the conserved vector components are obtainable from Noether's Theorem [10] which requires, firstly, the construction of variational symmetries (vector fields) $X = \xi^{s^i}\frac{\partial}{\partial s^i} + \eta^{u^{\alpha}}\frac{\partial}{\partial u^{\alpha}}$ that leave the action integral invariant. The conservation laws may be expressed as conserved forms [11]. It is well known that the vector fields that leave the system of differential equations invariant (generators of Lie point symmetries) contain the algebra of variational symmetries, if the latter exists [12–14].

2 The Pdes of the 'Denoising Algorithm'

The cases studied below are the various nonlinear pdes that, in the first two cases, are variational and are pdes in space variables only. The first of these variational pdes is a parabolic type one that is independent of a 'source term' whilst the second contains a linear source term in the dependent variable. When the latter of these two contains a time variable as an evolution parameter, the resulting pde mimics a Klein-Gordon equation.

2.1 Parabolic 'Denoising Pde'

Let $u(x, y)$ denote the 'clean image' for $(x, y) \in \mathbb{R}$. Then, a constrained minimization problem is given by minimizing [1] $\int L_1 \mathrm{d}x\mathrm{d}y$, where the Lagrangian, L_1, is given by

$$L_1 = \sqrt{u_x^2 + u_y^2} \tag{7}$$

subject to some constraints. The corresponding Euler or Euler-Lagrange operator on L_1 is $\frac{\delta L_1}{\delta u} = -\frac{u_{xx}{u_y}^2 - 2u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}}$. The variational symmetries (vector fields) that leave the functional $\int L_1 \mathrm{d}x\mathrm{d}y$ invariant with the corresponding conserved forms $\omega = T^x \mathrm{d}x + T^y \mathrm{d}y$ (where $(T^x, -T^y)$ is the conserved flow/vector)).

(i) $-x\partial_x - y\partial_y + \partial_u$:

$$\frac{-y{u_x}^2 + u_y + u_x x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{{u_y}^2 - u_x - u_y y u_x}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y;$$

(ii) ∂_x:

$$\frac{-u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{-u_y^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y;$$

(iii) ∂_y:

$$\frac{u_x^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y;$$

(iv) $-y\partial_x + x\partial_y$:

$$\frac{u_x(u_x x + u_y y)}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{u_y(u_x x + u_y y)}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y.$$

Here, (ii) and (iii) represent the conservation of linear momentum in x and y, respectively, and (iv) is the angular momentum.

If we consider the Euler-Lagrange equation corresponding to L_1

$$-\frac{u_{xx}{u_y}^2 - 2u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}} = 0, \tag{8}$$

it turns out that its algebra of point symmetry generators include generalisations of the vectors fields above. That is,

$$\left[\frac{1}{2}xf_4(u)y + f_7(u)y + x^2 f_2(u) + f_8(u)x + f_9(u)\right]\partial_x$$
$$+ \left[xf_2(u)y + xf_3(u) + \frac{1}{2}f_4(u)y^2 + f_5(u)y + f_6(u)\right]\partial_y + f_1(u)\partial_u,$$

or

$$\begin{array}{lll} f_1\partial_u, & f_2\big(x^2\partial_x + xy\partial_y\big), & f_3x\partial_y, \\ f_4\big(xy\partial_x + y^2\partial_y\big), & f_5y\partial_y, & f_6\partial_y, \\ f_7y\partial_x, & f_8x\partial_x, & f_9\partial_x \end{array}$$

(where the f_i's are arbitrary functions of u).

2.2 The 'Denoising Pde' with a Linear Source Term

With a linear and nonlinear constraint, we have the Euler-Lagrange equation

$$\lambda_1 + \lambda_2 u - \frac{u_{xx}{u_y}^2 - 2\,u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}} = 0 \tag{9}$$

with Lagrangian given by

$$L_2 = \lambda_1 u + \frac{1}{2}\lambda_2 u^2 + \sqrt{u_x^2 + u_y^2} \tag{10}$$

which lead to the following variational symmetries and corresponding conserved forms.

(i) ∂_x (linear momentum in x):

$$\frac{-u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x - \frac{1}{2}\frac{2\sqrt{{u_x}^2 + {u_y}^2}\lambda_1 u + \sqrt{{u_1}^2 + {u_y}^2}\lambda_2 u^2 + 2\,{u_y}^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y;$$

(ii) ∂_y (linear momentum in y):

$$\frac{1}{2}\frac{2\sqrt{{u_x}^2 + {u_y}^2}\lambda_1 u + \sqrt{{u_1}^2 + {u_y}^2}\lambda_2 u^2 + 2\,{u_x}^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y;$$

(iii) $-y\partial_x + x\partial_y$ (angular momentum):

$$\frac{1}{2}\frac{2x\sqrt{{u_x}^2 + {u_y}^2}\lambda_1 u + x\sqrt{{u_1}^2 + {u_y}^2}\lambda_2 u^2 + 2xu_x{}^2 + 2\,u_y y u_x}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x$$
$$+ \frac{1}{2}\frac{2\,y\sqrt{{u_x}^2 + {u_y}^2}\lambda_1 u + y\sqrt{{u_1}^2 + {u_y}^2}\lambda_2 u^2 + 2\,yu_y{}^2 + 2\,u_x x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y.$$

The Euler equation (9) generates the symmetries

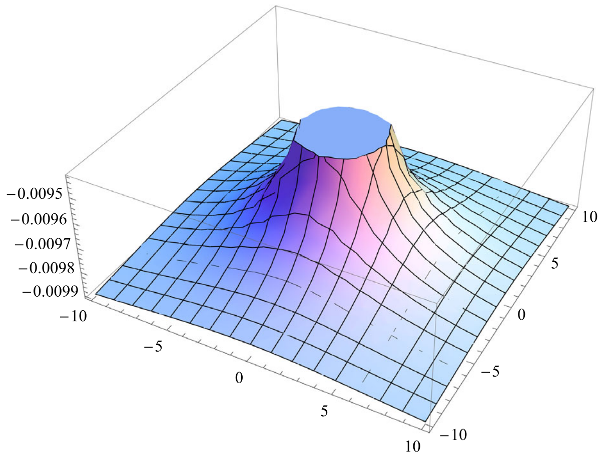
$$\left[-\frac{f_1(u)\lambda_2 x}{\lambda_1 + \lambda_2 u} + f_2(u)y + f_3(u)\right]\partial_x$$
$$+ \left[-x\,f_2(u) - \frac{f_1(u)\lambda_2\,y}{\lambda_1 + \lambda_2\,u} + f_4(u)\right]\partial_y + f_1(u)\partial_u,$$

or

$$-f_1\left(\frac{\lambda_2}{\lambda_1 + \lambda_2\,u}x\partial_x + \frac{\lambda_2}{\lambda_1 + \lambda_2\,u}y\partial_y + \partial_u\right), \quad f_2(y\partial_x - y\partial_y), \quad f_3\partial_x, \quad f_4\partial_y$$

(where the f_i's are arbitrary functions of u).

Fig. 1 $u(x, y) : -10 \le x, y \le 10$



The rotation symmetry $-y\partial_x + x\partial_y$ lead to the reduced equation

$$\lambda_1 + \lambda_2 u - \sqrt{2}/\sqrt{\alpha} = 0, \quad \alpha = x^2 + y^2$$

so that

$$u(x, y) = \frac{1}{\lambda_2}\left[\frac{\sqrt{2}}{\sqrt{x^2 + y^2}} - \lambda_1\right]. \tag{11}$$

A profile of the solution for some choices of the parameters of (11) is given in Fig. 1, i.e.,

The translation symmetries do not produce nontrivial solutions.

2.3 The Time Evolution 'Denoising Pde'

In the parabolic equation with 'time' as an evolution parameter, we obtain the evolution equation

$$u_t + \lambda_1 + \lambda_2 u - \frac{u_{xx}{u_y}^2 - 2u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}} = 0 \tag{12}$$

whose algebra of Lie symmetries is generated by

$$\begin{aligned}
&\partial_t, && \partial_x, \quad \partial_y, \\
&y\partial_x - x\partial_y, && e^{-\lambda_2 t}\partial_u, \quad -x\partial_x - y\partial_y + \frac{\lambda_1 + \lambda_2 u}{\lambda_2}\partial_u, \\
&e^{-\lambda_2 t}\partial_t - \lambda_2 u e^{-\lambda_2 t}\partial_u.
\end{aligned}$$

Here, the rotation symmetry lead to, again, $\alpha = x^2 + y^2$ and $u = v(t, \alpha)$ so that

$$v_t + \lambda_1 + \lambda_2 v = \sqrt{2}/\sqrt{\alpha}$$

which is a first order linear pde. By the method of invariants, it can be shown that

$$u(x, y, t) = -\frac{1}{\lambda_2}\left[\lambda_1 - \frac{\sqrt{2}}{\sqrt{x^2 + y^2}}\right]e^{-\lambda_2 t}. \tag{13}$$

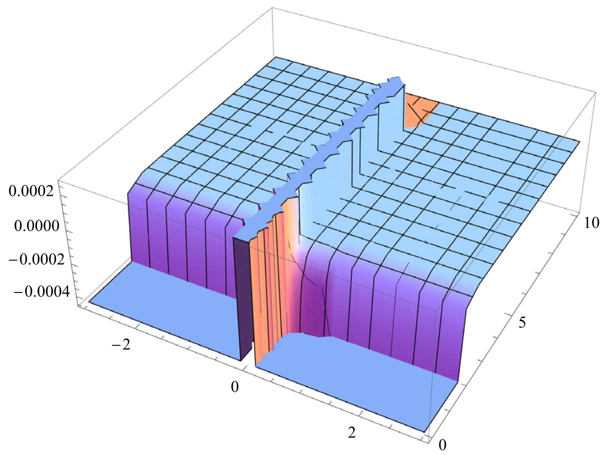


Fig. 2 $u(x,t): -3 \le x \le 3$, $0 \le t \le 10$

A projection of the solution in (u, x, t) $(y = 0)$ system for (12) given by (13) has the profile in Fig. 2.

The multiplier approach on (12) leads to a single multiplier $Q = e^{-\lambda_2 t}$ with the corresponding conserved 2-form

$$
\begin{aligned}
&\frac{1}{t^2\lambda_2{}^3(u_x{}^2+u_y{}^2)^{3/2}}\big[2u_{yy}u_x{}^2 - 4u_xu_yu_{xy} + 2u_{xx}u_2{}^2 - 2\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_x{}^2\\
&-2\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 - 2\mathrm{e}^{\lambda_2 t}u_{yy}u_x{}^2 - \mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_{yy}u_x{}^2 + 2\mathrm{e}^{\lambda_2 t}\lambda_2 t u_{yy}u_x{}^2\\
&-4\mathrm{e}^{\lambda_2 t}\lambda_2 t u_xu_yu_{xy} + 2\mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_xu_yu_{1,2} + 4\mathrm{e}^{\lambda_2 t}u_xu_yu_{xy} + 2\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_x{}^2\\
&+2\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 - \mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_{xx}u_y{}^2 - 2\mathrm{e}^{\lambda_2 t}u_{xx}u_y{}^2 - 2\mathrm{e}^{\lambda_2 t}\lambda_1 t\lambda_2\sqrt{u_x{}^2+u_2{}^2}u_x{}^2\\
&-2\mathrm{e}^{\lambda_2 t}\lambda_1 t\lambda_2\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 + 2\mathrm{e}^{\lambda_2 t}\lambda_2 t u_{xx}u_y{}^2 + \mathrm{e}^{\lambda_2 t}\lambda_1 t^2\lambda_2{}^2\sqrt{u_x{}^2+u_y{}^2}u_x{}^2\\
&+\mathrm{e}^{\lambda_2 t}\lambda_1 t^2\lambda_2{}^2\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 + u\mathrm{e}^{\lambda_2 t}t^2\lambda_2{}^3\sqrt{u_x{}^2+u_y{}^2}u_x{}^2\\
&+u\mathrm{e}^{\lambda_2 t}t^2\lambda_2{}^3\sqrt{u_x{}^2+u_y{}^2}u_y{}^2\big]\mathrm{d}x\mathrm{d}y\\
&-\frac{y}{t^3\lambda_2{}^3(u_x{}^2+u_y{}^2)^{3/2}}\big[2u_{yy}u_x{}^2 - 4u_xu_yu_{xy} + 2u_{xx}u_2{}^2 - 2\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_x{}^2\\
&-2\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 - 2\mathrm{e}^{\lambda_2 t}u_{yy}u_x{}^2 - \mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_{yy}u_x{}^2 + 2\mathrm{e}^{\lambda_2 t}\lambda_2 t u_{yy}u_x{}^2\\
&-4\mathrm{e}^{\lambda_2 t}\lambda_2 t u_xu_yu_{xy} + 2\mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_xu_yu_{1,2} + 4\mathrm{e}^{\lambda_2 t}u_xu_yu_{xy} + 2\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{u_x{}^2+u_2{}^2}u_x{}^2\\
&+2\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 - \mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_{xx}u_y{}^2 - 2\mathrm{e}^{\lambda_2 t}u_{xx}u_y{}^2 - 2\mathrm{e}^{\lambda_2 t}\lambda_1 t\lambda_2\sqrt{u_x{}^2+u_2{}^2}u_x{}^2\\
&-2\mathrm{e}^{\lambda_2 t}\lambda_1 t\lambda_2\sqrt{u_x{}^2+u_y{}^2}u_y{}^2 + 2\mathrm{e}^{\lambda_2 t}\lambda_2 t u_{xx}u_y{}^2 + \mathrm{e}^{\lambda_2 t}\lambda_1 t^2\lambda_2{}^2\sqrt{u_x{}^2+u_y{}^2}u_x{}^2\\
&+\mathrm{e}^{\lambda_2 t}\lambda_1 t^2\lambda_2{}^2\sqrt{u_x{}^2+u_y{}^2}u_y{}^2\big]\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$+\frac{x}{t^3\lambda_2{}^3(u_x{}^2+u_y{}^2)^{3/2}}\big[2u_{yy}u_x{}^2-4u_xu_yu_{xy}+2u_{xx}u_2{}^2-2\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_x{}^2$$

$$-2\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_y{}^2-2\mathrm{e}^{\lambda_2 t}u_{yy}u_x{}^2-\mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_{yy}u_x{}^2+2\mathrm{e}^{\lambda_2 t}\lambda_2tu_{yy}u_x{}^2$$

$$-4\mathrm{e}^{\lambda_2 t}\lambda_2tu_xu_yu_{xy}+2\mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_xu_yu_{1,2}+4\mathrm{e}^{\lambda_2 t}u_xu_yu_{xy}+2\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{u_x{}^2+u_2{}^2}u_x{}^2$$

$$+2\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{u_x{}^2+u_y{}^2}u_y{}^2-\mathrm{e}^{\lambda_2 t}\lambda_2{}^2t^2u_{xx}u_y{}^2-2\mathrm{e}^{\lambda_2 t}u_{xx}u_y{}^2-2\mathrm{e}^{\lambda_2 t}\lambda_1t\lambda_2\sqrt{u_x{}^2+u_2{}^2}u_x{}^2$$

$$-2\mathrm{e}^{\lambda_2 t}\lambda_1t\lambda_2\sqrt{u_x{}^2+u_y{}^2}u_y{}^2+2\mathrm{e}^{\lambda_2 t}\lambda_2tu_{xx}u_y{}^2+\mathrm{e}^{\lambda_2 t}\lambda_1t^2\lambda_2{}^2\sqrt{u_x{}^2+u_y{}^2}u_x{}^2$$

$$+\mathrm{e}^{\lambda_2 t}\lambda_1t^2\lambda_2{}^2\sqrt{u_x{}^2+u_y{}^2}u_y{}^2\big]\mathrm{d}y\mathrm{d}t$$

in which the term $T^t\mathrm{d}x\mathrm{d}y$ leads to the 'conserved density'.

3 Conclusions

We have shown that the pdes that arise in the various models of noise removal algorithms are abundant in symmetries that leave the systems invariant. These have applications in, inter alia, the reduction of the systems.

References

1. Rudin, L.I., Osher, S., Fatemi, E.: Nonlinear total variation based noise removal algorithms. Phys. D, Nonlinear Phenom. **60**, 259–268 (1992)
2. Esedolu, S., Osher, S.J.: Decomposition of images by the anisotropic Rudin-Osher-Fatemi model. Commun. Pure Appl. Math. **57**, 1609–1626 (2004)
3. Osher, S., Solé, A., Vese, L.: Image decomposition and restoration using total variation minimization and the H^{-1}-norm. Multiscale Model. Simul. **1**, 349–370 (2003)
4. Vese, L.A., Osher, S.J.: Modeling textures with total variation minimization and oscillating patterns in image processing. J. Sci. Comput. **19**, 553–572 (2003)
5. Shi, J., Osher, S.: A nonlinear inverse scale space method for a convex multiplicative noise model. SIAM J. Imaging Sci. **1**, 294–321 (2008)
6. Chang, S.G., Yu, B., Vetterli, M.: Adaptive wavelet thresholding for image denoising and compression. IEEE Trans. Image Process. **9**, 1532–1546 (2000)
7. Elad, M., Aharon, M.: Image denoising via sparse and redundant representations over learned dictionaries. IEEE Trans. Image Process. **15**, 3736–3745 (2006)
8. Mahmoudi, M., Sapiro, G.: Fast image and video denoising via nonlocal means of similar neighborhoods. IEEE Signal Process. Lett. **12**, 839–842 (2005)
9. Portilla, J., Strela, V., Wainwright, M.J., Simoncelli, E.P.: Image denoising using scale mixtures of Gaussians in the wavelet domain. IEEE Trans. Image Process. **12**, 1338–1351 (2003)
10. Noether, E.: Nachr. Akad. Wiss. Gött. Math.-Phys. Kl. **2**, 235 (1918), English translation in Transp. Theory Stat. Phys. **1**, 186 (1971)
11. Anderson, I.M., Pohjanpelt, o.J.: The cohomology of invariant variational bicomplexes. In: Geometric and Algebraic Structures in Differential Equations, pp. 3–19. Springer, Amsterdam (1995)
12. Olver, P.: Application of Lie Groups to Differential Equations. Springer, New York (1986)
13. Ibragimov, N.H. (ed.): CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1. CRC Press, Boca Raton (1996)
14. Stephani, H.: Differential Equations: Their Solution Using Symmetries. Cambridge Univ. Press, Cambridge (1989)