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On New Filters in Ordered Semigroups

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Abstract: Ordered semigroups are understood through their subsets. The aim of this article is to study ordered semigroups through some new substructures. In this regard, quasi-filters and (m,n)-quasi-filters of ordered semigroups are introduced as new types of filters. Some properties of the new concepts are investigated, different examples are constructed, and the relations between quasi-filters and quasi-ideals as well as between (m,n)-quasi-filters and (m,n)-quasi-ideals are discussed.

Keywords: ordered semigroup; (m,n)-quasi-ideal; quasi-filter; (m,n)-quasi-filter; generalized (m,n)-quasi-ideal

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1. Introduction and Preliminaries

Kehayopulu [1] was the first to investigate filters in *poe*-semigroups. Lee et al. [2] introduced and described the notion of left (resp. right) filters in *po*-semigroups in terms of prime right (resp. left) ideals. The notion of Γ-filters in ordered Γ-semigroups was developed by Hila [3], while Tang et al. [4] proposed the concept of filters in ordered semihypergroups. Khan et al. [5] introduced the notions of left-m-filters, right-n-filters, and (m,n)-filters in ordered semigroups as a generalization of the concept of left (right) filters of ordered semigroups. Fuzzy set theory was applied to filters of ordered semigroups by Kehayopulu and Tsingelis [6], and the notion of fuzzy filters in ordered semigroups was established. By generalizing the notion of fuzzy filters, Davvaz et al. [7] established the concept of $(\in,\in\vee q)$-fuzzy filters in ordered semigroups. Ali [8] developed generalized rough approximations for fuzzy filters in ordered semigroups; in addition, in [9], Ali et al. proposed the notion of soft filters in soft ordered semigroups.

As novel forms of filters and in continuation of the work initiated in this regard, quasi-filters and (m,n)-quasi-filters of ordered semigroups are introduced herein. Some new concepts and characteristics are studied. Furthermore, relationships between quasi-filters (resp. (m,n)-quasi-filters) and quasi-ideals (resp. (m,n)-quasi-ideals) are discussed.

An ordered semigroup $(\Omega,\cdot,\leq)$ is a semigroup with a partial order relation $\leq$ that is compatible, i.e., $\vartheta\leq\gamma$ implies $\vartheta\kappa\leq\gamma\kappa$ and $\kappa\vartheta\leq\kappa\gamma$ for all $\vartheta,\gamma,\kappa\in\Omega$. For $\Upsilon\neq\emptyset\subseteq\Omega$, we denote $(\Upsilon]=\{t\in\Omega\ :\ t\leq a \text{ for some } a\in\Upsilon\}$ and $[\Upsilon)=\{t\in\Omega\ :\ a\leq t \text{ for some } a\in\Upsilon\}$.

Ordered semigroups have been studied through their subsets (see [5,10–13].) A subset $\Upsilon\neq\emptyset$ of Ω is called a *subsemigroup* of Ω if $\Upsilon\Upsilon\subseteq\Upsilon$, and Υ is called the *left (resp. right) ideal* of Ω if $\Omega\Upsilon\subseteq\Upsilon$ ($\Upsilon\Omega\subseteq\Upsilon$) and $(\Upsilon]\subseteq\Upsilon$. If a subset Υ is both a left ideal and a right ideal of

Ω, it is called an *ideal* of Ω. A subsemigroup Υ of Ω is called a *bi-ideal* of Ω if $\Upsilon\Omega\Upsilon \subseteq \Upsilon$ and $(\Upsilon] \subseteq \Upsilon$. A non-empty subset Υ of Ω is called a *quasi-ideal* of Ω if $(\Upsilon\Omega] \cap (\Omega\Upsilon] \subseteq \Upsilon$ and $(\Upsilon] \subseteq \Upsilon$. Furthermore, a subsemigroup Υ of Ω is called a left filter (resp. right filter) of Ω if for all $a, b \in \Omega$, $ab \in \Upsilon$ implies $a \in \Upsilon$ (resp. $b \in \Upsilon$) and $[\Upsilon) \subseteq \Upsilon$. It is a filter if it is both a left filter and a right filter of Ω. For positive integers m and n, a subsemigroup Q of Ω is an (m, n)-quasi-ideal of Ω of $(Q] \subseteq Q$ and $(Q^m\Omega] \cap (\Omega Q^n] \subseteq Q$.

An ordered semigroup Ω is called (m, n)*-regular* if for all $\vartheta \in \Omega$, there exists $\gamma \in \Omega$ such that $\vartheta \leq \vartheta^m\gamma\vartheta^n$. For more related details, we refer to [14].

2. Quasi-Filters of Ordered Semigroups: Redefined

In [15], Jirojkul and Chinram introduced quasi-filters of ordered semigroups. In addition, in [16], Yaqoob and Tang used a similar definition to introduce quasi-hyperfilters. Their definition was based on a non-general definition of a quasi-ideal. In this section, we redefine quasi-filters of ordered semigroups in a more general way. Furthermore, we explore some of their properties and relate them to quasi-ideals.

Definition 1. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q \neq \emptyset \subseteq \Omega$. Then, Q is a quasi-filter of Ω if the following conditions hold for all $\alpha, \beta, \gamma, \delta \in \Omega$.*

(1) $Q \cdot Q \subseteq Q$;

(2) $[Q) \subseteq Q$;

(3) *If $x \in Q$ and for some $\alpha, \beta, \gamma, \delta \in \Omega$, $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$, then, $\{\alpha, \delta\} \cap Q \neq \emptyset$ and $\{\beta, \gamma\} \cap Q \neq \emptyset$.*

If we drop the subsemigroup condition in Definition 1, we obtain Q as a *generalized quasi-filter* of Ω.

Remark 1. *A quasi-filter Q in a semigroup Ω is a subsemigroup of Ω satisfying the following condition for all $\alpha, \beta, \gamma, \delta \in \Omega$.*

$$\alpha \cdot \beta = \gamma \cdot \delta \in Q \text{ implies } \{\alpha, \delta\} \cap Q \neq \emptyset \text{ and } \{\beta, \gamma\} \cap Q \neq \emptyset.$$

Example 1. *Let $(\mathbb{Z}^+ \cup \{0\}, \cdot, \leq)$ be the semigroup of non-negative integers under standard multiplication and the usual order of numbers. Then, $\mathbb{Z}^+$ is a proper quasi-filter of $\mathbb{Z}^+ \cup \{0\}$.*

Example 2. *Consider the ordered semigroup $\Omega_1 = \{\vartheta, \kappa, \varpi\}$, with operation "$\cdot_1$" and order "$\leq_1$" described as follows:*

$\cdot_1$	ϑ	κ	ϖ
ϑ	ϑ	ϑ	ϑ
κ	κ	κ	κ
ϖ	ϖ	ϖ	ϖ

$$\leq_1 := \{(\vartheta, \vartheta), (\kappa, \kappa), (\gamma, \gamma), (\vartheta, \kappa), (\vartheta, \varpi)\}.$$

One can easily see that $\{\kappa, \varpi\}$ is a quasi-filter of Ω_1.

Example 3. *Let $(M_2(\mathbb{Z}), \cdot, \leq_t)$ be the ordered semigroup of two by two matrices with integer coefficients under the standard multiplication of matrices and trivial order. Then, $\mho = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} :$ $(b, c, d) \neq (0, 0, 0)\}$ is a generalized quasi-filter of $M_2(\mathbb{Z})$, and it is not a quasi-filter of $M_2(\mathbb{Z})$ as $\mho^2$ is not a subset of $\mho$. This is clear as $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \notin \mho$.*

Now, to show that $\mho$ is a generalized quasi-filter of $M_2(\mathbb{Z})$, it suffices to show that for all $\alpha, \beta, \gamma, \delta \in M_2(\mathbb{Z})$, if $\alpha\beta = \gamma\delta \in \mho$, thus, $\{\alpha, \delta\} \cap \mho \neq \emptyset$ and $\{\beta, \gamma\} \cap \mho \neq \emptyset$. Without *loss of generality, suppose that* $\{\alpha, \delta\} \cap \mho = \emptyset$. *Then, there exist* $a, b \in \mathbb{Z}$ *with* $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\delta = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$.

Let $\beta = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$ *and* $\gamma = \begin{pmatrix} g & h \\ i & j \end{pmatrix}$. *Having* $\alpha\beta = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} gb & 0 \\ ib & 0 \end{pmatrix} = \gamma\delta$ *implies that* $ad = ib = 0$. *The latter implies that* $\alpha\beta = \gamma\delta \notin \mho$.

Lemma 1. *Let* $(\Omega, \cdot, \leq)$ *be an ordered semigroup and* $Q \neq \emptyset \subseteq \Omega$. *If* Q *is a (generalized) quasi-filter of* Ω, *then,* Q *is prime.*

Proof. Let $\alpha, \beta \in \Omega$ with $\alpha \cdot \beta \in Q$. By setting $x = \alpha \cdot \beta$, we see that $x \in Q$ satisfies $x \leq \alpha \cdot \beta$. Having Q a (generalized) quasi-filter of Ω implies that $\{\alpha, \beta\} \cap Q \neq \emptyset$ and, hence, $\alpha \in Q$ or $\beta \in Q$. □

Lemma 2. *Let* $(\Omega, \cdot, \leq)$ *be an ordered semigroup,* $p_1, p_2, \ldots, p_k \in \Omega$ *and* $Q \neq \emptyset \subseteq \Omega$ *be a (generalized) quasi-filter of* Ω. *If* $p_1 p_2 \ldots p_k \in Q$, *then,* $\{p_1, p_2, \ldots, p_k\} \cap Q \neq \emptyset$.

Proof. Let $p_1(p_2 \ldots p_k) \in Q$. Having Q be a quasi-filter of Ω implies that Q is prime (by Lemma 1) and, hence, $p_1 \in Q$ or $p_2 \ldots p_k \in Q$. If $p_1 \in Q$, we are finished. Otherwise, $p_2(p_3 \ldots p_k) \in Q$ implies that $p_2 \in Q$ or $p_3 \ldots p_k \in Q$. If $p_2 \in Q$, we are finished. Otherwise, $p_3 \ldots p_k \in Q$. Repeating the same procedure, we see that $\{p_1, p_2, \ldots, p_k\} \cap Q \neq \emptyset$. □

Proposition 1. *Let* $(\Omega, \cdot, \leq)$ *be a commutative ordered semigroup and* $Q \neq \emptyset \subseteq \Omega$. *Then,* Q *is a quasi-filter of* Ω *if and only if* Q *is a filter of* Ω.

Proof. Let Q be a filter of Ω and $\alpha, \beta, \gamma, \delta \in \Omega$. Let $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$, with $x \in Q$. Then, $\alpha \cdot \beta \in Q$ and $\gamma \cdot \delta \in Q$ as $[Q) \subseteq Q$. Having Q be a filter of Ω implies that $\alpha, \beta, \gamma, \delta \in Q$ and, hence, Q is a quasi-filter of Ω.

Conversely, let Q be a quasi-filter of Ω and $\alpha \cdot \beta \in Q$. By setting $x = \alpha \cdot \beta$ and $x \in Q$, we see that $x \leq \alpha \cdot \beta$ and $x \leq \beta \cdot \alpha$. aving Q be a quasi-filter of Ω implies that $\{\alpha\} \cap Q \neq \emptyset$ and $\{\beta\} \cap Q \neq \emptyset$. The latter implies that $\{\alpha, \beta\} \subseteq Q$ and, hence, Q is a filter of Ω. □

Proposition 2. *Let* $(\Omega, \cdot, \leq)$ *be an ordered semigroup and* $Q \neq \emptyset \subseteq \Omega$. *If* Q *is a left filter (right filter) of* Ω, *then,* Q *is a quasi-filter of* Ω.

Proof. Let Q be a left filter of Ω and $\alpha, \beta, \gamma, \delta \in \Omega$. Let $x \in Q$ with $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$. Having Q be a left filter of Ω implies that $\alpha, \gamma \in Q$ and, hence, $\{\alpha, \delta\} \cap \Omega \neq \emptyset$, and $\{\beta, \gamma\} \cap \Omega \neq \emptyset$. Therefore, Q is a quasi-filter of Ω. The case Q is a right filter of Ω can be handled similarly. □

Proposition 3. *Let* $(\Omega, \cdot, \leq)$ *be an ordered semigroup and* $Q, F \neq \emptyset \subseteq \Omega$. *If* Q *is a (generalized) quasi-filter of* Ω, *and* F *is a filter of* Ω, *then,* $Q \cap F$ *is either empty or a (generalized) quasi-filter of* Ω.

Proof. Let $x \in Q \cap F \neq \emptyset$. One can easily see that $Q \cap F$ is a subsemigroup of Ω and that $[Q \cap F) \subseteq Q \cap F$. Suppose that there exist $\alpha, \beta, \gamma, \delta \in \Omega$ with $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$. Having $x \in F$ and $[F) \subseteq F$ implies that $\alpha \cdot \beta \in F, \gamma \cdot \delta \in F$ and, hence, $\{\alpha, \beta, \gamma, \delta\} \subseteq F$. In addition, having Q be a quasi-filter of Ω implies that $\{\alpha, \delta\} \cap Q \neq \emptyset$ and $\{\beta, \gamma\} \cap Q \neq \emptyset$. We see now that $\{\alpha, \delta\} \cap (Q \cap F) \neq \emptyset$ and $\{\beta, \gamma\} \cap (Q \cap F) \neq \emptyset$. Therefore, $Q \cap F$ is a quasi-filter of Ω. □

Lemma 3. *Let* $(\Omega, \cdot, \leq)$ *be an ordered semigroup and* $Q_1, Q_2 \neq \emptyset \subseteq \Omega$ *be (generalized) quasi-filters of* Ω. *Then,* $Q_1 \cup Q_2$ *is a generalized quasi-filter of* Ω.

Proof. One can easily see that $[Q_1 \cup Q_2) \subseteq Q_1 \cup Q_2$. Let $x \in Q_1 \cup Q_2$, with $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$ for some $\alpha, \beta, \gamma, \delta \in \Omega$. We have two cases: $x \in Q_1$ and $x \in Q_2$. We deal with the case $x \in Q_1$, and the case $x \in Q_2$ is handled similarly. Since Q_1 is a (generalized) quasi-filter of Ω, it follows that $\{\alpha, \delta\} \cap Q_1 \neq \emptyset$ and $\{\beta, \gamma\} \cap Q_1 \neq \emptyset$. The latter implies that $\{\alpha, \delta\} \cap (Q_1 \cup Q_2) \neq \emptyset$ and $\{\beta, \gamma\} \cap (Q_1 \cup Q_2) \neq \emptyset$ and, hence, $Q_1 \cup Q_2$ is a generalized quasi-filter of Ω. $\square$

Lemma 4. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q_1, Q_2 \neq \emptyset \subseteq \Omega$ be quasi-filters of Ω. Then, $Q_1 \cup Q_2$ is a quasi-filter of Ω if and only if $Q_1 \cup Q_2$ is a subsemigroup of Ω.*

Proof. The proof can be easily executed using Lemma 3. $\square$

Lemma 5. *Let $(\Omega, \cdot, \leq)$ be an ordered monoid with identity 1 and $Q \neq \emptyset \subseteq \Omega$ be a (generalized) quasi-filter of Ω. Then, $1 \in Q$.*

Proof. Let $x \in Q \neq \emptyset$; then, $x \leq x \cdot 1$ and $x \leq 1 \cdot x$. Since Q is a (generalized) quasi-filter of Ω, it follows that $1 \in Q$. $\square$

Corollary 1. *Let $(\Omega, \cdot, \leq)$ be an ordered group. Then, $Q \neq \emptyset \subseteq \Omega$ is a (generalized) quasi-filter of Ω if and only if $Q = \Omega$.*

Proof. Let $Q \neq \emptyset$ be a (generalized) quasi-filter of Ω. Lemma 5 asserts that $1 \in Q$. Having $1 = x \cdot x^{-1}$ for all $x \in \Omega$ implies that $1 \leq x \cdot x^{-1}$ and $1 \leq x^{-1} \cdot x$. Since Q is a (generalized) quasi-filter of Ω, it follows that $x \in Q$ and, hence, $Q = \Omega$. $\square$

Proposition 4. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup with $0 \in \Omega$ satisfying $0 \cdot \theta = \theta \cdot 0 = 0$ for all $\theta \in \Omega$ and Q be a (generalized) quasi-filter of Ω. If $0 \in Q$, then, $Q = \Omega$.*

Proof. For all $x \in \Omega$, we have $0 \leq 0 \cdot x$ and $0 \leq x \cdot 0$. Having Q be a (generalized) quasi-filter of Ω implies that $x \in Q$ and, hence, $Q = \Omega$. $\square$

Lemma 6. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q \neq \emptyset$ be a proper subset of Ω. Then, $[Q) \subseteq Q$ if and only if $(\Omega - Q] \subseteq \Omega - Q$.*

Lemma 7. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q \neq \emptyset$ be a proper subset of Ω. Then, Q is a subsemigroup of Ω if and only if $\Omega - Q$ is a prime subset of Ω.*

In [2], Lee SK and Lee SS proved that a non-empty proper subset F of Ω was a left (right) filter of Ω if and only if $\Omega - F$ was a right (left) ideal of Ω. The following theorem presents a similar result regarding quasi-filters.

Theorem 1. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q \neq \emptyset$ be a proper subset of Ω. Then, Q is a quasi-filter of Ω if and only if $\Omega - Q$ is a prime quasi-ideal of Ω.*

Proof. Let Q be a quasi-filter of Ω and $x \in ((\Omega - Q)\Omega] \cap (\Omega(\Omega - Q])$. If $x \notin \Omega - Q$, then, $x \in Q$ and, hence, there exist $\alpha, \delta \in \Omega - Q, \beta, \gamma \in \Omega$ such that $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$. Having Q be a quasi-filter of Ω implies that $\{\alpha, \delta\} \cap Q \neq \emptyset$ and $\{\beta, \gamma\} \cap Q \neq \emptyset$. The latter contradicts the fact that $\{\alpha, \delta\} \subseteq \Omega - Q$. Lemmas 6 and 7 complete the proof.

Conversely, suppose that $\Omega - Q$ is a quasi-ideal of Ω, and let $\alpha, \beta, \gamma, \delta \in \Omega$. Let $x \leq \alpha\beta$ and $x \leq \gamma \cdot \delta$, with $x \in Q$. Then, $\{\alpha, \delta\} \cap Q = \emptyset$ or $\{\beta, \gamma\} \cap Q \neq \emptyset$. Otherwise, $x \in ((\Omega - Q)\Omega] \cap (\Omega(\Omega - Q]) \subseteq \Omega - Q$ contradicts the fact that $x \in Q$. Thus, Q is a quasi-filter of Ω. Lemmas 6 and 7 complete the proof. $\square$

Corollary 2. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup. Then, Ω has no proper quasi-filters if and only if Ω has no proper prime quasi-ideals.*

Proof. The proof is an immediate consequence of Theorem 1. □

Remark 2. *An ordered semigroup may have proper quasi-ideals but still has no proper quasi-filters. See Example 4.*

Example 4. *Let $\Omega_2 = \{0, \mathring{a}\}$, with operation "$\cdot_2$" and order "$\leq_2$" described as follows:*

$\cdot_2$	0	$\mathring{a}$
0	0	0
$\mathring{a}$	0	0

$$\leq_2 := \{(0,0), (0,\mathring{a}), (\mathring{a},\mathring{a})\}.$$

It is clear that Ω_2 is an ordered semigroup, and that $\{0\}$ is a proper quasi-ideal of Ω_2. Moreover, Ω_2 has no proper quasi-filters.

Theorem 2. *Let $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q \neq \emptyset$ be a proper subset of Ω. Then, Q is a generalized quasi-filter of Ω if and only if $\Omega - Q$ is a quasi-ideal of Ω.*

Proof. The proof is similar to that of Theorem 1. □

3. (m,n)-Quasi-Filters of Ordered Semigroups

In this section, we generalize the concept of (generalized) quasi-filters of ordered semigroups to (generalized) (m,n)-quasi-filters of ordered semigroups. Moreover, we present some non-trivial examples of the new concept and relate them to (generalized) (m,n)-quasi-ideals of ordered semigroups.

Throughout this section, m and n are positive integers.

Definition 2. *Let $m, n > 0$ be integers, $(\Omega, \cdot, \leq)$ be an ordered semigroup, and $Q \neq \emptyset \subseteq \Omega$. Then, Q is an (m,n)-quasi-filter of Ω if the following conditions hold for all $p_1, \ldots, p_{m+1}, q_1, \ldots, q_{n+1} \in \Omega$.*

(1) $Q \cdot Q \subseteq Q$;

(2) $[Q) \subseteq Q$;

(3) *If $x \in Q$, $x \leq p_1 \ldots p_m p_{m+1}$ and $x \leq q_1 \ldots q_n q_{n+1}$, then, $\{p_1, \ldots, p_m, q_2, \ldots, q_{n+1}\} \cap Q \neq \emptyset$, and $\{p_2, \ldots, p_{m+1}, q_1, \ldots, q_n\} \cap Q \neq \emptyset$.*

If we drop the subsemigroup condition in Definition 2, we obtain Q as a *generalized (m,n)-quasi-filter*.

Proposition 5. *Let $m > 0$ be an integer, $(\Omega, \cdot, \leq)$ be an ordered semigroup, and $Q \neq \emptyset \subseteq \Omega$ be an (m,m)-quasi filter (generalized (m,m)-quasi filter) of Ω. Then, for all $p_1, \ldots, p_m, p_{m+1} \in \Omega$, $p_1 \ldots p_m p_{m+1} \in Q$ implies that $p_i \in Q$ for some $i \in \{1, \ldots, m+1\}$.*

Proof. Let $x = p_1 \ldots p_m p_{m+1} \in Q$. Then, $x \leq p_1 \ldots p_m p_{m+1} \in Q$ implies that $\{p_1, \ldots, p_m, p_{m+1}\} \cap Q \neq \emptyset$. □

Proposition 6. *Let $m, n > 0$ be integers, $(\Omega, \cdot, \leq)$ be an ordered semigroup and $Q \neq \emptyset \subseteq \Omega$ be a (generalized) quasi-filter of Ω. Then, Q is a (generalized) (m,n)-quasi-filter of Ω.*

Proof. Let $x \in Q$, $x \leq p_1 \ldots p_m p_{m+1}$, and $x \leq q_1 \ldots q_n q_{n+1}$. Having Q be a (generalized) quasi-filter of Ω, $x \leq p_1(p_2 \ldots p_m p_{m+1})$, and $x \leq q_1(q_2 \ldots q_n q_{n+1})$ implies that $\{p_1, q_2 \ldots q_n q_{n+1}\} \cap Q \neq \emptyset$ and $\{q_1, p_2 \ldots p_m p_{m+1}\} \cap Q \neq \emptyset$. We have the following four cases.

Case $p_1, q_1 \in Q$. Having

$$q_1 \in \{q_1, \ldots, q_n, p_2, \ldots p_m, p_{m+1}\} \cap Q \text{ and } p_1 \in \{p_1, \ldots, p_m, q_2, \ldots, q_n, q_{n+1}\} \cap Q$$

implies that

$$\{p_1,\ldots,p_m,q_2,\ldots,q_n,q_{n+1}\}\cap Q\neq\emptyset \text{ and } \{q_1,\ldots,q_n,p_2,\ldots p_m,p_{m+1}\}\cap Q\neq\emptyset.$$

Case $p_1, p_2\ldots p_m p_{m+1}\in Q$. Having $p_2\ldots p_m p_{m+1}\in Q$ and Q be a (generalized) quasi-filter of Ω, we see that $\{p_2,\ldots,p_m,p_{m+1}\}\cap Q\neq\emptyset$ (by Lemma 2). We see now that

$$p_1\in\{p_1,\ldots,p_m,q_2,\ldots,q_{n+1}\}\cap Q \text{ and } \{p_2,\ldots,p_m,p_{m+1}\}\cap Q.$$

The latter implies that

$$\{p_1,\ldots,p_m,q_2,\ldots,q_n,q_{n+1}\}\cap Q\neq\emptyset \text{ and } \{q_1,\ldots,q_n,p_2,\ldots p_m,p_{m+1}\}\cap Q\neq\emptyset.$$

Case $q_1, q_2\ldots q_n q_{n+1}\in Q$. Having $q_2\ldots q_n q_{n+1}\in Q$ and Q a (generalized) quasi-filter of Ω, we see that $\{q_2,\ldots,q_n,q_{n+1}\}\cap Q\neq\emptyset$. We see now that

$$\{q_2,\ldots,q_n,q_{n+1}\}\cap Q\subseteq\{p_1,\ldots,p_m,q_2,\ldots,q_{n+1}\}\cap Q \text{ and } q_1\in\{q_1,\ldots,q_n,p_2,\ldots,p_{m+1}\}.$$

The latter implies that

$$\{p_1,\ldots,p_m,q_2,\ldots,q_n,q_{n+1}\}\cap Q\neq\emptyset \text{ and } \{q_1,\ldots,q_n,p_2,\ldots p_m,p_{m+1}\}\cap Q\neq\emptyset.$$

Case $p_2\ldots p_m p_{m+1}, q_2\ldots q_n q_{n+1}\in Q$. Having $p_2\ldots p_m p_{m+1}, q_2\ldots q_n q_{n+1}\in Q$ and Q a (generalized) quasi-filter of Ω, we see that $\{p_2,\ldots,p_{m+1}\}\cap Q\neq\emptyset$ and $\{q_2,\ldots,q_{n+1}\}\cap Q\neq\emptyset$. We see now that

$$\{q_2,\ldots,q_n,q_{n+1}\}\cap Q\subseteq\{p_1,\ldots,p_m,q_2,\ldots,q_{n+1}\}\cap Q$$

and

$$\{p_2,\ldots,p_{m+1}\}\cap Q\subseteq\{q_1,\ldots,q_n,p_2,\ldots,p_{m+1}\}\cap Q.$$

The latter implies that

$$\{p_1,\ldots,p_m,q_2,\ldots,q_n,q_{n+1}\}\cap Q\neq\emptyset \text{ and } \{q_1,\ldots,q_n,p_2,\ldots p_m,p_{m+1}\}\cap Q\neq\emptyset. \quad \square$$

Remark 3. *An (m,n)-quasi-filter of an ordered semigroup may fail to be a quasi-filter. See Example 5.*

Example 5. *Let $\Omega_3=\{\dot{a},\dot{b},\dot{c},\dot{d}\}$, with operation "$\cdot_3$" and order relation "$\leq_3$" described as follows:*

$\cdot_3$	$\dot{a}$	$\dot{b}$	$\dot{c}$	$\dot{d}$
$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$	$\dot{a}$
$\dot{b}$	$\dot{b}$	$\dot{b}$	$\dot{b}$	$\dot{b}$
$\dot{c}$	$\dot{c}$	$\dot{c}$	$\dot{c}$	$\dot{c}$
$\dot{d}$	$\dot{a}$	$\dot{b}$	$\dot{b}$	$\dot{a}$

$\leq_3:=\{(\dot{a},\dot{a}),(\dot{a},\dot{b}),(\dot{b},\dot{b}),(\dot{b},\dot{c}),(\dot{c},\dot{c}),(\dot{d},\dot{d})\}$. *One can easily see that $(\Omega_3,\cdot_3,\leq_3)$ is an ordered semigroup, and that $\{\dot{b},\dot{c}\}$ is a $(2,2)$-quasi-filter of Ω_3 that is not a quasi-filter of Ω_3. This is clear because $\dot{b}\leq\dot{b}\cdot_3\dot{d}=\dot{d}\cdot_3 b\in\{b,c\}$ and $d\notin\{b,c\}$.*

Theorem 3. *Let $m,n>0$ be integers, $(\Omega,\cdot,\leq)$ be an ordered semigroup, and $Q\neq\emptyset$ be a proper subset of Ω. Then, Q is an (m,n)-quasi-filter of Ω if and only if $\Omega-Q$ is a prime generalized (m,n)-quasi-ideal of Ω.*

Proof. Let Q be an (m,n)-quasi-filter of Ω and $x\in((\Omega-Q)^m\Omega]\cap(\Omega(\Omega-Q)^n]$. Then, there exist $p_1,\ldots,p_m,q_1,\ldots,q_n\in\Omega-Q$, $\alpha,\beta\in\Omega$ such that $x\leq p_1\ldots p_m\alpha$ and $x\leq\beta q_1\ldots q_n$. If $x\notin\Omega-Q$, then, $x\in Q$. Having Q an (m,n)-quasi-filter of Ω implies that $\{p_1,\ldots,p_m,q_1,\ldots,q_n\}\cap Q\neq\emptyset$ and $\{p_2,\ldots,p_m,\alpha,\beta,q_1,\ldots,q_{n-1}\}\cap Q\neq\emptyset$. Having

$\{p_1, \ldots, p_m, q_1, \ldots, q_n\} \cap Q \neq \emptyset$ contradicts the fact that $\{p_1, \ldots, p_m, q_1, \ldots, q_n\} \subseteq \Omega - Q$. Lemmas 6 and 7 complete the proof.

Conversely, let $\Omega - Q$ be a prime generalized (m,n)-quasi-filter of Ω and $x \in Q$, with $x \leq p_1 \ldots p_m p_{m+1}$ and $x \leq q_1 \ldots q_n q_{n+1}$ for some $p_1, \ldots, p_m, p_{m+1}, q_1, \ldots, q_n, q_{n+1} \in \Omega$. Suppose that $\{p_1, \ldots, p_m, q_2, \ldots, q_{n+1}\} \cap Q = \emptyset$ or $\{p_2, \ldots, p_{m+1}, q_2, \ldots, q_n\} \cap Q = \emptyset$. Then, $p_1 \ldots p_m p_{m+1} \in ((\Omega - Q)^m \Omega]$, and $q_1 \ldots q_n q_{n+1} \in (\Omega(\Omega - Q)^n]$. Having $\Omega - Q$ be a prime generalized (m,n)-quasi-ideal of Ω implies that $x \in ((\Omega - Q)^m \Omega] \cap (\Omega(\Omega - Q)^n] \subseteq \Omega - Q$, which contradicts the fact that $x \in Q$. Lemmas 6 and 7 complete the proof. □

Theorem 4. *Let $m, n > 0$ be integers, $(\Omega, \cdot, \leq)$ be an ordered semigroup, and $Q \neq \emptyset$ be a proper subset of Ω. Then, Q is a generalized (m,n)-quasi-filter of Ω if and only if $\Omega - Q$ is a generalized (m,n)-quasi-ideal of Ω.*

Proof. The proof is similar to that of Theorem 3. □

Theorem 5. *Let $m, n > 0$ be integers, $(\Omega, \cdot, \leq)$ be an (m,n)-regular semigroup, and $Q \neq \emptyset \subseteq \Omega$. Then, Q is a (generalized) (m,n)-quasi-filter of Ω if and only if Q is a (generalized) quasi-filter of Ω.*

Proof. Let Q be a (generalized) (m,n)-quasi-filter of Ω and $\alpha, \beta, \gamma, \delta \in \Omega$. Since Ω is an (m,n)-regular semigroup, it follows that there exist $z_1, z_2, z_3, z_4 \in \Omega$ such that $\alpha \leq \alpha^m z_1 \alpha^n$, $\beta \leq \beta^m z_2 \beta^n$, $\gamma \leq \gamma^m z_3 \gamma^n$, and $\delta \leq \delta^m z_4 \delta^n$. Let $x \in Q$, with $x \leq \alpha \cdot \beta$ and $x \leq \gamma \cdot \delta$. Then, $x \leq \alpha^m (z_1 \alpha^n \beta^m z_2) \beta^n$, and $x \leq \gamma^m (z_3 \gamma^n \delta^m z_4) \delta^n$. Having Q be a (generalized) (m,n)-quasi-filter of Ω implies that $\{\alpha, \delta\} \cap Q \neq \emptyset$ and $\{\beta, \gamma\} \cap Q \neq \emptyset$ and, hence, Q is a (generalized) quasi-filter of Ω. □

4. Conclusions and Future Directions

This paper introduced new types of filters in ordered semigroups. More precisely, it discussed quasi-filters and (m,n)-quasi-filters of ordered semigroups by exploring their properties and finding their relationships with quasi-ideals and (m,n)-quasi-ideals. The main results were formulated in five theorems, i.e., Theorems 1–5.

For future work, we raise the following ideas:

1. Study quasi-filters in certain special ordered semigroups, such as regular semigroups and intra-regular semigroups.
2. Define quasi-filters and (m,n)-quasi-filters in other ordered algebraic structures, such as ordered semirings.

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