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Exactly solvable new classes of potentials with finite discrete energies

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ABSTRACT

In this work, we propose more realistic models with discrete and finite number of energy levels that could fit well to molecules with potentials that were modeled previously as harmonic oscillator. The considered potentials could be also used as good models in quantum physics, statistical and condensed matter physics, atomic physics, nuclear physics, particle physics, high energy physics, mathematical physics, as well as in chemistry of complex molecules. More precisely, we derive the solutions of two families of Schrödinger equations using supersymmetric quantum mechanics technique for superpotentials having shape invariance properties, and where their eigenvalues and eigenfunctions are exactly determined. The range of their finite number of bound states is given explicitly. Furthermore, this result will contribute in extending the already small list of exactly solvable Schrödinger equations, where we have summarized in a table all well-known potentials having exact solutions and their superpotentials, their partner potentials, and their energies, as well as, the newly discovered potentials proposed here.

1. Introduction

It is almost a century since Schrödinger equation was formulated to describe the wavefunction of a quantum mechanical system, however, the solutions are limited to a list of a few known potentials, and due to this fact all physical quantum systems are modeled [43] with potentials from this list. To overcome this limitation, many techniques were developed to derive approximate solutions of the Schrödinger equation such as variational and perturbation theories, diagram methods, WKB, SWKB and ERS methods [5,20,23,24]. Exact solutions for defined potential are still an ultimate goal in quantum physics and connected fields of science. Supersymmetry is one of the powerful techniques allowing to determine exactly the energy and the wavefunctions of quantum systems with defined superpotentials [6,5].

After Nicolai, who introduced in 1976 supersymmetry in a non-relativistic quantum mechanics, as a generalization of Poincare symmetry, Witten in 1981 [53] introduced “supersymmetric quantum mechanics” and constructed a non-trivial system with a spin-$\frac{1}{2}$ particle moving in 1D. Its elegant way in analyzing the properties of a quantum system and its effectiveness in obtaining exact results, has catapulted the supersymmetry method to the forefront in many field of science, for example, in physics: quantum mechanics, atomic, nuclear, mathematical, statistical, and optics; in chemistry: molecular chemistry, and so on, as well as other fields of science, see [1–54].

Here, we determine the exact expressions of the wavefunctions as well as the eigenvalues for a new class of shape invariant potentials. In particular, we propose two more realistic models that would fit well, for example, to molecules with potentials modeled previously as harmonic oscillator.

We introduce supersymmetric approach in solving Schrödinger equation in Section 2, followed by the shape invariance method in Section 3. The two new potentials and their graphs, their finite bound states eigenvalues, as well as their excited eigenstates are introduced in Section 4. We present numerical approximations of three well known methods in Section 5 and compare them to the exact closed form solutions. A conclusion is given in Section 6.

We would like to mention that in the appendix, we include the conditions on the parameters that determine the exact number of the bound states. In addition, we summarize most of the all known shape invariant exactly solvable potentials which are available in the literature, with their superpotentials, their partner potentials, and their bound state energies, and we complement it by the new two solvable potentials proposed here.

2. Supersymmetry

Given a potential $V_-(x)$, we seek to built a partner potential $V_+(x)$, where these two potentials have the same energy eigenvalues, except for the ground state.

These partner potentials are defined by

$$V_\pm(x) = W^2(x) \pm W'(x) \tag{1}$$

where $W(x)$ is called the superpotential.

The associated Hamiltonian to these partner potentials are given by

$$H_- = A^\dagger A, \qquad H_+ = AA^\dagger \tag{2}$$

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where

$$A^{\dagger} = -\frac{d}{dx} + W(x), \qquad A = \frac{d}{dx} + W(x) \tag{3}$$

The Schrödinger equation with eigenstate E and potential $V_-(x)$ is defined by

$$\left(-\frac{d^2}{dx^2} + V_-(x)\right)\Psi = (A^{\dagger}A)\Psi = E\Psi$$

and the two Hamiltonians are written in the form

$$H_- = -\frac{d^2}{dx^2} + V_-(x), \quad H_+ = -\frac{d^2}{dx^2} + V_+(x), \tag{4}$$

These two Hamiltonians (4) have their eigenvalues (energy levels) and their eigenfunctions (wavefunctions) intertwined. That is, if $E_0^{(-)}$ is an eigenstate of the Hamiltonian H_- and its associated eigenfunction $\Psi_0^{(-)}$, then the Hamiltonian H_+ will have the same eigenstate $E_0^{(-)}$ and its eigenfunction is given by $A\Psi_0^{(-)}$, and vice versa if we change H_+ by H_-.

The two Hamiltonians are both positive semi-definite operators, so their energies are greater than or equal to zero.

For the Hamiltonian H_-, we have

$$H_-\Psi_0^{(-)} = A^{\dagger}A\Psi_0^{(-)} = E_0^{(-)}\Psi_0^{(-)} \tag{5}$$

So, by multiplying on the left side of Eq. (5) by the operator A, we obtain

$$AA^{\dagger}A\Psi_0^{(-)} = E_0^{(-)}(A\Psi_0^{(-)}),$$

so that

$$H_+(A\Psi_0^{(-)}) = E_0^{(-)}(A\Psi_0^{(-)}).$$

Similarly, for H_+

$$H_+\Psi_0^{(+)} = AA^{\dagger}\Psi_0^{(+)} = E_0^{(+)}\Psi_0^{(+)} \tag{6}$$

and multiplying the left side of (6) by $A^{\dagger}$ we obtain

$$H_-(A^{\dagger}\Psi_0^{(+)}) = E_0^{(+)}(A^{\dagger}\Psi_0^{(+)}). \tag{7}$$

The eigenfunctions of the two Hamiltonians and their exact relationships depend on whether the quantity $A\Psi_0^{(-)}$ is zero or nonzero, i.e. if $E_0^{(-)}$ is zero or nonzero, which means an unbroken supersymmetric system or a broken one. For a full discussion of this cases, see the references[49,50].

Thus, here we will consider only the case where $A_-\Psi_0^{(-)} = 0$. In this case, this state has no SUSY partner since the ground state wavefunction of H_- is annihilated by the operator A_-, and in this case $E_0^{(-)} = 0$.

It is then clear that the eigenstates and eigenfunctions of the two Hamiltonians H_- and H_+ are related by (for $n = 0, 1, 2, \ldots$)

$$E_0^{(-)} = 0, \quad E_n^{(+)} = E_{n+1}^{(-)}, \tag{8}$$

$$\Psi_n^{(+)} = (E_{n+1}^{(-)})^{-1/2}A\Psi_{n+1}^{(-)} \tag{9}$$

$$\Psi_{n+1}^{(-)} = (E_n^{(+)})^{-1/2}A^{\dagger}\Psi_n^{(+)}. \tag{10}$$

This process is obtained, as in the case of the harmonic oscillator, by applying the creation and annihilation operators..

By knowing $W(x)$, then the ground state wavefunction $\Psi_0^{(-)}$ can be expressed by

$$\Psi_0^{(-)}(\mathrm{x}) = N\ e^{-\int W(x)dx} \tag{11}$$

where N is the normalized constant, while by knowing the ground state wavefunction $\Psi_0^{(-)}$ the superpotential W can be expressed in the form

$$W(x) = -\frac{d}{dx}\log(\Psi_0^{(-)}(x)) \tag{12}$$

We can see that by normalizing the eigenfunction Ψ_n of H, the wavefunction Ψ_{n+1} is also normalized. Also, the operator A (as well as $A^{\dagger}$) converts an eigenfunction of H_- (H_+) into an eigenfunction of H_+ (H_-) with the same energy.

The bound state wavefunctions must converge to zero at the two ends of its domain interval, therefore the two statements $A\Psi_0^{(-)} = 0$ and $A^{\dagger}\Psi_0^{(+)} = 0$ cannot be satisfied at the same time, only one of the two ground state energies, $E_0^{(-)}$ and $E_0^{(+)}$, can be zero, while the other bound state energy is positive. So, we will define by convention W such that Ψ_0 is normalized with $E_0^{(-)} = 0$, and $E_0^{(+)} > 0$.

3. Shape invariance

We define potentials to be shape invariant if their dependence on x is similar and they only differ on some parameters appearing in their expressions and this similarity is described by the following relation:

$$V_-(x, a_1) + h(a_1) = V_+(x, a_0) + h(a_0) \tag{13}$$

where the parameter a_1 depends on a_0, i.e. $a_1 = f(a_0)$, and then,

$a_2 = f(a_1) = f^2(a_0)$, and by recurrence $a_k = f^k(a_0)$ where $a_0 \in \mathbb{R}^m$, and $f: \mathbb{R}^m \to \mathbb{R}^m$, is called a parameter change function, see [4,10,13,15,26,30].

In shape invariance method, the parameters of the superpotentials are changed, but the partner potentials have similar shapes and differ by a constant.

It is still challenging to find solutions to the Eq. (13) of the shape invariant condition, and only the already known solvable potentials, namely, harmonic oscillator, Coulomb, Morse [42], Rosen-Morse [45], Eckart [22], Pőschl-Teller [44], Scarf [47,41], etc. (see the list in Section B), which were discovered in the 30's and 50's of last century, have this condition satisfied. Indeed, It was noted in [10] that if the parameters are related by a linear translation then all well known exactly solvable potentials in nonrelativistic quantum mechanics can be solved using shape invariance methods.

It was suggested by [5] that there are no other shape invariant potentials found with linear translation of the parameters of the shape invariance condition (13). However, this statement was limited to two free parameters in the superpotential as a linear combination of two functions with the same argument and it was just a conjecture without a supporting proof.

Here, we introduce new scaling parameter for the argument that allowed us to extend this list to more new interesting potentials that could have application in complex molecules and other fields. In the Section B, a table gives a summary of all known potentials with their superpotentials, partner potentials, and energies and their corresponding coefficients a_k obtained using our new technique, followed by the two new potentials proposed in this paper.

All the bound state energies, as well as, the expression of their wavefunctions are given explicitly.

From now on, we will consider the partner potential $V_-(x, a_0)$ as a shape-invariant potential. Therefore, the two potentials $V_-(x, a_1)$ and $V_+(x, a_0)$ have the same dependence on x, up to the change in their parameters, and their Hamiltonians $H_-(x, a_1)$ and $H_+(x, a_0)$ will only differ by a vertical shift given by $C(a_0) = h(a_1) - h(a_0)$,

$$V_+(x, a_0) = V_-(x, a_1) + C(a_0) \tag{14}$$

where the partner potentials are defined by

$$V_-(x, a_1) = W^2(x, a_1) - W'(x, a_1), \tag{15}$$

$$V_+(x, a_0) = W^2(x, a_0) + W'(x, a_0) \tag{16}$$

Their common ground state wavefunction is given by

$$\Psi_0^{(+)}(x, a_0) = \Psi_0^{(-)}(x, a_1) \propto e^{-\int W(x, a_1)} \tag{17}$$

The first excited state $\Psi_1^{(-)}$ of $H_-(x, a_1)$ is given here, where we are omitting the normalization constant,

$$\Psi_1^{(-)}(x, a_0) = A^\dagger(x, a_0)\Psi_0^{(+)}(x, a_0) = A^\dagger(x, a_0)\Psi_0^{(-)}(x, a_1). \tag{18}$$

The eigenvalue associated to this Hamiltonian is

$$E_1^{(-)} = C(a_0) = h(a_1) - h(a_0) \tag{19}$$

The two Hamiltonians H_+ and H_- have the same eigenvalues except for additional zero energy eigenvalue of the lower ladder Hamiltonian H_-. They are related by

$$E_0^{(-)} = 0, \quad E_{n+1}^{(-)} = E_n^{(+)}, \tag{20}$$

$$\Psi_n^{(+)} \propto A\Psi_{n+1}^{(-)}, \quad A^\dagger\Psi_n^{(+)} \propto \Psi_{n+1}^{(-)}, \quad n = 0, 1, 2, \ldots \tag{21}$$

where we have iterated this procedure to construct a hierarchy of Hamiltonians

$$H_\pm^{(n)} = -\frac{d^2}{dx^2} + V_\pm(x, a_n) + \sum_{k=0}^{n-1} C(a_k) \tag{22}$$

and then derive the n^{th} excited eigenfunction and eigenvalues by

$$\Psi_n^{(-)}(x, a_0) \propto A^\dagger(x, a_0)A^\dagger(x, a_1)\ldots A^\dagger(x, a_n)\Psi_0^{(-)}(x, a_n) \tag{23}$$

$$\begin{aligned} E_0^{(-)} &= 0, \\ E_n^{(-)} &= \sum_{k=0}^{n-1} C(a_k) = \sum_{k=0}^{n-1} h(a_{k+1}) - h(a_k) \\ &= h(a_n) - h(a_0), \quad \text{for } n \geqslant 1. \end{aligned} \tag{24}$$

where $a_k = f(f(\ldots f(a_0)) = f^k(a_0)$, $k = 0, 1, 2, \ldots, n-1$.

Therefore, knowing the superpotential not only we know the potential, but also its ground state and from the algorithm above, the whole spectrum of the Hamiltonian H_- (H_+ as well) can be derived by supersymmetry quantum mechanics method.

In this work, we propose to extend this technique by scaling the argument of the superpotential.

4. New classes of exactly solvable potentials

We present two cases of argument scaling that generate new families of exactly solvable Schrödinger equations.

We consider the first potential U_- as follows

$$U_-(x, b, p) = \frac{1}{4}b(b-2p)(\tanh px - 2\tanh 2px)^2 - \frac{1}{2}(3b - 8p)p \tag{25}$$

with $b > 2p > 0$.This potential was obtained by introducing the ansatz

$$W(x, b, p) = \left(-\frac{b}{2} + p\right)\tanh px + b\tanh 2px \tag{26}$$

for the superpotential, with the partner potentials satisfying the shape invariance condition (13),

$$\begin{aligned} U_-(x, b, p) &= -\tfrac{1}{2}(3b - 8p)p + \tfrac{1}{4}b(b-2p)U_1(x), \\ U_+(x, b, p) &= \tfrac{1}{2}(3b + 2p)p + \tfrac{1}{4}b(b-2p)U_1(x), \end{aligned} \tag{27}$$

where $U_1(x)$ is the nonnegative function defined by

$$U_1(x) = (\tanh px - 2\tanh 2px)^2 \tag{28}$$

where it is easily seen that the value $(U_-)_{\min} = -\frac{1}{2}(3b - 8p)p$ is the minimum of $U_-(x)$ attained at $x = 0$.

The first energy level is,

$$E_1^{(-)} = E_0^{(+)} = C(b_0) = c_0^2 - c_1^2, \tag{29}$$

and by recurrence, we have

$$C(b_k) = c_k^2 - c_{k+1}^2, \quad k = 0, 1, 2, \ldots \tag{30}$$

where, $b_0 = b$, $b_k = b - 2kp$ for $k = 1, 2, \ldots$ and

$$c_k = \begin{cases} \frac{1}{2}b + p & \text{if} \quad k = 0 \\ \frac{1}{2}b - (k+1)p & \text{if} \quad k = 1, 2, .. \end{cases} \tag{31}$$

Therefore, for $n \geqslant 1$, the n^{th} bound energy level of the potential partner U_- is given by

$$\begin{aligned} E_n^{(-)} &= \sum_{k=0}^{n-1} C(a_k) = c_0^2 - c_n^2, \\ &= \left(\tfrac{1}{2}b + p\right)^2 - \left(\tfrac{1}{2}b - (n+1)p\right)^2, \; n = 1, 2, .. \end{aligned} \tag{32}$$

The potential U_- therefore has an energy $E_n^{(-)}$ given by

$$E_n^{(-)} = \begin{cases} 0 & \text{if} \quad n = 0 \\ \left(\frac{1}{2}b + p\right)^2 - \left(\frac{1}{2}b - (n+1)p\right)^2 & \text{if} \quad n = 1, 2, \ldots \end{cases} \tag{33}$$

For $n \geqslant 1$, the energy can be written as

$$E_n^{(-)} = p(b - np)(n + 2). \tag{34}$$

We can see that the energy expression of the n^{th} bound state is pseudo-linear in n for small p. The considered potential $U_-(x)$ tends asymptotically as $p \longrightarrow 0$, to $\frac{9}{4}(b - 2p)bp^2x^2 - \frac{1}{2}(3b - 8p)p$, which is a shifted harmonic oscillator.

We also observe that the ground wavefunction expression, obtained using (17) is given by

$$\Psi_0^{(-)}(x, b, p) = \cosh px^{-1+\frac{b_0}{2p}}\cosh 2px^{-\frac{b_0}{2p}}, \tag{35}$$

which is asymptotically convergent to a Gaussian for small values of p, more precisely,

$$\lim_{p \longrightarrow 0} \frac{\Psi_0^{(-)}(x)}{e^{-\frac{1}{4}(3b_0 + 2p)px^2}} = 1. \tag{36}$$

The first excited state

$$\begin{aligned} \Psi_1^{(-)}(x, b_0, p) &= \left(-\frac{d}{dx} + W(x, b_0, p)\right)\Psi_0^{(-)}(x, b_1, p) \\ &= \frac{1}{2}(b_0 - p)\cosh px^{-1+\frac{b_0}{2p}}\cosh 2px^{-\frac{b_0}{2p}}(3\sinh px + \sinh 3px) \end{aligned} \tag{37}$$

and the other eigenfunctions are obtained by recurrence using the formula

$$\Psi_{n+1}(x, b_0, p) = \left(-\tfrac{d}{dx} + W(x, b_0, p)\right)\Psi_n(x, b_1, p), \quad \text{for } n = 1, 2, .. \tag{38}$$

five excited states are plotted in Fig. 2.

We observe from Fig. 2 that the first bound state eigenfunctions of our proposed potential have similar shapes as the harmonic oscillator. This potential could be better used for realistic systems with discrete and finite number of energy levels by choosing suitable parameters.

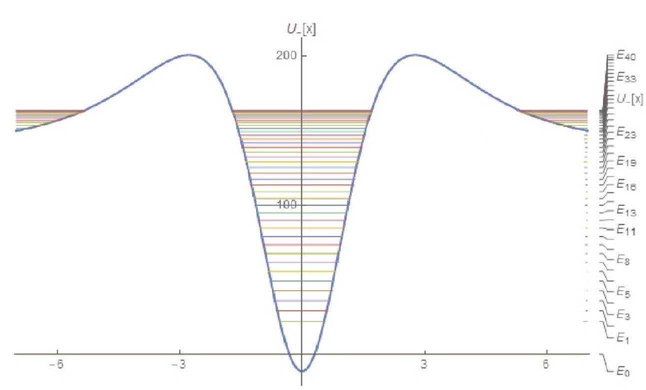

Fig. 1. Potential $U_-[x]$ and energy levels $E_1, \cdots, E_{40}$ for $b = 25$, $p = 0.3$.

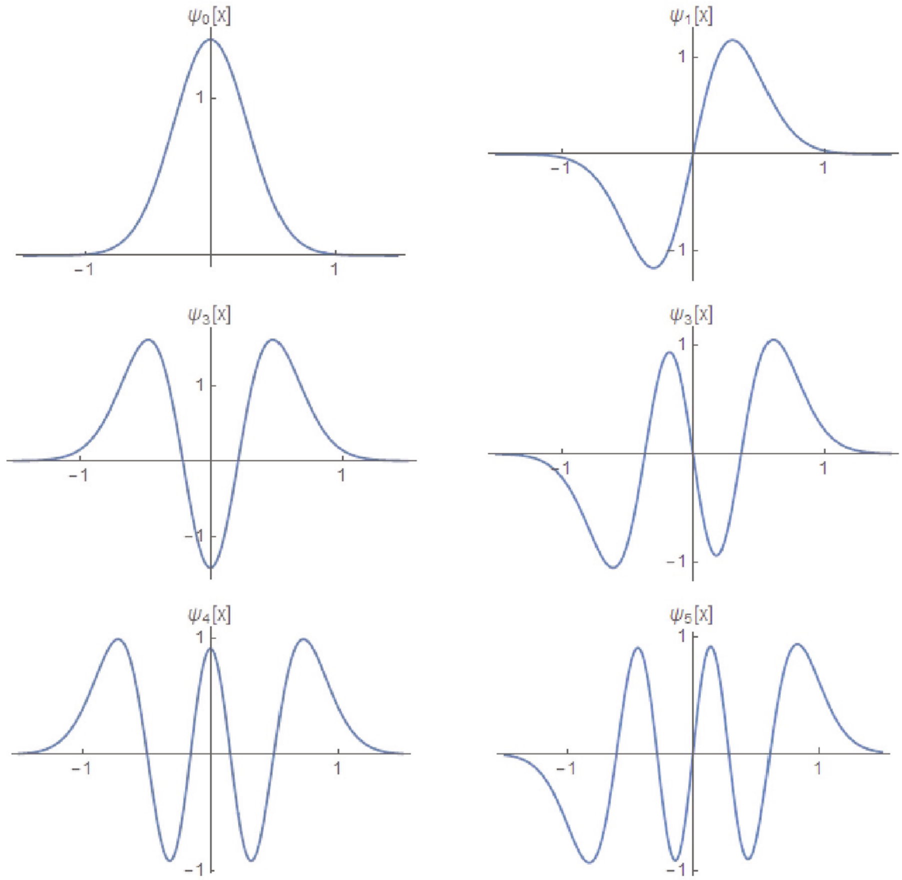

Fig. 2. The ground state and the first 5 excited states of the potential $U_{-}[x]$ with $b = 25$, $p = 0.3$.

Although the harmonic oscillator is useful at the first few energy levels, it fails miserably when the number n, the quantum number, increases by failing to model the atomic bonds and dissociations, and also it exhibits additional bound states that do not match reality. In the potential (25) and its potential partners (27), we can see in Fig. 1 that the first bound states levels match those of the harmonic oscillator, but contrary to the harmonic oscillator which has an infinite number of bound states, this potential has a finite number of state levels. An explication is given below.

The potential (25) has a local maximum of

$$(U_{-})_{\max} = U_{-}(x_{\pm}) = \frac{3}{4}(-3 + 2\sqrt{3})(b - 2p)p + \frac{1}{2}(3b - 8p)p \tag{39}$$

attained at the two symmetric points with x-coordinates given by,

$$x_{\pm} = \frac{\log(1 \pm \sqrt{2}\sqrt[4]{3} + \sqrt{3})}{p}. \tag{40}$$

There is a finite number of discrete bound states n for the nonnegative potential U_{-} which can be computed exactly by considering the following physical constraints on the energies E_n (33):

$$0 < E_n < E_{n+1} \leqslant (U_{-})_{\max}. \tag{41}$$

Since these inequalities are quadratic in n, requiring the study of different cases, hence all possible solutions will be included in the appendix. Here, we will just mention one simples case in which, if $b \geqslant \frac{p}{2}(5 + 3\sqrt{3})(-2 + 3\sqrt{3} + \sqrt{61 - 30\sqrt{3}}) \approx 31.6211\, p$, then, $1 \leqslant n \leqslant \left\lfloor \frac{b-3p}{2p} \right\rfloor = n_{1,\max}$, where $\lfloor . \rfloor$ represents the floor function, which gives the integer part of the argument. Here $n_{1,\max}$ is the maximum number of bound states allowed by the physical constraints.

In Fig. 1, we took the following parameters, $b = 25$, $p = 0.3$, and in this case, the highest bound state is given by,

$$n_{1,\max} = \left\lfloor \frac{b - 3p}{2p} \right\rfloor = 40. \tag{42}$$

The second potential that we propose in this paper is given by

$$V_{-}(x, b, p) = -\frac{1}{4}(15b - 64p)p + \frac{1}{16}b(b - 4p)(\tanh px - 4\tanh 4px)^2 \tag{43}$$

for $b > 4p > 0$, with the potential partners, defined by

$$\begin{aligned} V_{-}(x, b, p) &= -\tfrac{1}{4}(15b - 64p)p + \tfrac{1}{16}b(b - 4p)U_2(x) \\ V_{+}(x, b, p) &= \tfrac{1}{4}(15b + 4p)p + \tfrac{1}{16}b(b - 4p)U_2(x) \end{aligned} \tag{44}$$

where

$$U_2(x) = (\tanh px - 4\tanh 4px)^2 \tag{45}$$

the ansatz used for these superpotentials is

$$W(x, b, p) = \frac{1}{4}(-b + 4p)\tanh px + b\tanh 4px \tag{46}$$

It can be deduced that the value $(V_{-})_{\min} = -\frac{1}{4}(15b - 64p)p$ is the minimum of $V_{-}(x)$ attained at $x = 0$.

The first energy level,

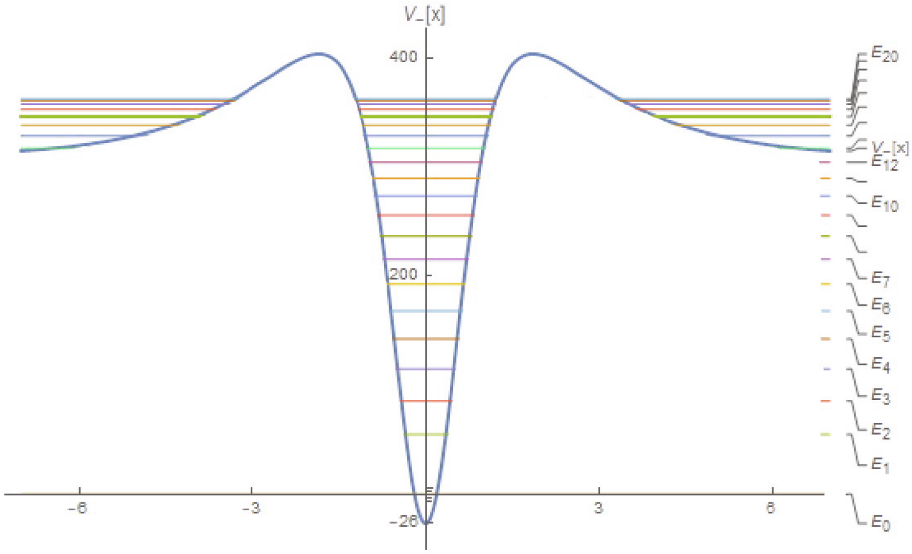

Fig. 3. Potential $V_{-}[x]$, and energy levels E_1, E_2,⋯,E^{20} for $b = 25$, $p = 0.3$.

$$E_1^{(-)} = E_0^{(+)} = C(b_0) = c_0^2 - c_1^2 \tag{47}$$

and by recurrence,

$$C(b_k) = c_k^2 - c_{k+1}^2, \quad k = 0, 1, 2, \ldots \tag{48}$$

where, $b_0 = b$, $b_k = b - 4kp$ and

$$c_k = \begin{cases} \frac{3}{4}b + p & \text{if} \quad k = 0 \\ \frac{3}{4}p - (3k+1)p & \text{if} \quad k = 1, 2, 3, .., n-1. \end{cases} \tag{49}$$

The n^{th} bound energy levels are given by

$$E_n^{(-)} = \sum_{k=0}^{n-1} C(a_k) = c_0^2 - c_n^2$$
$$= \left(\tfrac{3}{4}b + p\right)^2 - \left(\tfrac{3}{4}b - (3n+1)p\right)^2 \tag{50}$$

the energy $E_n^{(-)}$ of V_{-} is then given by

$$E_n^{(-)} = \begin{cases} 0 & \text{if} \quad n = 0 \\ \left(\frac{3b}{4} + p\right)^2 - \left(\frac{3b}{4} - (3n+1)p\right)^2 & \text{if} \quad n = 1, 2, \ldots \end{cases} \tag{51}$$

Therefore, for $n \geqslant 1$, it can be written in a simple form

$$E_n^{(-)} = \frac{3}{2}(3n+2)(b - 2np)p. \tag{52}$$

For the energies (51), we can also observe that their n^{th} bound state is pseudo-linear in n for small p and the considered potential $V_{-}(x)$ (43) tends asymptotically to

$$\frac{225}{16}(b - 4p)bp^2x^2 - \frac{1}{4}(15b - 64p)p, \tag{53}$$

a shifted harmonic oscillator. The unormalized ground state,

$$\Psi_0(x, b, p) = \cosh px^{\frac{b}{4p}} \cosh 4px^{1-\frac{b}{4p}}, \tag{54}$$

is again asymptotically equivalent to a Gaussian, more precisely,

$$\frac{\Psi_0(x)}{e^{-\left(\frac{-15b+64p}{8}\right)px^2}} = 1 + O(x^4) \tag{55}$$

The first excited state

$$\Psi_1(x, b_0, p) = \left(-\frac{d}{dx} + W(x, b_0, p)\right)\Psi_0(x, b_1, p)$$
$$= \frac{1}{4}(b_0 - 2p)\cosh px^{-1+\frac{b_0}{4p}} \cosh 4px^{-\frac{b_0}{4p}}(5\sinh 3px + 3\sinh 5px). \tag{56}$$

and recursively using (38), the other excited states can be derived automatically. We plotted the first six eigenstates in Fig. 4 and observed their similarities to the first model above.

The derivative of the potential (43) is given by

$$V_{-}'(x, b, p) = \frac{b(b-4p)p}{8}(\mathrm{sech}^2 px - 16\,\mathrm{sech}^2 4px)(\tanh px - 4\tanh 4px) \tag{57}$$

After converting each factor of this expression to an exponential form, we obtain from the second factor a local extrema at $x = 0$, (a zero local minimum) and from the first factor two symmetrical local maxima at $x_{\pm} = \frac{\log(t_{1,2})}{p}$, where $t_{1,2}$ are the two reciprocal positive real roots of the equation $1 - 4z^3 - 4z^5 + z^8 = 0$, and $z = e^{px}$.

So, the even function V_{-} has a local maximum $(V_{-})_{\max}$ at the symmetric points $x_{\pm}$, given by

$$(V_{-})_{\max} = V_{-}(x_{\pm}) = 16p^2 + \frac{1}{16}b^2\rho_2 + bp\rho_1 \tag{58}$$

where $\rho_{1,}$ is the real root of the equation $-7875 - 3300\rho_1 - 380\rho_1^2 + 4\rho_1^3 + 2\rho_1^4 = 0$, and ρ_2 the real root of the equation $-84375 + 22500\rho_2 - 2050\rho_2^2 + 52\rho_2^3 + \rho_2^4 = 0$.

Here again the maximum number of discrete bound states n of the potential V_{-} is finite and can be computed exactly by considering the following physical constraints on the energies $E_n^{(-)}$ (51):

$$0 < E_n^{(-)} < E_{n+1}^{(-)} \leqslant (V_{-})_{\max}. \tag{59}$$

Under these constraints on the system of inequalities of quadratic polynomials in the variable n, we can find among the different cases, in one simple case, the parameters p and b are such that $b > 48.6537\,p$, then the number of bound states satisfies, $1 \leqslant n \leqslant \left\lfloor \frac{3b - 10p}{12p} \right\rfloor = n_{2,\max}$. We have omitted here the discussion of the other cases for its lengthiness and refer you to the appendix for the discussion as we have done in the previous potential.

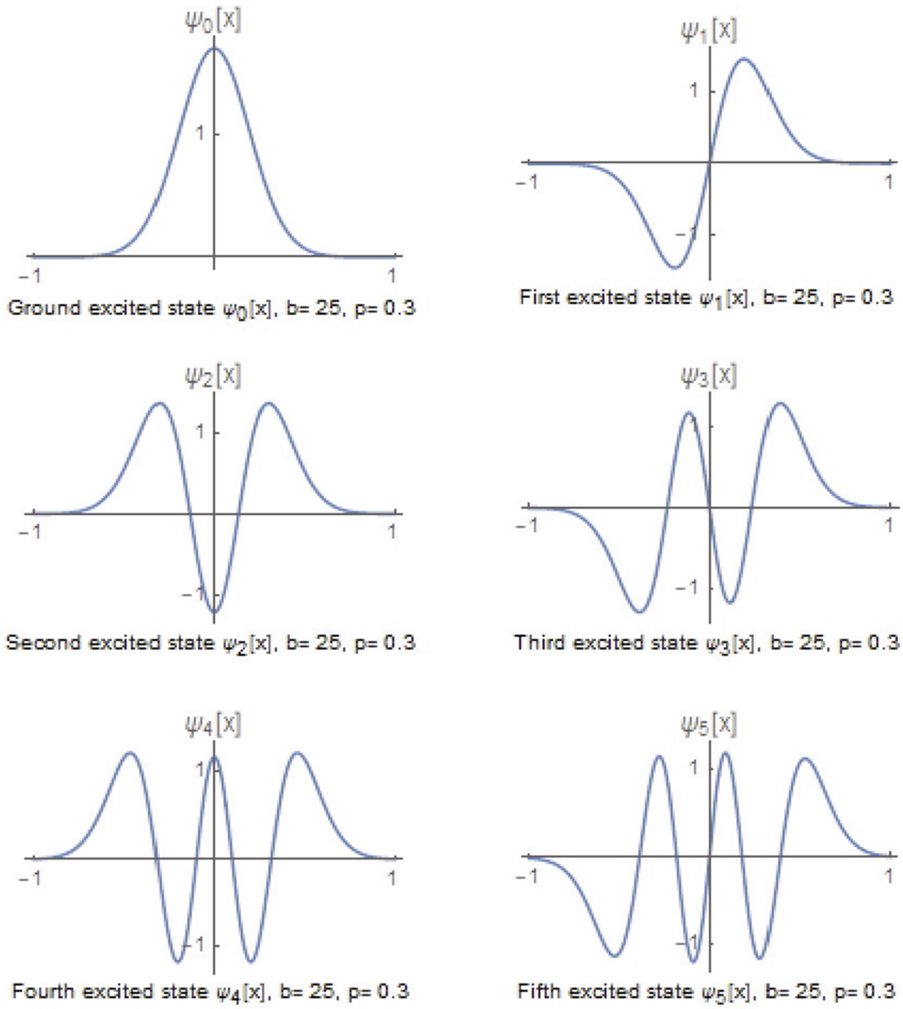

Fig. 4. The ground state and the first 5 excited states of the potential $V_{-}[x]$ with b = 25, p = 0.3.

In Fig. 3, we considered the same parameters $b = 25,\ p = 0.3$, where the highest number of bound states is given by

$$n_{2,\max} = \left\lfloor \frac{3b - 10p}{12p} \right\rfloor = 20. \tag{60}$$

We can observe some similarities between the two potentials proposed here, U_{-} and $V,_{-}$ and notice that the first potential (25) has more bound states (42) than the second potential (60) for the same chosen parameters, this is due to the inequality $\frac{b-3p}{2p} > \frac{3b-10p}{12p}$, when $b > 4p$, and the fact that the floor function is an increasing function.

Therefore the first potential will be more appropriate when the system has more bound states and the second potential will be better option if it has fewer bound states.

5. Exact versus numerical computations

We will compare in this section the exact values of the energies given explicitly in closed form by the two formulas (33) and (51), to the numerical values computed using three approximation methods of these eigen-energies of the two newly solved Schrödinger equations. These methods are: the Shooting method, also known as "Wag-the-Dog" method [29], SWKB method and WKB method [14].

Table 1 gives numerical comparison of the potential (25) with energy (33).

For the other potential (43) with energy (51), the Table 2 gives the numerical values.

Using numerical approximation, we have discovered that many of the bound states are skipped by the numerical methods, and the fact that these methods failed to detect some of the energies was mentioned in many papers and by different authors, see for example [2,48]. One of the reason behind this failure is that near a turning point, it is well known for example that WKB approximation method is not valid. We can see it in the first model where around the intersection of the 22^{th} bound state and the potential U_{-}, there are four turning points there, the WKB method stopped capturing any result, and therefore failed to account for all bound states from 22 to 40. These missed bound states are those around and above $\lim_{x\longrightarrow\pm\infty} U_{-}(x, b, p) = \frac{1}{4}(b^2 - 8bp - 4p^2) = 141.16$. The same can be said for the second model where the turning points occur around the 12^{th} bound state and where WKB method did not detect any state with energy around and above $\lim_{x\longrightarrow\pm\infty} V_{-}(x, b, p) = \frac{1}{16}(3b - 16p)^2 = 308.0$, which are the exact bound states from 12 to 20.

Table 1
Potential U_{-} energies: exact vs numerical approximations.

n	Exact	Shooting	SWKB	WKB
1	22.23	21.32	22.23	21.00
2	29.28	27.90		
3	36.15			
4	42.84	41.73	43.55	41.40
5	49.35			
6	55.68			
7	61.83	61.19	63.96	60.87
8	67.8			
9	73.59			
10	79.2	79.72		79.38
11	84.63	84.05	83.43	
12	89.88	89.62		
13	94.84	95.07		96.92
14	99.84	97.26	101.95	
15	104.55			
16	109.08	109.37		
17	113.43	113.81		113.47
18	117.6	115.58	119.49	
19	121.59	123.17		
20	125.4			
21	129.03	129.33		128.99
22	132.48	131.53		
23	135.75		136.04	
24	138.84	136.80		
25	141.75			
26	144.48	143.78		
27	147.03	146.05		
28	149.4			
29	151.59		151.5	
30	153.6	153.73		
31	155.43	155.87		
32	157.08	157.12		
33	158.55			
34	159.84			
35	160.94			
36	161.88			
37	162.63			
38	163.2	168.05		
39	163.59		163.8	
40	163.8	169.29		

One advantage of the shooting method is its flexibility to choose the initial conditions of the wavefunction, which allowed us to recover more bound states compared to SWKB and WKB methods.

6. Conclusion

In this paper we have used the supersymmetry technique to generate new family of potentials with exact energy and wavefunctions. It turns out that these potentials have a prospect of many applications in several fields of physics and chemistry.

In fact, the potential of the atomic molecule is usually modeled by the simplest and universally accepted harmonic oscillator, due to its exact solutions, however, in reality there are only a finite number of bound states in a quantum system, which makes this model fail miserably, for example, when the dissociation energy is high.

Table 2
Potential V_{-} energies: Exact vs numerical approximations.

n	Exact	Shooting	SWKB	WKB
1	54.90	51.82	54.90.	52.18
2	85.68			
3	114.84	100.6	106.73	100.90
4	142.38	146.18		146.50
5	168.3		155.47	
6	192.6	188.6	201.09	188.95
7	215.28			
8	236.34	227.91	243.55	228.20
9	255.78	264.0		264.31
10	273.6	271.13		
11	289.8	296.63	282.82	296.88
12	304.38			
13	317.34		318.85	
14	328.68	326.0		
15	338.4			
16	346.5			
17	352.98	351.61	351.55	
18	357.84			
19	361.08			
20	362.7	373.6	380.84	

The proposed potentials are generated by a superpotential as a combination of hyperbolic functions, and using an argument scaling, which has brought additional forms of supersymmetric potentials with a more realistic model of potentials having exact expression of the wavefunctions and energies of the bound states. Moreover, the proposed models have the desired requirement of a finite number of bound states which the harmonic oscillator lacks.

For small values of the argument, these models tend to be equivalent to the harmonic oscillator. Motivated by the particular circumstances of a given system and by other physical considerations, these new potentials could be used for realistic systems with discrete and finite number of energy levels, by choosing the adapted parameters.

We have compared our exact solutions with numerical well known methods, namely, “Wag-the-Dog”, SWKB, and WKB methods, for determining the bound state energies. The numerical methods have failed to detect some of the energies given by the exact formulas.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Bound states and parameters conditions

To discuss the number of bound states of the first Schrödinger Eq. (25), we solve the inequalities in (41) The energy $E_n^{(-)}$ is given by (33) and $(U_{-})_{\max}$ is given by (39), therefore we need to solve the system of inequalities

$$\left(\tfrac{1}{2}b+p\right)^2-\left(\tfrac{1}{2}b-(n+2)p\right)^2>\left(\tfrac{1}{2}b+p\right)^2-\left(\tfrac{1}{2}b-(n+1)p\right)^2>0$$
$$\left(\tfrac{1}{2}b+p\right)^2-\left(\tfrac{1}{2}b-(n+1)p\right)^2\leqslant\tfrac{3}{4}(-3+2\sqrt{3})(b-2p)p+\tfrac{1}{2}(3b-8p)p$$

In Table 3, we summarize the range of all positive integer values that the number of bound states n of this system can have under conditions on the parameters p and b, of the potential U_{-}, and the same is done for the potential V_{-} by solving the system of inequalities

Table 3
Potential U: Range of number of bound states.

Number n of bound states	Parameters conditions
$1 \leqslant n \leqslant \lfloor \zeta_1 \rfloor$	$\alpha_1 \leqslant \frac{b}{p} < \beta_1$
$1 \leqslant n \leqslant \lfloor \zeta_1 \rfloor$ or $\zeta_2 \leqslant n \leqslant \left\lfloor \frac{1}{2}\frac{b}{p} - \frac{3}{2} \right\rfloor$	$\beta_1 \leqslant \frac{b}{p} < \gamma_1$
$1 \leqslant n \leqslant \left\lfloor \frac{1}{2}\frac{b}{p} - \frac{3}{2} \right\rfloor$	$\frac{b}{p} \geqslant \gamma_1$

$$\left(\frac{3b}{4} + p\right)^2 - \left(\frac{3b}{4} - (3n+4)p\right)^2 > \left(\frac{3b}{4} + p\right)^2 - \left(\frac{3b}{4} - (3n+1)p\right)^2 > 0$$
$$\left(\frac{3b}{4} + p\right)^2 - \left(\frac{3b}{4} - (3n+4)p\right)^2 \leq 16p^2 + \frac{1}{16}b^2\rho_2 + bp\rho_1.$$

and summarizing the results of the discussion in Table 4.

We summarize in the Table 3 the number of bound states of potential U_-. where

$$\zeta_1 = \frac{b}{2p} - 2 - \frac{1}{2}\sqrt{(5 - 3\sqrt{3})\left(\frac{b}{p}\right)^2 + 2(-2 + 3\sqrt{3})\frac{b}{p} - 6}$$
$$\zeta_2 = \frac{b}{2p} - 2 + \frac{1}{2}\sqrt{2(5 - 3\sqrt{3})\left(\frac{b}{p}\right)^2 + 2(-2 + 3\sqrt{3})\frac{b}{p} - 6}$$
$$\alpha_1 = \frac{2}{9}(21 + 11\sqrt{3} + \sqrt{6(89 + 41\sqrt{3})}) \approx 15.5897$$
$$\beta_1 = \frac{1}{2}(17 + 9\sqrt{3} + \sqrt{467 + 267\sqrt{3}}) \approx 31.5377$$
$$\gamma_1 = \frac{1}{2}(17 + 9\sqrt{3} + \sqrt{472 + 270\sqrt{3}}) \approx 31.6211.$$

A quick computation shows that the number of bound states is at least 14 when $\frac{b}{p} \geqslant \gamma_1$. Also, we can see that $n_{1,\max} = \lfloor \zeta_1 \rfloor$ cannot exceed 13,.when.$\alpha_1 \leqslant \frac{b}{p} < \beta_1$.

The number of bound state levels for potential V_- is summarize in Table 4. where

$$\eta_1 = \frac{1}{4}\frac{b}{p} - \frac{4}{3} - \sqrt{-240 - 2.57688\left(\frac{b}{p}\right)^2 + 130.308\frac{b}{p}}$$
$$\eta_2 = \frac{1}{4}\frac{b}{p} - \frac{4}{3} + \sqrt{-240 - 2.57688\left(\frac{b}{p}\right)^2 + 130.308\frac{b}{p}}.$$

For the potential V_-, we have reduced to numerical values, the coefficients in the algebraic expressions found by solving the system of inequalities, in order to avoid having cumbersome mathematical expressions and to make it easier for the reader to follow the main idea which is finding the range of possible number of bound states.

We can notice from the last row in Table 4 where $\frac{b}{p} \geqslant 48.6537$, that the maximum number of bound states, $n_{2,\max}$, is at least 11. Also, we can observe that if we fix b and let p tend to zero, or if we let $\frac{b}{p}$ tends to infinity, then we will obtain an infinite number of bound states like in the case of the harmonic oscillator.

Table 4
Potential V: Range of number of bound states.

Number n of bound states	Parameters conditions
$1 \leqslant n \leqslant \lfloor \eta_1 \rfloor$	$21.689 \leqslant \frac{b}{p} < 48.3529$
$1 \leqslant n \leqslant \lfloor \eta_1 \rfloor$ or $\eta_2 \leqslant n \leqslant \left\lfloor \frac{1}{4}\frac{b}{p} - \frac{5}{6} \right\rfloor$	$48.3529 \leqslant \frac{b}{p} < 48.6537$
$1 \leqslant n \leqslant \left\lfloor \frac{1}{4}\frac{b}{p} - \frac{5}{6} \right\rfloor$	$\frac{b}{p} \geqslant 48.6537$

Appendix B. Summary of well known shape invariant potentials and the two new exactly solvable potentials

Many authors have compiled and summarized lists of well known shape invariant potentials and exactly solvable, from the time of Schrödinger almost a century ago, that were available at the time of their published papers, so we would like to contribute in this work by compiling in Table 5 the most available and widely used potentials at this time, in a way to emphasize the importance of these results, and to make it available to a wide range of scientists working in the field. Furthermore, we add to this list the newly discovered exactly solvable potentials.

Table 5: List of well known exactly solvable potentials.

Table 5
Well know exactly solvable potentials with the two newly discovered potentials.

Name	a_0	$a_1 = f(a_0)$	Superpotential $W(x)$	Partner $V_-(x)$	Energy $E_n^{(-)}$
Harmonic Osc	(A, B)	(A, B)	$Ax - B$	$(Ax - B)^2 - A$	$2\ A\ n$
Coulomb	(A, B)	$\left(\frac{AB}{1+B}, 1 + B\right)$	$A - \frac{B}{r}$	$A^2 + \frac{B(B-1)}{r^2} - \frac{2AB}{r}$	$A^2 - \left(\frac{AB}{B+n}\right)^2$
3D-Oscillator	(A, B)	$(A, 1 + B)$	$Ar - \frac{B}{r}$	$A^2r^2 + \frac{B(B-1)}{r} - A(2B + 1)$	$4\ A\ n$
Morse	(A, B)	$(A - p, B)$	$A - Be^{-px}$	$A^2 + B^2e^{-2px} - 2B(A + \frac{p}{2})e^{-px}$	$A^2 - (A - pn)^2$
Rosen-Morse I	(A, B)	$\left(A - p, \frac{AB}{1-B}\right)$	$A\text{cot}px + B$	$-A^2 + B^2 + A(A + p)\text{csc}^2px + 2B\text{cot}px$	$B^2 - A^2 - [\frac{A^2B^2}{(A-pn)^2} - (A - pn)^2]$
Rosen-Morse II	(A, B)	$\left(A - p, \frac{AB}{1-B}\right)$	$A\text{tanh}px + B$	$A^2 + B^2 - A(A + p)\text{sech}^2 px + 2B\text{tanh}px$	$A^2 + B^2 - [\frac{A^2B^2}{(A-pn)^2} + (A - pn)^2]$
Eckart	(A, B)	$\left(A - p, \frac{AB}{1-B}\right)$	$A\text{coth}px + B$	$A^2 + B^2 + A(A + p)\text{csch}^2px + 2AB\coth px$	$A^2 + B^2 - [(A - pn)^2 + \frac{A^2B^2}{(A-pn)^2}]$
Scarf I	(A, B)	$(A + p, B)$	$A\text{tan}px + B\text{sec}px$	$-A^2 + (A^2 + B^2 - pA)\text{sec}^2 px + B(2A - p)\text{sec}px\text{tan}px$	$(A + pn)^2 - A^2$
Scarf II	(A, B)	$(A - p, B)$	$A\text{tanh}px + B\text{sech}px$	$A^2 + (B^2 - A^2 - pA)\text{sech}^2 px - B(2A + p)\text{sech}\, px \tanh px$	$A^2 - (A - pn)^2$
Poschl-Teller I	(A, B)	$(A - p, B)$	$A\text{coth}px - B\text{csch}px$	$A^2 + (A^2 + B^2 + pA)\text{csch}^2px - B(2A + p)\coth px\ \text{csch}px$	$A^2 - (A - pn)^2$
Poschl-Teller II	(A, B)	$(A - p, B + p)$	$A\text{tanh}px - B\text{coth}px$	$(A - B)^2 + B(B - p)\text{csch}^2px - A(A + p)\text{sech}^2 px$	$(A - B)^2 - (A - B - 2pn)^2$
New potentials					
New Potential I	(A, B)	$(A - p, B - 2p)$	$A\text{tanh}px + B\text{tanh}2px$	$-3p\left(A + B - \frac{7p}{3}\right) + \frac{p}{2}(A + B - 2p)(\text{tanh}px - 2\text{tanh}2px)$	$(A + B)^2 - (A + B - 3np)^2$
New Potential II	(A, B)	$(A - p, B - 4p)$	$A\text{tanh}px + B\text{tanh}4px$	$\frac{-49p}{12}\left(A + B - \frac{241p}{49}\right) + \frac{p}{12}(A + B - 4p)(\tanh px - 4\tanh 4px)$	$(A + B)^2 - (A + B - (3n + 2)p)^2$

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