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Skew information correlations beyond entanglement in dissipative two Su (2)-systems

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ABSTRACT

We investigate the dynamics of quantum correlations [including skew information correlations beyond entanglement] of two isolated Su(2)-systems that are initially in a maximum Bell state or in a separable state. Each Su(2)-system is interacting with two-mode Su(1, 1) coherent states. The time evolution of the density matrix of the two Su(2)-systems with the effect of intrinsic decoherence is analytically explored. The robustness of the initial quantum correlations for several initial coherence intensities, detunings, and intrinsic decoherences, is investigated. The robustness analysis of various correlation functions is based on the concurrence and quantum skew information. It is shown that the phenomena of sudden death and birth entanglement, and the sudden changes of local quantum uncertainty can be controlled. The correlations of the skew information present good robustness against the intrinsic decoherence and the detuning.

Introduction

Many spatially separated quantum systems have been proposed, theoretically [1] and realized experimentally [2,3], in particular, for the distribution of the coherent information across macroscopic distances in quantum network [4–6]. Spatially separated cavities could be a major components for the conception of quantum network based on photons [7,8]. There are numerous applications in teleportation [9] and quantum cryptography [10].

There is a growing interest on the dynamics of quantum correlations (QCs) [11]. Quantum entanglement (QE) is a special kind of QCs. It is playing a fundamental role in quantum information processing [12–14], and also providing a new method to study interesting quantum phenomena [15–17]. However, QE is not the only type of QCs [18]. New types of QCs was introduced beyond QE [19] as: measurement-induced disturbance [20], measurement-induced nonlocality (MIN) [21], quantum discord (QD) [22] and its geometric (GQD) [23]. Where, the GQD is derived using the minimal distance from the given states and the classical states set by the *p*-norms as: Hilbert-Schmidt norm, Schatten one-normand Bures norm.

Furthermore, new quantum correlation measures based on the quantum skew information (QSI) [24] was proposed such as: local quantum uncertainty (LQU) [25], uncertainty-induced quantum nonlocality (UIN) [26] and quantum coherence [27]. The quantum coherence allows the analysis of the quantum coherence distribution. If the state is pure, the QSI is non-negative and reduces to the variance. The LQU is useful to quantify the quantum uncertainty minimum and for which there is a unique closed form for qubit-qudit systems [25]. It has zero-value for all vanishing discord states and satisfies the complete physical requirements for a measure of quantum correlations. For general cases [28], LQU was studied numerically, and analytically investigated for some particular cases [29–31]. Based on skew information, UIN presents other type of QCs that is updated version of the MIN [32]. It eliminates the non-contractivity problem (UIN will be vanished for the states without MIN). Recently, several investigations, on other types of quantum correlations, beyond the entanglement were published [33,34].

For realistic physical models, decoherence process is an important factor to consider as it handicaps the quantum information control [35]. The intrinsic decoherence of an isolated quantum system under its own dynamics was studied recently in [36].

Motivated by the importance of the intrinsic decoherence in isolated quantum systems, we explore the quantum correlations for two isolated Su(2)-systems, which are initially started in a maximum Bell state or in a separable state. Each Su(2)-system is interacting with two-mode Su(1, 1) coherent states. This paper is structured as follows: In Section

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"The physical model", the physical model is presented. In Section "Non-classical correlation functions", a brief review of the quantum correlation functions is given. Section "Numerical results" shows the numerical results and discussions. The conclusion is in Section "Conclusion".

The physical model

Here, we consider a system of two Su(2)-systems, A and B, which are initially started in a maximum symmetric Bell state or in a separable state and having no mutual interaction. The two Su(2) quantum systems are within two spatially isolated Su(1, 1)-cavities. Each Su(2) quantum system is a two-level atom interacting with two coupled fields in the non-degenerate parametric amplifier. In the rotating wave approximation, the total Hamiltonian of the system is given by

$$\widehat{H} = \sum_{r=A,B} \widehat{H}_r. \tag{1}$$

While, the Hamiltonian of the each Su(2) quantum system is given by [37]

$$\widehat{H}_r = \omega_r\left(\widehat{a}_r^\dagger \widehat{a}_r + \widehat{b}_r^\dagger \widehat{b}_r + 1\right) + \frac{\omega_\circ^r}{2}\widehat{\sigma}_z^r \qquad + \lambda_r\left(\widehat{a}_r \widehat{b}_r \widehat{\sigma}_+^r + \widehat{a}_r^\dagger \widehat{b}_r^\dagger \widehat{\sigma}_-^r\right)\Bigg], \tag{2}$$

where $\widehat{a}_r(\widehat{a}_r^\dagger)$ and $\widehat{b}_r\left(\widehat{b}_r^\dagger\right)$ are the annihilation and (creation) operators for the field modes, while $\widehat{\sigma}_-^r$, $\widehat{\sigma}_+^r$ and $\widehat{\sigma}_z^r$ are the usual Pauli spin matrices. λ_r represents the coupling parameter, ω_r and $\omega_\circ^r$ are, respectively, frequencies of the two cavity modes and the two-level atom. In terms of Su(1, 1) Lie algebra, the Hamiltonian (2) has the following expression:

$$\widehat{H}_r = \frac{1}{2}\omega_\circ^r \widehat{\sigma}_z^r + \omega \widehat{K}_0^r + \lambda_r\left(\widehat{K}_-^r \widehat{\sigma}_+^r + \widehat{K}_+^r \widehat{\sigma}_-^r\right), \quad r = A, B... \tag{3}$$

where we have defined the following two-mode field operators:

$$\widehat{K}_0^r = \widehat{a}_r^\dagger \widehat{a}_r + \widehat{b}_r^\dagger \widehat{b}_r + 1, \quad \widehat{K}_+^r = \widehat{a}_r^\dagger \widehat{b}_r^\dagger, \quad \widehat{K}_-^r = \widehat{a}_r \widehat{b}_r. \tag{4}$$

where,

$$[\widehat{K}_o^r, \widehat{K}_\pm^r] = \pm K_\pm^r, \quad [\widehat{K}_-^r, \widehat{K}_+^r] = 2\widehat{K}_o^r, \quad r = A, B. \tag{5}$$

The eigenstates of the Hamiltonian of Eq. (3) are given by

$$\left|\psi_1^n\right\rangle_r = \frac{1}{\sqrt{2}}\left[\left|1, n_r, k_r\right\rangle + \left|0, n_r+1, k_r\right\rangle\right], \tag{6}$$

$$\left|\psi_2^n\right\rangle_r = \frac{1}{\sqrt{2}}\left[\left|1, n_r, k_r\right\rangle - \left|0, n_r+1, k_r\right\rangle\right]. \tag{7}$$

To derive the time-dependent density matrix, we use the Hamiltonian (4) that describes the interaction between an Su(1, 1) and an Su(2) Lie algebra. Where, we have the following relations:

$$\begin{aligned}\widehat{K}_-^r|n_r, k_r\rangle &= \sqrt{n_r(n_r+2k_r-1)}\,|n_r-1, k_r\rangle, \widehat{K}_+^r|n_r, k_r\rangle \\ &= \sqrt{(n_r+1)(n_r+2k_r)}\,|n_r+1, k_r\rangle, \widehat{K}_0^r|n_r, k_r\rangle = (n_r+k_r) \\ &\quad |n_r, k_r\rangle.\end{aligned} \tag{8}$$

k_r is the Bargmann number, it comes from the Casimir operator defined as $\widehat{C}_r^2 = (\widehat{K}_0^r)^2 - \frac{1}{2}(\widehat{K}_+^r \widehat{K}_-^r + \widehat{K}_-^r \widehat{K}_+^r) = k_r(k_r-1)\widehat{I}$. In this paper we set $k_r = \frac{1}{4}$.

Here, we study the effect of the intrinsic decoherence. The condition of the intrinsic decoherence is for short time steps, the Su(2)-Su(1, 1) system developed by stochastic sequences of identical unitary transformations rather than by a continuous unitary evolution [38]. Therefore, the master equation that describes the time evolution for the $Su(2)$-Su(1, 1)-cavity quantum system is given by [38]

$$\frac{d}{dt}\rho(t) = -i\left[\widehat{H}, \rho\right] - \frac{1}{2\widetilde{\gamma}}\left[\widehat{H}, \left[\widehat{H}, \rho\right]\right], \tag{9}$$

where $\widetilde{\gamma}$ designs the intrinsic decoherence parameter. In numerical computation, we use $\gamma = \frac{1}{\widetilde{\gamma}}$.

Since we are investigating the robustness of QCs against the intrinsic decoherence, we assume that each Su(1, 1)-cavity is initially in the coherent states that can be expressed as:

$$\left|\alpha\right\rangle_r = \sum_{n_r=0}^{\infty} \eta_r^n \left|n_r\right\rangle = \sum_{n_r=0}^{\infty} e^{-\frac{1}{2}\left|\alpha_r\right|^2} \frac{\alpha_r^{n_r}}{\sqrt{n_r!}}\left|n_r\right\rangle. \tag{10}$$

While the two Su(2)-system A and B are initially in two cases: **(i)** The two Su(2)-system are initially in maximal correlated sate, symmetric Bell state, that is known by $\rho^{AB}(0) = |\phi\rangle_{AB}\langle\phi|$, where $\left|\phi\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|1_A 1_B\right\rangle - \left|0_A 0_B\right\rangle\right)$. Therefore, the dynamical evolution of the Su(2)-Su(1, 1) system is governed by

$$\widehat{\rho}(t) = \frac{1}{2}\sum_{k,l=0,1}(-1)^{k+l}\rho_{kl}^A(t) \otimes \rho_{kl}^B(t). \tag{11}$$

(ii) They are initially in pure state, $\rho^{AB}(0) = |1_A, 0_B\rangle\langle 1_A, 0_B|$. Using this initial state and the Hamiltonian of Eq. (3), the dynamical evolution of the Su(2)-Su(1, 1) system is governed by

$$\widehat{\rho}(t) = e^{\widehat{L}t}\widehat{\rho}(0) = \rho_{11}^A(t) \otimes \rho_{00}^B(t). \tag{12}$$

The matrices $\rho_{kl}^r(t)\,(r = A, B)$ are given by

$$\rho_{11}^r(t) = \frac{1}{4}\sum_{m,n=0}\sum_{k,l=1,2}\eta_r^{m,n} h_{kl}^{11}\left|\phi_k^m\rangle_r\left\langle\phi_l^m\right|\right., \tag{13}$$

$$\begin{aligned}\rho_{00}^r(t) = &(\eta_r^0)^2\left|\phi_2^{-1}\right\rangle_r\left\langle\phi_2^{-1}\right| + \sum_{m=0}\eta_r^{m+1,0}e^{-\frac{1}{2\gamma_r}\nu_r^{m2}t}\Big[(-i\sin\nu_r^m t|\phi_1^m\rangle_r\langle\phi_2^{-1}| \\ &+ \cos\nu_r^m t|\phi_2^m\rangle_r\langle\phi_2^{-1}|) + h.\,c.\Big] + \frac{1}{4}\sum_{m,n=0}\sum_{k,l=1,2}\eta_r^{m+1,n+1}h_{kl}^{00} \\ &\left|\phi_k^m\right\rangle_r\left\langle\phi_l^n\right|,\end{aligned} \tag{14}$$

$$\begin{aligned}\rho_{10}^r(t) = &(\rho_{01}^r(t))^\dagger = \sum_{m=0}\eta_r^{n,0}e^{-\frac{1}{2\gamma_s}\nu_r^{m2}t}\left(\cos\nu_r^m t\left|\phi_1^m\right\rangle_r\left\langle\phi_2^{-1}\right|\right. \\ &\left.- i\sin\nu_r^m t\left|\phi_2^m\right\rangle_r\left\langle\phi_2^{-1}\right|\right) + \frac{1}{4}\sum_{m,n=0}\sum_{k,l=1,2}\eta_r^{m,n+1}h_{kl}^{10}\left|\phi_k^m\right\rangle_r \\ &\left\langle\phi_l^n\right|,\end{aligned} \tag{15}$$

where $|\phi_1^n\rangle_r = |1_r, n_r\rangle$, $|\phi_2^n\rangle_r = |0_r, n_r+1\rangle$, $\eta_r^{m,n} = \eta_r^m(\eta_r^n)^*$ and $\nu_r^m = \lambda_r\sqrt{\delta_r^2 + (S_{i_r+1}^r)^2}$ with the abbreviations:

$$\begin{aligned}h_{11}^{11} &= h_{22}^{00} = h_{12}^{10} = a_r^{++} + a_r^{+-} + a_r^{-+} + a_r^{-}, h_{12}^{11} = h_{21}^{00} = h_{11}^{10} \\ &= a_r^{++} - a_r^{+-} + a_r^{-+} - a_r^{-}, h_{21}^{11} = h_{12}^{00} = h_{22}^{10} = a_r^{++} + a_r^{+-} - a_r^{-+} \\ &\quad - a_r^{-}, h_{22}^{11} = h_{11}^{00} = h_{21}^{10} = a_r^{++} - a_r^{+-} - a_r^{-+} + a_r^{-},\end{aligned} \tag{16}$$

where $a_r^{mn} = \exp[-i(\varepsilon_{i_r}^m - \varepsilon_{j_r}^n)t - \frac{\gamma_r}{2}(\varepsilon_{i_r}^m - \varepsilon_{j_r}^n)^2 t]$ and the $\varepsilon_{i_r}^{\pm}$ are the eigenvalues of H_r, which are given by: $\varepsilon_{i_r}^{\pm} = \omega(i_r + 0.5) \pm \sqrt{\delta^2 + (i_r+1)^2(i_r+2k_r)^2}$.

Since, the behavior of the non-classical correlations in two Su(2)-system is our goal, we find its reduced density matrix $\rho^{AB}(t)$ by tracing out the states of the cavity modes $|m\rangle_A \otimes |n\rangle_B$ from the final state of Eq. (12). Then, the final two Su(2)-system states are given by,

$$\rho^{AB}(t)=\frac{1}{2}\sum_{k,l=1,0}(-1)^{k+l}\sum_{m=0}^{\infty}u_m^A\otimes\sum_{n=0}^{\infty}u_n^B. \tag{17}$$

where $u_m^r=\langle i|\rho_{kl}^r(t)|i\rangle$ $(i=m,n$ and $r=A,B)$. Therefore, we can quantify the non-classical correlations of the state Eq. (17) by different correlation functions as: the local quantum uncertainty, the uncertainty-induced non-locality and the concurrence,

Non-classical correlation functions

We consider the two Su(2)-systems are initially in maximum correlated states, which may turn into partially correlated states or separable states due to the time evolution of the Su(2)-Su(1, 1) interactions. By using different skew information functions and concurrence, we investigate the robustness of the initial quantum correlations functions for several initial coherence intensities, resonance detunings, and intrinsic decoherences. These two measures, LQN and UIN, will be compared by the concurrence. The measure functions are described in details as following:

LQU

The Local quantum uncertainty (LQU) [24] may be used to distinguish quantum correlated states from separable ones [39]. If ρ^{AB} is the density matrix of a bipartite quantum state and is non-degenerate spectrum, then the quantum skew information (QSI) is given by

$$\left[I\left(\rho^{AB},K^{\Lambda}\right)=-\frac{1}{2}Tr[\sqrt{\rho^{AB}},K^{\Lambda}]^2,\right. \tag{18}$$

where K^{Λ} is a local observable defined by: $K^{\Lambda}=K_A^{\Lambda}\otimes\Pi_B$ with hermitian operator K_A^{Λ} on the subsystem A. The QSI is presented to quantify of the quantum information in the density matrix of a bipartite quantum state ρ^{AB} to a quantity K^{Λ}. The QSI is used to quantify of the uncertainty of the K^{Λ} in ρ_{AB}. It quantifies the non-commutativity between ρ^{AB} and K^{Λ}. Based on the definition of QSI, the LQU is introduced as a new quantum correlation for ρ_{AB} [25], it is defined as:

$$\mathscr{L}_A(\rho^{AB})=\min_{K^{\Lambda}}\left\{I\left(\rho,K^{\Lambda}\right)\right\}, \tag{19}$$

where the minimum is over all local maximally informative observable. The $\mathscr{L}_A(\rho^{AB})$ measures the minimum amount of the uncertainty of the observable K^{Λ} in the bipartite quantum state ρ_{AB}. The interesting properties of $\mathscr{L}_A(\rho^{AB})$ are: (i) it vanishes for all zero discord states, i.e., $\mathscr{L}_A(\rho^{AB})$ may be used as measure of discord-like quantum correlation. (ii) $\mathscr{L}_A(\rho^{AB})$ is invariant under local unitary operation on the another subsystem B. (iii) the advantage of the LQU has an explicit expression [40]. For a density matrix of bipartite quantum state, $\rho^{AB}(t)$, the local quantum uncertainty is defined by:

$$L(t)=1-\lambda_{max}(W_{AB}), \tag{20}$$

where λ_{max} is the biggest eigenvalue of the 3×3 symmetric matrix W_{AB}, the elements of the matrix W_{AB} are given by

$$w_{ij}=\mathrm{Tr}\{\sqrt{\rho^{AB}(t)}(\sigma_i\otimes I)\sqrt{\rho^{AB}(t)}(\sigma_j\otimes I)\}, \tag{21}$$

where σ_i, $i=1,2,3$ represent the Pauli operators. In our work, we use MATLAB program to numerically calculate the elements, w_{ij}, of the matrix W_{AB} by using Eq. (17)

UIN

Uncertainty-induced quantum non-locality (UIN) is another new quantum correlation. For a density matrix of bipartite quantum state, ρ^{AB}, UIN is defined as [26]:

$$\mathscr{U}_c(\rho)=\max_{K^{\Lambda}}I\left(\rho^{AB},K^{\Lambda}\right). \tag{22}$$

The UIN can be considered as the updated version of the original measurement-induced non-locality. The UIN vanishes for the separable states, $\rho^{AB}=\rho_A\otimes\rho_B$.

For a bipartite quantum state $\rho^{AB}(t)$, the expression of UIN is [26]:

$$U(t)=\begin{cases}1-\lambda_{\min}(W_{AB}), & \vec{\mathbf{x}}=0;\\ 1-\frac{1}{\|\vec{\mathbf{x}}\|}\vec{\mathbf{r}}\,W_{AB}\,\vec{\mathbf{x}}^T, & \vec{\mathbf{x}}\neq0.\end{cases} \tag{23}$$

where $\|\vec{\mathbf{x}}\|$ is the vector norm of the Bloch vector $\vec{\mathbf{x}}$. The elements of symmetric matrix W_{AB} are given in Eq. (21). In the space states $\{|1\rangle=|1_A1_B\rangle,|2\rangle=|1_A0_B\rangle,|3\rangle=|0_A1_B\rangle,|4\rangle=|0_A0_B\rangle\}$, if $\langle i|\rho^{AB}|j\rangle=I_{ij}+iR_{ij}(i,j=1-4)$ are the elements of ρ^{AB}, then the vector $\vec{\mathbf{x}}$ is given by

$$\vec{\mathbf{x}}=(2I_{13}+2I_{24},\,2R_{31}+2R_{42},\,2\rho_{11}+2\rho_{22}-1)^t. \tag{24}$$

The vector $\vec{\mathbf{x}}$ and the matrix W_{AB} will be calculated numerically by using Eq. (17).

Concurrence measure

We consider here the concurrence [41] to quantify the quantum entanglement. For a general bipartite state, ρ_{AB}, the concurrence function is defined by

$$C(t)=\max\{\mu_1-\mu_2-\mu_3-\mu_4,0\}, \tag{25}$$

where $\mu_i(i=1,2,3,4)$ are the square nonzero roots of the eigenvalues for the matrix: $\rho^{AB}(\sigma^y\otimes\sigma^y)(\rho^{AB})^*(\sigma^y\otimes\sigma^y)$ in descending order. The values of the concurrence varies from, $C=0$, (for unentangled state) to $C=1$ (for a maximally entangled state).

Numerical results

In this part, we investigate the dynamics of LQU and UIN dynamics of the two Su(2) quantum systems, and compare their dynamics with that QE via the concurrence. First, we consider the resonance case, $\delta_i=0$. In Fig. 1, the quantum correlation functions $L(t)$, $U(t)$ and $C(t)$ are plotted when the two Su(2) quantum systems are started in the Bell state, $\frac{1}{\sqrt{2}}\left(\left|1_A1_B\right\rangle-\left|0_A0_B\right\rangle\right)$, with various values of (α_A,α_B). Here, the intrinsic decoherence parameters are zero. In our numerical simulation we consider that all the coupling constants λ_r are equal and we represent them by λ.

For large initial coherence intensities of two $Su(1,1)$-cavities, $(\alpha_A,\alpha_B)=(5,4)$, the correlation functions $L(t)$, $U(t)$ and $C(t)$ are more robust at points $\lambda t=n\pi(n=1,2,...)$ see Fig. 1a. From Fig. 1a, we observe that: (i) The function of local quantum uncertainty, $L(t)$ exhibits periodic evolution with period π. From Fig. 1a, we observe that: (i) The function of local quantum uncertainty, $L(t)$, only exhibits more sudden changes periodically during its time evolution with period π. This phenomenon has been observed in the nuclear magnetic resonance experiments [42]. In periods of the sudden changes, at first the local quantum uncertainty decreases to its minimum value L_{min}^{sud} in which the first sudden change occurs, where it momentary increases abruptly to its maximum value of L_{max}^{sud}. After $L(t_{max})$, it abruptly changes (second sudden) to vanish at points $\lambda t=(n+0.5)\pi(n=0,1,2,...)$ and grows exponentially to the same value of L_{max}^{sud}. Also, the third and fourth sudden changes occur when $L(t)$ have L_{min}^{sud} and $L(t_{max})$ respectively. The correlation function $U(t)$ decreases exponentially to vanish at points $\lambda t=(n+0.5)\pi(n=0,1,2,...)$ and also grows exponentially increases to its maximum values at points $\lambda t=n\pi(n=1,2,...)$. In the intervals between two sudden changes at L_{max}^{sud}, the correlation functions $L(t)$ and $U(t)$ have the same curve, and they represent the same quantum correlation, which may be beneficial to the quantum information

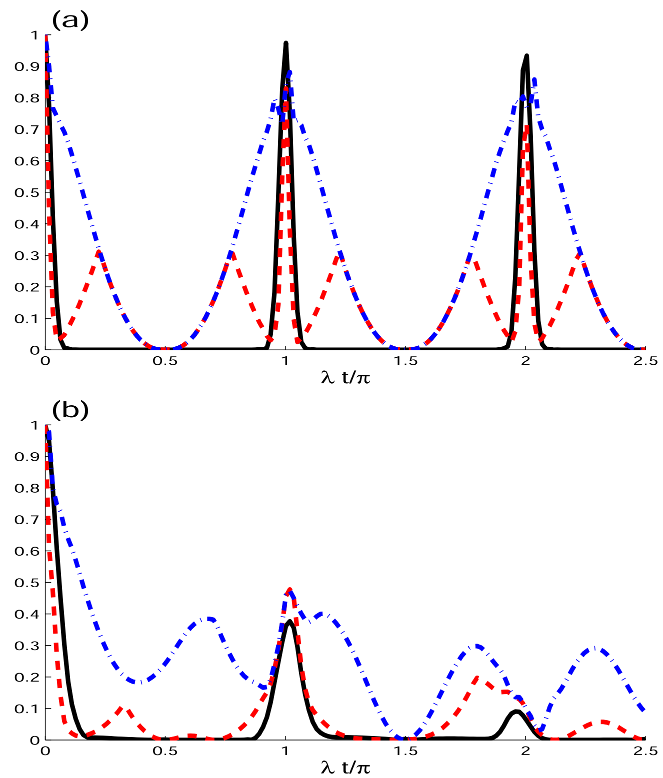

Fig. 1. Time evolutions of $L(t)$ (dashed plots), $U(t)$ (dashed dotted plots) and $C(t)$ (solid plots) when the two Su(2) systems are started in the Bell state and in resonance case, $\delta_i = 0$, with $k_i = \frac{1}{4}$, $\gamma_i = 0$ and for different values of $(\alpha_A, \alpha_B) = (5, 4)$ in (a) and $(\alpha_A, \alpha_B) = (0, 3)$ in (b).

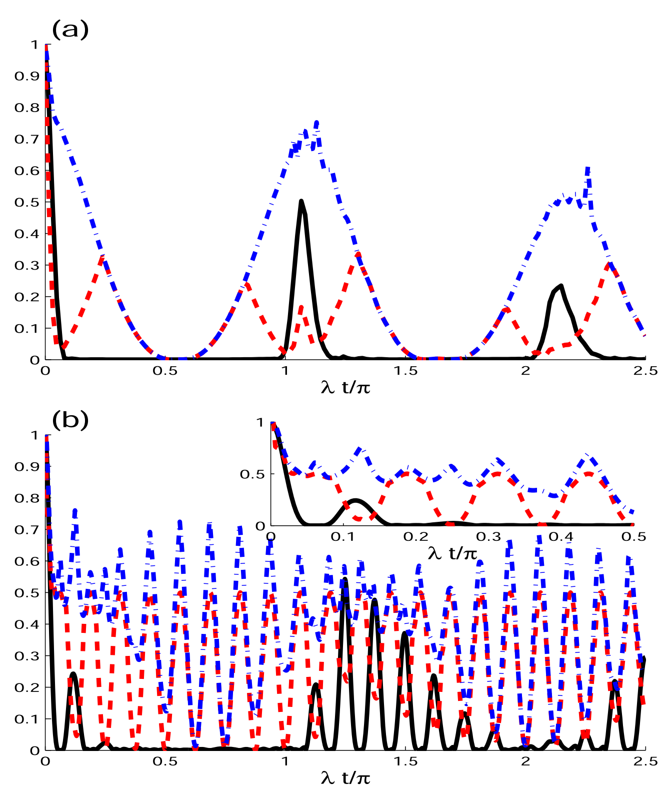

Fig. 2. As Fig. 1, but for the off-resonance case, $\delta_i = 8\lambda$.

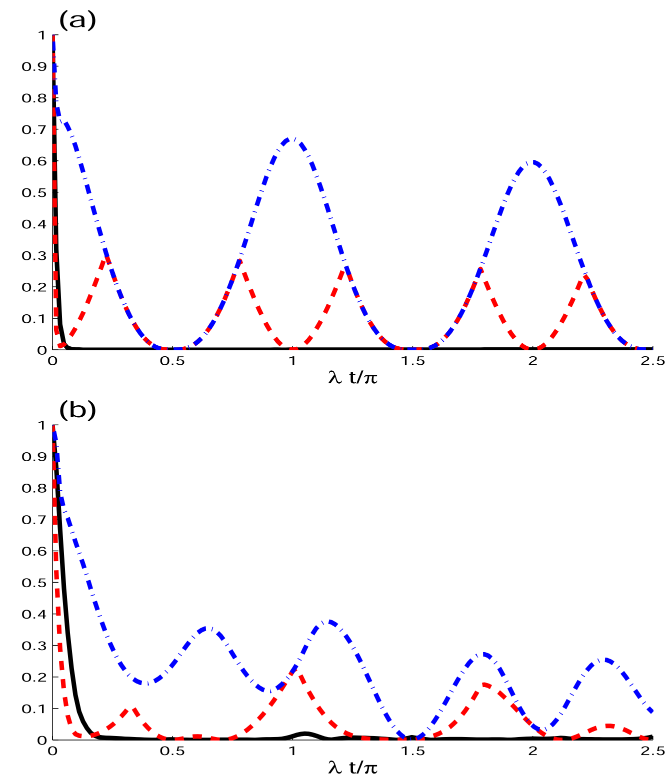

Fig. 3. As Fig. 1, but for $\gamma_A = \gamma_B = 0.01\lambda$.

processing. (ii) The correlation of UIN is always stronger than LQU and its time evolution does not have any sudden transition. (iii) The phenomena of sudden death and birth of concurrence appear with the same period, π, of $L(t)$ and $U(t)$. The concurrence function $C(t)$ has periodic sudden death and sudden birth entanglement [15,16].

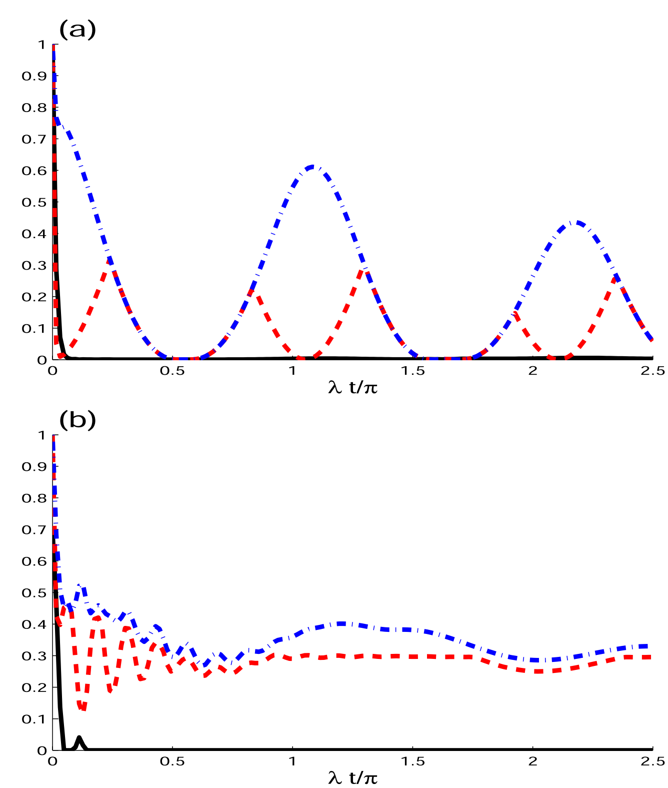

Fig. 4. As Fig. 2, but for $\gamma_A = \gamma_B = 0.01\lambda$.

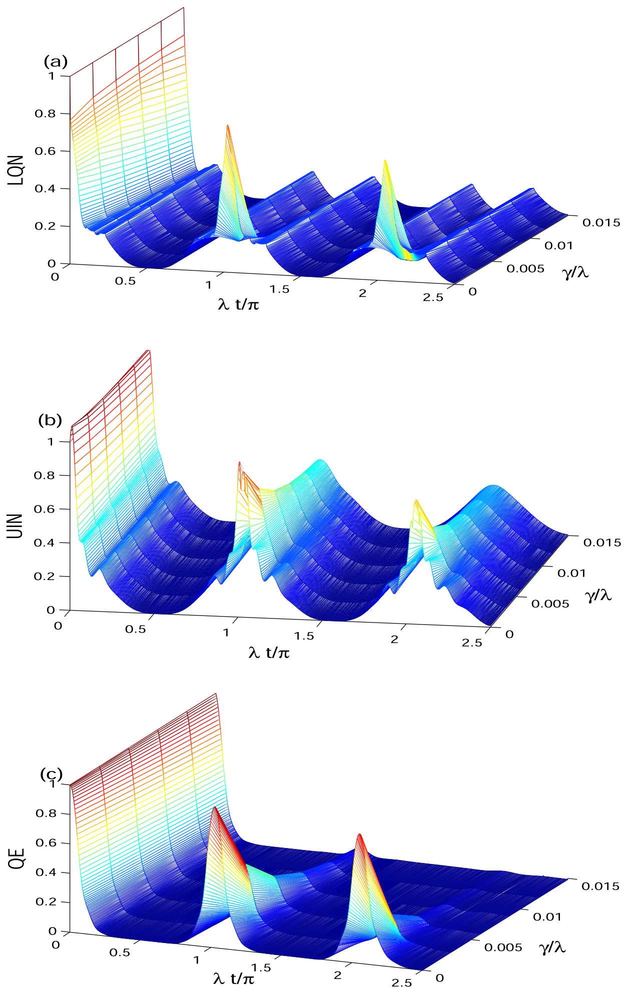

Fig. 5. The correlation functions $L(t)$, $M(t)$ and $C(t)$ in off-resonance case, $\delta_i = \lambda$, with different values of $\gamma_i = \gamma \in [0, 0.015\lambda]$, at fixed values for $\alpha_A = \alpha_B = 4$ and $k = \frac{1}{2}$.

In Fig. 1b, the correlation functions $L(t)$, $U(t)$ and $C(t)$ are plotted for the small initial coherence intensities of the two $Su(1, 1)$-cavities, $(\alpha_A, \alpha_B) = (0, 3)$. This means that one of the cavities is in a vacuum state. In this case, the dynamic robustness of QCs is very weak compared to the one of the Fig. 1a. The regularity of oscillating $L(t)$ and $U(t)$ disappears completely.

We can note that the upper bounds of $L(t)$, $U(t)$ and $C(t)$ are smaller compared to the case where we have larger initial coherence intensities. The phenomena of the sudden death and birth of $C(t)$ have notable changes.

In Fig. 2, the dynamic evolution of the correlation functions $L(t)$, $U(t)$ and $C(t)$ for the non-resonant case ($\delta_i = 8\lambda$), is depicted for different values of α_i.

In the off-resonant case, with large initial coherence intensities $(\alpha_A, \alpha_B) = (5, 4)$, we observe that: (i) The robustness of the functions $L(t)$, $U(t)$ and $C(t)$ depends on the detuning. The upper bounds of the correlation functions are smaller compared with that ones in the resonant case, see Fig. 2a. The correlation functions dissipate. (ii) The periodicity of the dynamics for the correlation functions $L(t)$, $U(t)$ and $C(t)$ is destroyed. In Fig. 2b, the $L(t)$, $U(t)$ and $C(t)$ are plotted for the case of small initial coherence intensities, $(\alpha_A, \alpha_B) = (0, 3)$. The off-resonant case leads to disappearance of the sudden changes, whereas, the sudden birth and death of the concurrence still occur. With the small values of (α_A, α_B), the oscillations of the correlation functions are very faster and lose their regularity. Moreover, the extreme values of the correlation functions change. We conclude that the robustness of the correlation functions $L(t)$, $U(t)$ and $C(t)$ depend not only on the initial coherence intensities but also on the detuning. In Fig. 3, we show the robustness of QCs against the phase decoherence, $\gamma_A = \gamma_B = 0.01\lambda$. According to Fig. 3a, the behavior of the correlation functions has notable changes compared to the case of $\gamma_A = \gamma_B = 0.0$. After short time the concurrence vanishes completely due to the intrinsic decoherence. In Fig. 3b, the correlation functions $L(t)$, $U(t)$ and $C(t)$ are plotted with small initial coherence intensities, $(\alpha_A, \alpha_B) = (0, 3)$ and intrinsic decoherence parameters $\gamma_A = \gamma_B = 0.01\lambda$. In this case, the intrinsic decoherence effect is more pronounced. The correlation functions have damped oscillatory dynamics and their upper bounds are smaller than those of the case with $\gamma_i = 0$.

In Fig. 4, the effect of the damping parameters combined with the off-resonance case, where $\delta_i = 8\lambda$ and $\gamma_A = \gamma_B = 0.01\lambda$ is explored. For large values of $(\alpha_A, \alpha_B) = (5, 4)$, the correlation functions $L(t)$, $U(t)$ and $C(t)$ are almost independent of the deutuning. However for small values

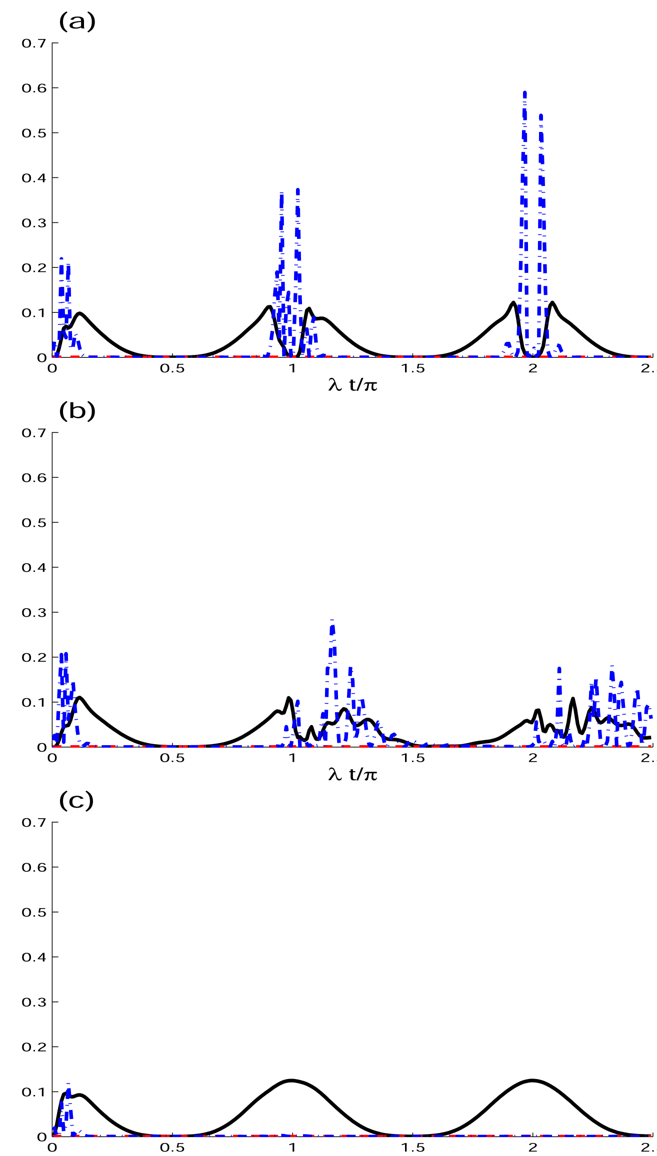

Fig. 6. Time evolutions of $L(t)$ (dashed plots), $U(t)$ (dashed dotted plots) and $C(t)$ (solid plots) when the two atoms are initially in a separable state and for the cases; $(\delta_i, \gamma_i) = (0, 0)$ in (a), $(\delta_i, \gamma_i) = (5\lambda, 0)$ in (b) and $(\delta_i, \gamma_i) = (0, 0.01\lambda)$ in (c) with $k_i = \frac{1}{4}$ and $(\alpha_A, \alpha_B) = (3, 4)$.

of $(\alpha_A, \alpha_B) = (0, 3)$, there are notable changes in the correlation functions. The quantum correlations disappear after short time. Therefore, the robustness of the quantum correlations is very sensitive to the intrinsic decoherence for small coherence amplitudes α_i.

In Fig. 5, the correlation functions $L(t)$, $U(t)$ and $C(t)$ are plotted for $\gamma_i = \gamma \in [0, 0.015\lambda]$. We observe that; (i) The stationary values of the correlation functions $L(t)$, $U(t)$ and $C(t)$ for particular values of γ do not depend on the intrinsic decoherence. (ii) We can find a large (t, γ)-space in which the state of the two Su(2) quantum systems are sparable (disentangled state) and non-vanishing correlation functions LQU and UIN. The results show that the geometric quantum discord and the measurement-induced nonlocality introduce other quantum correlations beyond the entanglement.

In this part, we investigate the dynamics of generated QCs via LQU, UIN and the concurrence when the two atoms are initially started from the separable state. In Fig. 6, the quantum correlation functions $L(t)$, $U(t)$ and $C(t)$ are plotted when the two $Su(2)$ quantum systems are initially started in the pure state, $|1_A, 0_B\rangle\langle 1_A, 0_B|$, for $(\delta_i, \gamma_i) = (0, 0)$ and $(\alpha_A, \alpha_B) = (3, 4)$.

In Fig. 6a, we consider the case, $(\delta_i, \gamma_i) = (0, 0)$ which describes the resonant case without intrinsic decoherence. The plots of the correlation functions show the ability of the unitary interaction to generate quantum correlations when the two Su(2)-systems started from a separable state. Where the partial QCs via the LQU and the concurrence can be generated beyond the UIN. The concurrence is generated periodically with period π whereas LQU has irregular oscillations.

In Fig. 6b, the off-resonant case is displayed, $(\delta_i, \gamma_i) = (5\lambda, 0)$, we observe that the correlation functions $L(t)$ and $C(t)$ oscillate and the their extreme values have notable changes. In Fig. 6c, the effect of the damping parameters is shown. After short time, the partial generated QC of LQU disappears completely while the behavior of the QE reaches its stationary regular oscillations.

Conclusion

We have analyzed the robustness control of the quantum correlations for two isolated Su(2)-systems, where, each one of the Su(2)-systems interacting with two-mode Su(1, 1) coherent states. In particular we have explored the effects of the intrinsic decoherence, initial coherence intensities, and the detuning. An analytical description of the final states of the two quantum Su(2)-systems are explicitly obtained. For the resonance case, the correlation functions are more robust and can be controlled by the initial coherence intensities and the detuning. The quantum correlations are indispensable for the quantum processing, which is of an extreme importance to be conserved for a large time. Here, we have explored the robustness control of the quantum correlations.

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Appendix A

Here, the Eq. (9) can be rewritten as:

$$\frac{d}{dt}\hat{\rho} = \hat{L}\hat{\rho}, \tag{26}$$

where the operator $\hat{L}$ is given by; $\hat{L}*{=}-i\left[\hat{H}, *\right] - \frac{1}{2\gamma}\left[\hat{H}^2 * {-}2\hat{H} * \hat{H} + *\hat{H}^2\right]$.Therefore,

$$\hat{\rho}(t) = e^{\hat{L}t}\hat{\rho}(0). \tag{27}$$

If the two Su(2)-system are initially in symmetric Bell state, $\rho^{AB}(0) = |\phi\rangle_{AB}\langle\phi|$, where $\left|\phi\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|1_A 1_B\right\rangle - \left|0_A 0_B\right\rangle\right)$, then the initial state of the whole system is given by

$$\begin{aligned}\hat{\rho}(0) = \frac{1}{2}\Big(&\left|1_A, \alpha\right\rangle\left\langle 1_A, \alpha\right| \otimes \left|1_B, \alpha\right\rangle\left\langle 1_B, \alpha\right| - \left|0_A, \alpha\right\rangle\left\langle 1_A, \alpha\right| \otimes \left|0_B, \alpha\right\rangle\left\langle 1_B, \alpha\right| - |1_A, \alpha\rangle\langle 0_A, \alpha| \otimes |1_B, \alpha\rangle\langle 0_B, \alpha| + |0_A, \alpha\rangle\langle 0_A, \alpha| \\ &\otimes |0_B, \alpha\rangle\langle 0_B, \alpha|\Big) = \frac{1}{2}\sum_{k,l=0,1}(-1)^{k+l}\rho^A_{kl}(0) \otimes \rho^B_{kl}(0).\end{aligned} \tag{28}$$

By using Eq. (27), the solution of Eq. 26 in terms the eigenstates of Eq. (6) is governed by

$$\hat{\rho}(t) = \frac{1}{2}\sum_{k,l=0,1}(-1)^{k+l}\rho^A_{kl}(t) \otimes \rho^B_{kl}(t). \tag{29}$$

To find the matrices $\rho^r_{ij}(t)$ $(r = A, B)$, we rewrite the initial density matrix of Eq. (28) in terms of the eigenstates of Eq. (6) as:

$$\rho^r_{11}(0) = \left|1_r, \alpha\right\rangle\left\langle 1_r, \alpha\right| = \frac{1}{2}\sum_{m,n=0}\eta^m_r\eta^{n*}_r\sum_{i,j=1,2}\left|\psi^m_i\rangle_r\left\langle\psi^n_j\right|_r, \tag{30}$$

$$\begin{aligned}\rho^r_{00}(0) = &\left|0, 0\right\rangle_r\left\langle 0, 0\right|_r e^{-|\alpha_r|^2} + e^{-\frac{1}{2}\left|\alpha_r\right|^2}\sum_{m=0}\frac{\eta^{m+1}_r}{\sqrt{2}}\left(\left|\psi^m_1\right\rangle_r\left\langle 0, 0\right|_r - \left|\psi^m_2\right\rangle_r\left\langle 0, 0\right|_r\right) + e^{-\frac{1}{2}\left|\alpha_r\right|^2}\sum_{n=0}\frac{\eta^{n+1}_r}{\sqrt{2}}\left(\left|0, 0\right\rangle_r\left\langle\psi^n_1\right|_r - \left|0, 0\right\rangle_r\left\langle\psi^n_2\right|_r\right) \\ &+ \frac{1}{2}\sum_{m,n=0}\eta^{m+1}_r(\eta^{n+1}_r)^*\sum_{i,j=1,2}(-1)^{i+j}\left|\psi^m_i\right\rangle_r\left\langle\psi^n_j\right|_r.\end{aligned} \tag{31}$$

$$\rho_{10}^{r}(0)=e^{-\frac{1}{2}|\alpha_r|^2}\sum_{m=0}\frac{\eta_r^m}{\sqrt{2}}\left(\left|\psi_1^m\right\rangle_r+\left|\psi_2^m\right\rangle_r\right)\left\langle 0,0\right|$$
$$+\frac{1}{2}\sum_{m,n=0}\eta_r^m\eta_r^{n+1*}\sum_{i,j=1,2}(-1)^{j+1}\left|\psi_i^m\right\rangle_r\left\langle\psi_j^n\right|_r.$$

With the super-operator $\hat{L}$, the time evolution of the density matrices $\left|\psi_i^m\right\rangle\left\langle\psi_j^n\right|$ is given by

$$e^{\hat{L}t}\left|\psi_i^m\right\rangle_r\left\langle\psi_j^n\right|_r=a_r^{mn}\left|\psi_i^m\right\rangle\left\langle\psi_j^n\right|=e^{\left[-i\left(\varepsilon_{ir}^m-\varepsilon_{jr}^n\right)t-\frac{\gamma_r}{2}\left(\varepsilon_{ir}^m-\varepsilon_{jr}^n\right)^2t\right]}\left|\psi_i^m\right\rangle_r\left\langle\psi_j^n\right|_r. \tag{32}$$

By using the Eqs. (27)–(32), we get the final density matrix, $\hat{\rho}(t)$, as in the Eq. (29). After that, we write the matrices $\rho_{ij}^r(t)$ $(r=A,B)$ of the final density matrix, $\hat{\rho}(t)$ in the space states $\{|\phi_1^n\rangle_r=|1_r,n_r\rangle, |\phi_2^n\rangle_r=|0_r,n_r+1\rangle\}$ by using Eq. (6).

Similarly, if the two atoms started initially from the separable state, $\rho^{AB}(0)=|1_A,0_B\rangle\langle 1_A,0_B|$, then

$$\hat{\rho}(0)=|1_A,0_B\rangle\langle 1_A,0_B|\otimes|\alpha\rangle_A\langle\alpha|\otimes|\alpha\rangle_B\langle\alpha|,$$
$$=\rho_{11}^A(0)\otimes\rho_{00}^B(0). \tag{33}$$

consequently we have

$$\hat{\rho}(t)=e^{\hat{L}t}\hat{\rho}(0)=\rho_{11}^A(t)\otimes\rho_{00}^B(t). \tag{34}$$

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