Density Functions for Entrainment and Deposition Rates of Uniform Sediment

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Abstract: A model has been developed for the prediction of the density functions of bed-elevation and entrainment and deposition rates of sediment in sand bed streams within the lower regime flow condition. The model incorporates both statistical and deterministic parameters in its form. A total of 46 experimental runs have been carried out in a recirculating tilting flume under the equilibrium flow condition using three grain sizes of uniform gradation to validate the model and estimate its parameters. The model parameters are related to the hydraulic conditions of flow and fluid and sediment properties through dimensional and regression analyses. The study has shown that the density functions of bed elevation and entrainment and deposition rates can be approximated quite satisfactorily with the normal distribution curve. Transformation of the density functions into the standardized normal distribution curve provides a unique pattern for all the experimental runs regardless of the sediment grain size, flow condition, and shapes and dimensions of the bed forms. The developed density functions have been utilized to provide a closure for the probabilistic Exner equation for uniform sediment.

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CE Database subject headings: Entrainment; Dunes; Probabilistic models; Bedforms; Sediment deposit.

Introduction

Studies which are concerned with the prediction of entrainment and deposition rates, bed geometry, and bed form movement can be classified into two major groups: Deterministic (e.g., Exner 1925; Ertel 1966; Kennedy 1969; Richards 1980) and probabilistic or stochastic (e.g., Nordin and Algert 1966; Engelund 1966a; Hino 1968; Jain and Kennedy 1971). Deterministic models provide relationships among relevant flow and sediment parameters while obscuring the details of the actual movement process and vertical exchange of sediment particles within the river bed. Probabilistic models can provide a better description of the sediment transport process, but they are not sufficiently generalized and do not typically account for variations in flow condition. Sediment transport and morphodynamics models should therefore incorporate both deterministic and statistical parameters in their forms (e.g., Einstein 1950; Engelund and Fredsoe 1976; Blom and Parker 2004; Elhakeem 2004). A significant amount of research has been carried out that is related to the probabilistic nature of bed load and bed-elevation fluctuation. Yet, few of these studies have led to predictive relations among sediment transport, bed-elevation variation, and stratigraphy (Parker et al. 2000). This can be attributed to the lack of a conceptual framework within which the probabilistic information could be incorporated. The difficulty of incorporating the hydraulic parameters in probabilistic models is perhaps another reason why only few probabilistic models of sediment transport are available in the literature compared to numerous models that are deterministic. Parker et al. (2000) proposed a new approach to a more general formulation of sediment continuity in which the probability density function of bed-elevation is used to derive a general Exner equation with no discrete layers. This formulation of sediment mass conservation requires more information than that of the active layer approach (Parker 1990). However, the new approach offers the prospect of a better understanding of the processes of entrainment, deposition, and sediment transport.

The main objective of this study is the development of a model describing the density functions of bed-level fluctuations and entrainment and deposition rates of uniform sediment within the conceptual framework proposed by Parker et al. (2000) for the modeling of the vertical exchange of sediment particles. In developing the model: (1) the continuous forms of the density functions described by Parker et al. (2000) are first converted into discrete forms; (2) general predictors for the density functions of bed-level fluctuations and entrainment and deposition rates are developed; (3) the model parameters are related to the hydraulic condition and fluid and sediment properties using dimensional and regression analyses; (4) experiments have been carried out to validate the model and estimate the model parameters under the equilibrium flow condition. The study also provides closure to the probabilistic form of the Exner equation by utilizing the developed density functions and the bed-load transport model developed by Elhakeem and Imran (2005). The present study is limited to uniform sediment.

Exner Equation of Sediment Continuity

In order to proceed with the theoretical development of this work, the probabilistic form of the Exner equation of sediment continu-

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ity proposed by Parker et al. (2000) is briefly revisited here. For uniform sediment, Parker et al. (2000) have expressed the Exner equation as

$$c_B p_z \frac{\partial \bar{z}_B}{\partial t} = D_z - E_z \tag{1}$$

in which c_B=volume fraction of the bed that is solid and is related to the porosity λ_p by $c_B = 1 - \lambda_p$; $\bar{z}_B$=mean bed-elevation averaged over local fluctuations associated with bed forms; E_z and D_z are elevation specific densities of entrainment and deposition rates, respectively; and p_z is the probability density function of bed-elevation fluctuations, that

$$E = \int_{-\infty}^{\infty} E_z dz \tag{2a}$$

$$D = \int_{-\infty}^{\infty} D_z dz \tag{2b}$$

$$P = 1 - \int_{-\infty}^{\infty} p_z dz \tag{2c}$$

where E and D=total rates of entrainment and deposition, respectively, in volume per unit area per unit time of bed sediment, and P=probability function of bed-elevation fluctuations over a number of bed forms. In Eq. (2), the lower limit of integration represents the lowest point on the bed at which the bed is neither degrading nor aggrading while the upper limit represents the highest point on the bed at which the bed is neither entraining nor depositing. Parker et al. (2000) speculated that the general forms for p_z, E_z, and D_z would decline smoothly toward zero as $z \to -\infty$ and as $z \to \infty$.

The deterministic form of the Exner equation can be obtained by integrating Eq. (1) over all bed elevations with the aid of Eq. (2) leading to

$$c_B \frac{\partial \bar{z}_B}{\partial t} = D - E \tag{3}$$

Present Approach for the Prediction of Density Functions

This section describes an approach for the prediction of the density functions described in Eq. (2) from the analysis of the bed-elevation fluctuations due to bed-form migration. Physical measurements of the elevation-specific densities of entrainment and deposition rates require the conversion of their continuous forms into discrete forms. Bagnold's (1941) approach is used here to describe the general movement of sediment particles within bed forms. A reach of stream is considered in equilibrium if the statistical properties of velocity, flow depth, channel energy slope, bed-load rate, and bed-form geometry do not change with either time or space where the variables are observed. Under this condition, Eq. (3) reduces to

$$E = D \tag{4}$$

Consider Fig. 1 which describes the profile $z_{B1}(x)$ at time t and the profile $z_{B2}(x)$ at time $t+\Delta t$ for a number of two-dimensional bed forms moving in the downstream direction. The difference of areas between the bed profiles $z_{B1}(x)$ and $z_{B2}(x)$ multiplied by c_B represents the net amount of sediment entrained from the bed, $c_B A_E$, and the net amount of sediment deposited into the bed, $c_B A_D$, within distance Δx. Under the equilibrium condition, these two quantities should be equal over the same distance. The areas of net entrainment and net deposition can easily be determined by the condition that if the difference in the bed-elevation at a given location, $\Delta z_B = z_{B2} - z_{B1}$, within Δx is negative, then this location on the stream bed is entraining, and if Δz_B is positive then this location on the stream bed is depositing. The volume rate of net sediment entrained from the bed per unit area, E, and the volume rate of net sediment deposited into the bed per unit area, D, can be expressed as

$$E = \frac{c_B A_E}{\Delta x \Delta t} \tag{5a}$$

$$D = \frac{c_B A_D}{\Delta x \Delta t} \tag{5b}$$

in which A_E=summation of the areas between the bed profiles $z_{B1}(x)$ and $z_{B2}(x)$ with negative values of Δz_B, and A_D=summation of the areas between the bed profiles $z_{B1}(x)$ and $z_{B2}(x)$ with positive values of Δz_B within the distance Δx (Fig. 1). If A_E is not equal to A_D, then the stream is not in equilibrium and there is sediment storage (positive or negative) within the reach under consideration.

The frequency histograms of E and D can be obtained by dividing the areas between the bed profiles $z_{B1}(x)$ and $z_{B2}(x)$ into a number of layers K of thickness Δz parallel to the x axis (Fig. 1). Multiplying the entraining areas in each layer by c_B gives the amount of net sediment entrained from the ith bed layer, $c_B A_{Eei}$, and multiplying the depositing areas in each layer by c_B gives the amount of net sediment deposited into the ith bed layer, $c_B A_{Dei}$, that

$$c_B A_E = c_B \sum_{i=1}^{K} A_{Eei} \tag{6a}$$

$$c_B A_D = c_B \sum_{i=1}^{K} A_{Dei} \tag{6b}$$

If Eqs. (6a) and (6b) are multiplied by $(1/\Delta x \Delta t)$ then

$$E = \sum_{i=1}^{K} E_{ei} \tag{7a}$$

$$D = \sum_{i=1}^{K} D_{ei} \tag{7b}$$

In Eq. (7), E_{ei} and D_{ei}=rates of sediment entrained from and deposited into the ith bed layer, respectively. The frequency histograms of E and D may have different shapes but the area under both curves should be the same if the stream is in equilibrium. Eqs. (7a) and (7b) are the discrete forms of Eqs. (2a) and (2b). The relationship between the elevation-specific density E_z or D_z at bed-elevation z_i and the histogram ordinate at the same level is given by

$$E_z(z_i) = \frac{E_{ei}}{\Delta z} \tag{8a}$$

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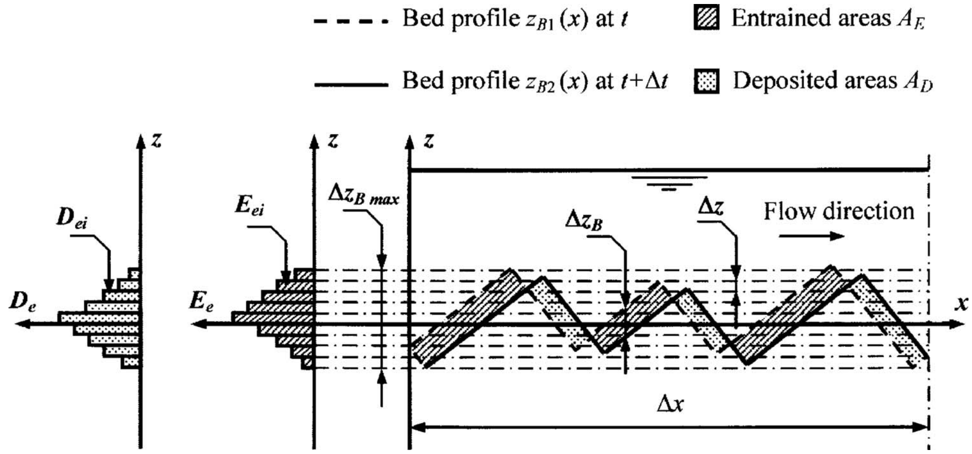

Fig. 1. Definition diagram for the derivation of the density functions of entrainment and deposition rates of uniform sediment

$$D_z(z_i) = \frac{D_{ei}}{\Delta z} \tag{8b}$$

The time interval Δt between the bed profiles $z_{B1}(x)$ and $z_{B2}(x)$ is chosen: (1) to be sufficiently short that a location along Δx has only one process: either entrainment or deposition, not the two processes together; and (2) to maximize the entrained and deposited volumes between the profiles. According to the first condition, the net entrainment or deposition rate will be equal to the absolute rate (Elhakeem and Imran 2005). Since entrainment, deposition, and bed load transport are interrelated, the second condition follows the same analogy used in the derivation of bed-load equations such that they predict the maximum bed-load rate that a stream in equilibrium can possibly carry at given hydraulic and sedimentary condition (Graf 1970). These two conditions can be satisfied when the bed forms travel a distance approximately equal to half of the average wave length.

The cumulative distribution P defined in Eq. (2) can be obtained from the measurement of bed level using the definition diagram shown in Fig. 2. Consider the distance Δx in the flow direction passing through a number of dunes. Let $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ represent the fractions of the distance Δx at elevation z that falls within the sediment bed rather than in the flowing water. The ratio $\Sigma_{i=1}^{n}\Delta x_i/\Delta x$ should approach unity at the lowest point on the bed and approach zero at the highest point on the bed (Parker et al. 2000). For three-dimensional bed features, the present approach can be applied by dividing the stream width into a number of intervals and obtaining the density functions of E, D, and P at each interval. The density functions of the stream reach can then be obtained by averaging over the stream width.

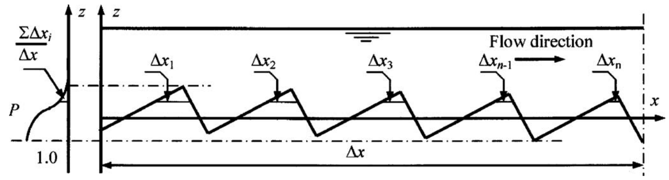

Fig. 2. Definition diagram for the derivation of the P function of Exner probabilistic equation of sediment continuity (after Parker et al. 2000)

Model Development

Preliminary analysis of the experimental data has shown that the density functions of D_z, E_z, and p_z can be fitted to the normal distribution curve. Accordingly, Eq. (2) can be written as

$$E = D = A\frac{1}{\sigma\sqrt{2\pi}}\int_{-B}^{B} e^{-1/2[(z_B-\mu)/\sigma]^2}dz \tag{9a}$$

$$P = 1 - \frac{1}{\sigma\sqrt{2\pi}}\int_{-B}^{B} e^{-1/2[(z_B-\mu)/\sigma]^2}dz \tag{9b}$$

where A=scale factor adjusting the area under the normal distribution curve that takes the hydraulic conditions of the flow into account; μ and σ, respectively=mean and the standard deviation of the density functions of E_z, D_z, and p_z; and z_B=departure of bed elevation from the mean bed elevation of the bed forms.

It should be noted that for the proposed distribution, the limits of the integration $-B$ and B are approximately equal to -3σ and $+3\sigma$, respectively. Here, B is equal to $\Delta z_{B\max}/2$, where $\Delta z_{B\max}$ represents the range of the bed surface layer (the difference between the lowest and highest points on bed) over which the processes of entrainment, deposition, and sediment movement occur (Fig. 1). The statistical parameters of an individual experimental run are represented by the mean μ and the standard deviation σ that are functions of $\Delta z_{B\max}$ such that

$$\mu = 0 \tag{10a}$$

$$\sigma \cong \Delta z_{B\max}/6 \tag{10b}$$

However, experimental results to be described later have shown that the mean is about $-0.05\Delta z_{B\max}$. This small deviation from the theoretical value given in Eq. (10a) can be attributed to the fact that the mean (the x axis through the bed forms) obtained from averaging over the bedform elevation does not coincide with the mean obtained from averaging with respect to the bed form areas. The latter should be lower as confirmed by experimental results.

In Eq. (9a), the value of

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{B=-3\sigma}^{B=3\sigma} e^{-1/2[(z_B-\mu)/\sigma]^2}dz$$

is approximately equal to 1.0 leading to

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$$E = D = A \tag{11}$$

Therefore, A=total rate of entrainment E or the total rate of deposition D for a reach of stream in equilibrium. Here, $\Delta z_{B\max}$ and A should be determined experimentally as functions of the hydraulic condition of flow, and the fluid and sediment properties. The range $\Delta z_{B\max}$ and the scale factor A can be expressed in nondimensional form as

$$\Delta z_* = \Delta z_{B\max} L/d_s^2 = f(\tau'_*) \tag{12a}$$

$$A_* = A/\sqrt{(S_s-1)gd_s} = f(\tau'_*) \tag{12b}$$

where A_*=dimensionless scale factor adjusting the area under the normal distribution curve; Δz_*=dimensionless range of the bed surface layer over which the processes of entrainment, deposition, and sediment movement occur; L=average length of bed forms; g=acceleration due to gravity; d_s=grain size diameter; τ'_*=dimensionless shear stress due to grain resistance only expressed as $R'_B S_f/(S_s-1)\ d_s$—in which S_f=slope of the energy grade line; R'_B=hydraulic radius of the channel bed corresponding to grain resistance only; and S_s=density of sediment relative to fluid expressed as ρ_s/ρ.

Experimental Setup and Procedure

A total of 46 runs have been conducted in the Hydraulics Laboratory of the Department of Civil and Environmental Engineering, at the University of South Carolina under the equilibrium flow condition in a recirculating tilting flume having a length of 10 m, width of 0.28 m, and depth of 0.4 m using natural sand of relative density equal to 2.65. The sand was carefully sieved using six standard sieves to obtain three different grain sizes of 0.25 mm, 0.44 mm, and 0.72 mm of uniform gradation. The flume was narrowed to 0.2 m, with a nonsealed wooden plate to fill and drain the water gradually from the side of the flume without affecting the bed-form patterns. Fig. 3 shows some patterns of bed forms developed with 0.44 mm sand. A closed operating system involves a submerged pump with a maximum discharge of 13 l/s to recirculate flow and sediment. The experimental setup also includes a tail gate, discharge meter, movable carriage with a point gauge to measure water depth and bed elevation, and a bed load trap with a sediment collector and scale.

The working section of the flume was filled with a 10 cm layer of the experimental sediment and the bed was leveled using a scraper. From the initial runs, the slope of the tilting flume was preadjusted to near equilibrium condition for a given depth and discharge. Water and sand were recirculated in the flume until the equilibrium condition was established. The flow was considered to be at equilibrium when the bed configuration was consistent in the full length of the flume, excluding the sections influenced by the entrance or the exit condition, and when the water surface slope and the mean bed slope remain constant and parallel with respect to time. Slope and depth reach equilibrium in accordance with the bed roughness for a specific flow condition.

Discharge and water depth were recorded after achieving the equilibrium condition, the experiment was stopped, and the water was carefully drained. The bed elevation was measured in the transverse and longitudinal directions of the flume. Five measurements were taken in the transverse direction at each section. Depending on the average bed-form wavelength, the step length in the longitudinal direction varied from 1 cm up to 6 cm for bed forms with average wave lengths ranging from 8 cm up to 120 cm, respectively. The reach Δx varied from 50 cm up to 300 cm for bed forms with average wave lengths ranging from 8 cm up to 120 cm, respectively. After taking the bed-elevation measurements, the flume was gradually filled with water and run again under the same flow condition until the bed forms travel nearly half their lengths. The experiment was then stopped again, and the water was carefully drained to measure the new bed elevation. The mean bed slope was also determined from bed-elevation measurements. These steps were repeated twice for each flow condition to obtain two successive trails of the density functions of entrainment and deposition rates. The rate of bed-load transport was obtained by using the sediment collector located near the end of the working section. Further details of the experimental conditions can be found in Elhakeem (2004).

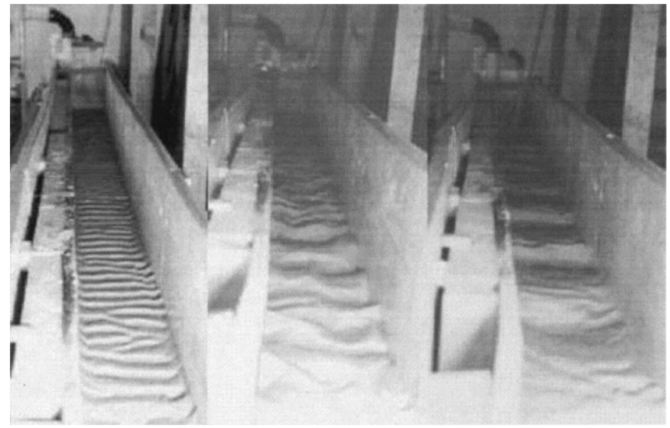

Fig. 3. Bedforms developed in uniform sand of 0.44 mm grain size diameter

Model Validation and Estimation of Parameters

The experimental results used to validate the model and estimate its parameters are presented in this section. Figs. 4(a), 5(a), and 6(a) show the density functions of entrainment and deposition rates and the probability density function of bed-elevation fluctuations of some of the runs fitted to the proposed theoretical curve (the normal distribution curve). Each experiment involved three sets of bed-elevation data. The bed-elevation data of the first and the second sets has been utilized in obtaining the plots designated as Run a while the second and the third sets have been utilized in obtaining the plots designated as Run b. The symbols represent the experimental results while the solid lines represent the theoretical curve. It is apparent from Figs. 4(a), 5(a), and 6(a) that the proposed normal distribution curve satisfactorily approximates the density functions of E_z, D_z, and p_z of the experimental runs. The Kolmogorov-Smirnov test for goodness of fit has shown that the normal distribution curve is a valid model for the density functions of all 46 experimental runs.

It can be noticed clearly from Figs. 4(a), 5(a), and 6(a) that the density functions have a large diversity in both shape and magnitude. With a new transformation variable $m=(z_B-\mu)/\sigma$, the density functions of E_z, D_z, and p_z can be transformed into the standardized normal distribution curve. In Figs. 4(a), 5(a), and 6(a), the E_z and D_z coordinates are transformed to the dimensionless coordinates $\sigma E_z/A$ and $\sigma D_z/A$, respectively, while the p_z coordinate is transformed to the dimensionless coordinate $\sigma\ p_z$. This transformation has two advantages: (1) It normalizes the

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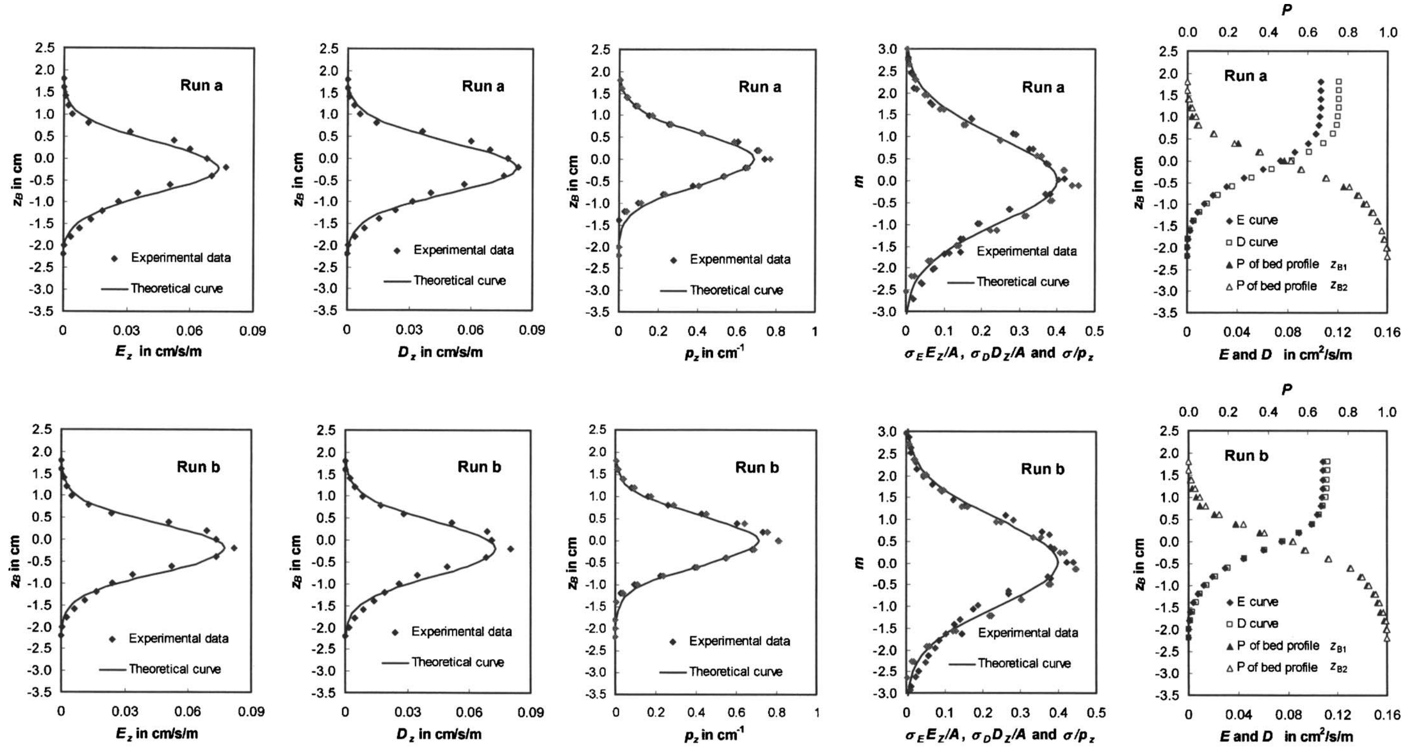

a. Density functions of entrainment, deposition and bed-level fluctuations.

b. Transformation of the density functions to the standardized normal distribution curve.

c. Cumulative curves of E_z, D_z and the P function.

Fig. 4. Experimental results and density functions of Runs 15 a and b under uniform flow condition, d_s=0.25 mm

density function coordinates; and (2) it gives a unique pattern for all the density functions under different hydraulic conditions. Figs. 4(b), 5(b), and 6(b) show the transformation of the density function to the standardized normal distribution.

Figs. 4(c), 5(c), and 6(c) show the cumulative curves of entrainment and deposition rates and the function P. The close match between the total entrainment and deposition rates indicates that the equilibrium condition was achieved in these runs. An individual run is considered to be in equilibrium if the absolute difference between entrainment and deposition rates divided by their average is less than 20%.

Three parameters control the shape and the magnitude of the density functions of entrainment and deposition rates of each individual run, namely, the mean μ, the standard deviation σ, and the scale factor A. The shape of the probability density function of bed-elevation fluctuations is controlled only by the mean and the standard deviation. The statistical parameters μ and σ are already related to the parameter $\Delta z_{B\max}$ by Eq. (10). However, mathematical relationships for A and $\Delta z_{B\max}$ are still needed to be expressed as functions of the hydraulic condition, and the fluid and sediment properties.

Fig. 7 shows the dimensionless parameter Δz_* versus the dimensionless grain shear stress resistance τ'_* for all the experimental runs. Engelund's (1966b) approach is applied to separate the grain resistance from the form drag due to bed forms. Vanoni and Brooks' (1957) approach to side-wall correction is also applied to the shear stress. Thus, the plotted values of shear stress correspond to grain resistance and bed roughness effect only. The symbols on the figure show the values obtained from experiments while the solid line represents the fitted line. As shown in Fig. 7, the relationship between Δz_* and τ'_* is not linear. The data was linearized by plotting Δz_{B*} versus $(\tau'_* - c)$ instead of τ'_* on log-log scale, where c is a constant. It has been found that a value of c equal to 0.033 gives the highest correlation coefficient between the experimental results and the fitted line (R^2=0.903). The relationship between Δz_* and τ'_* is obtained from regression analysis as

$$\Delta z_* = \Delta z_{B\max} L/d_s^2 = 4.611 \times 10^6 (\tau'_* - 0.033)^{1.315} \tag{13}$$

where L=bed form's average wave length which could be computed from the following relationship (Elhakeem and Imran 2005):

$$L_* = L/d_s = 7{,}400(\tau'_* - 0.033)^{0.737} \tag{14}$$

Fig. 8 shows the relationship between the dimensionless scale factor A_* and τ'_*. The value of the scale factor A in Fig. 8 represents the total rate of entrainment or the total rate of deposition. Since there is little difference between these rates in an individual run, the average values of the total entrainment rate E and total deposition rate D are used in plotting Fig. 8. The symbols show the values obtained from experiments while the solid line represents the fitted line obtained in the same manner described above. It has been found that the c value of 0.033 again gives the highest correlation coefficient between the experimental results and the fitted line (R^2=0.956). The relationship between A_* and τ'_* is given by

$$A_* = \frac{A}{\sqrt{(S_s - 1)gd_s}} = 0.0024(\tau'_* - 0.033)^{1.263} \tag{15}$$

The value 0.033 of the constant c is very close to the critical shear stress values of the experimental condition according to

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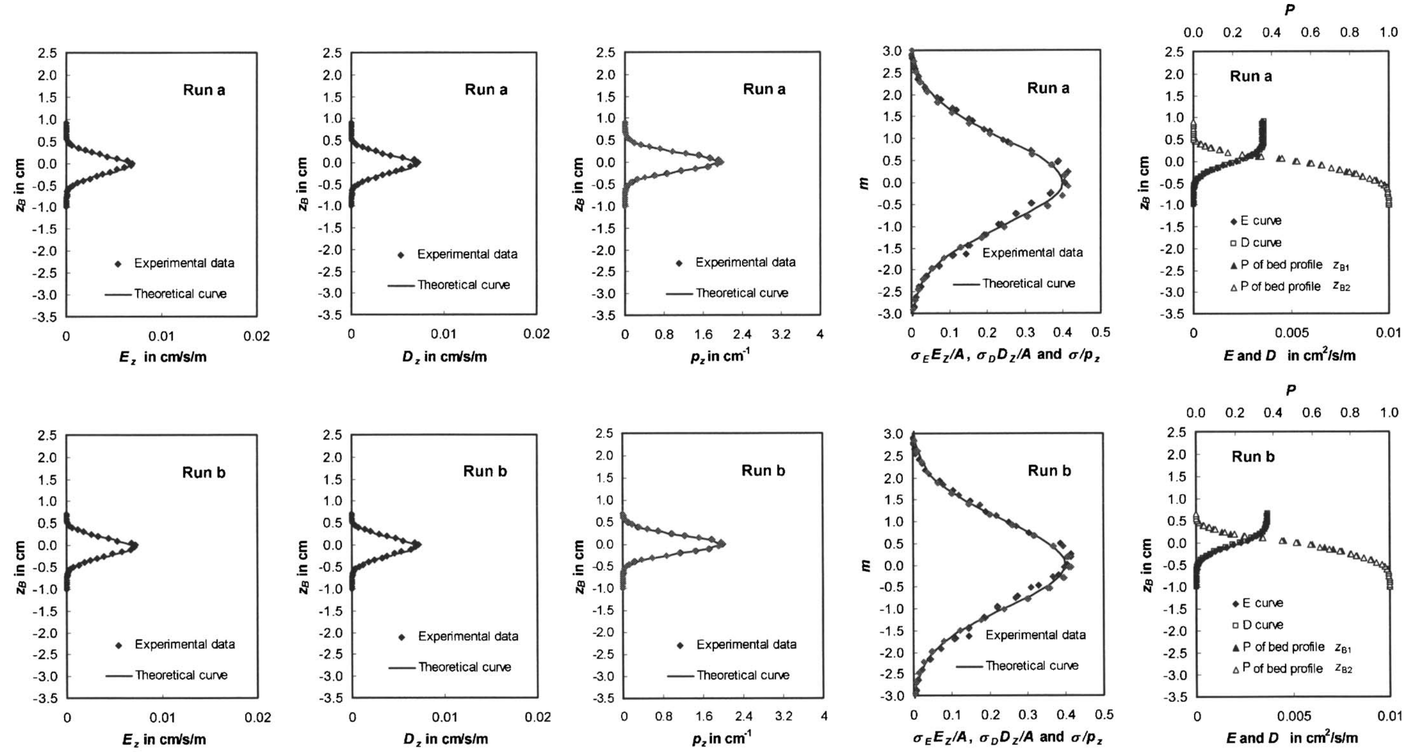
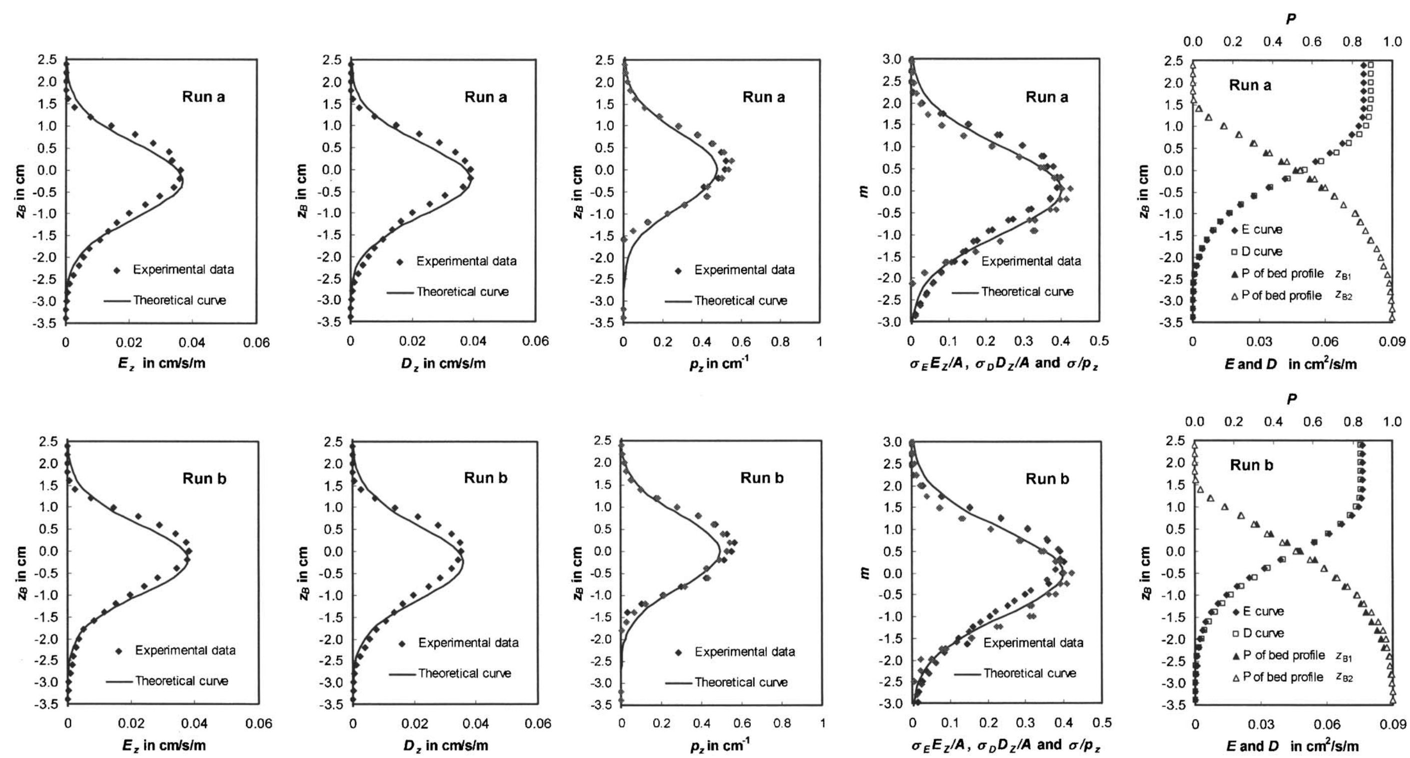


a. Density functions of entrainment, deposition and bed-level fluctuations.

b. Transformation of the density functions to the standardized normal distribution curve.

c. Cumulative curves of E_z, D_z and the P function.

Fig. 5. Experimental results and density functions of Runs 23 a and b under uniform flow condition, d_s=0.44 mm



a. Density functions of entrainment, deposition and bed-level fluctuations.

b. Transformation of the density functions to the standardized normal distribution curve.

c. Cumulative curves of E_z, D_z and the P function.

Fig. 6. Experimental results and density functions of Run 37 a and b under uniform flow condition, d_s=0.72 mm

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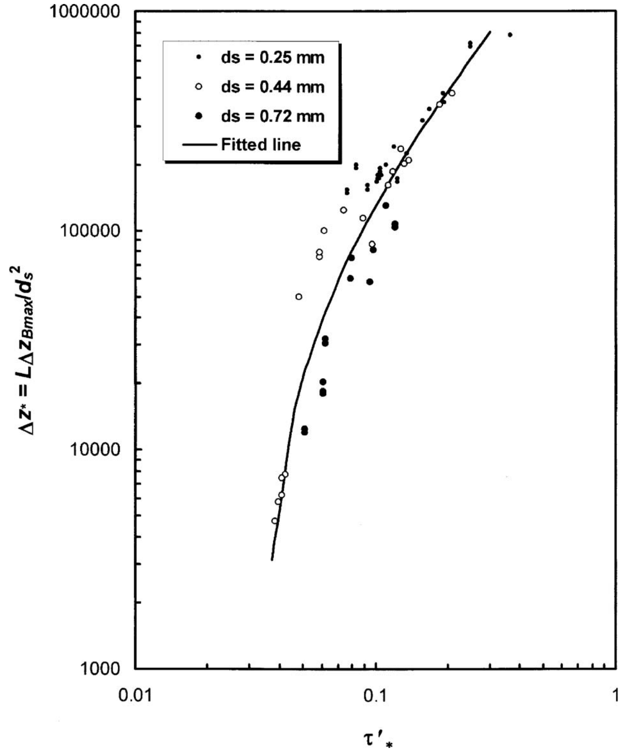

Fig. 7. Relationship between Δz_* and τ'_*

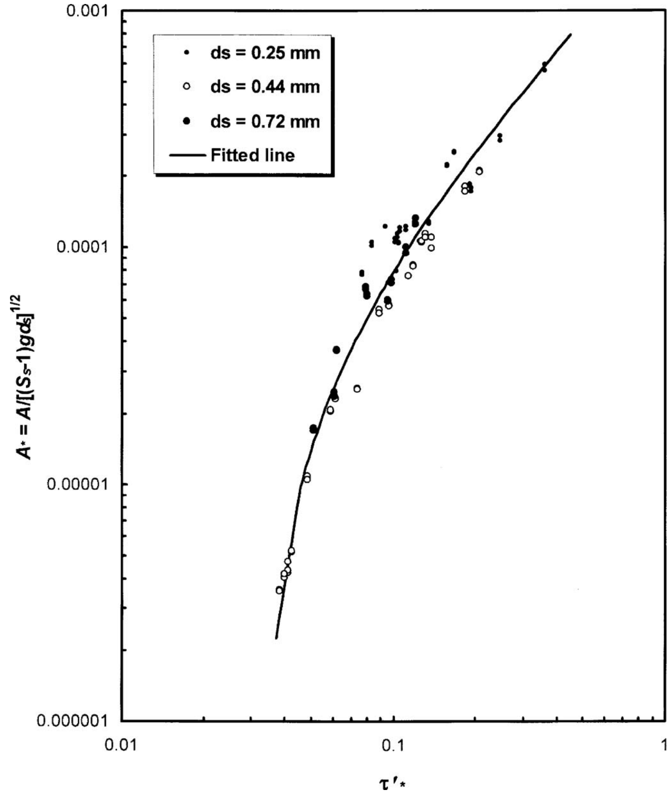

Fig. 8. Relationship between A_* and τ'_*

the Shields' criterion. Thus, the value of 0.033 in the abovementioned equations can be replaced by τ_{*c} which can be determined as a function of particle Reynolds number from the Shields diagram. Eqs. (13)–(15) allow for the determination of the statistical parameters given by Eq. (10) and the scale factor A that in turn can be used for the determination of the density functions of bed-elevation fluctuations and entrainment and deposition rates for a specific hydraulic condition and fluid and sediment property.

Application

In this section, a closure for the probabilistic Exner equation of sediment continuity is formulated utilizing the density functions developed here and the bed-load model proposed by Elhakeem and Imran (2005). The probabilistic form of the Exner equation of sediment continuity proposed by Blom and Parker (2004) is more general than the one proposed by Parker et al. (2000). Therefore, it will be considered here. Blom and Parker (2004) have expressed the Exner equation as

$$c_B\left(\frac{\partial P}{\partial t} + p_z\frac{\partial \bar{z}_B}{\partial t}\right) = D_z - E_z \tag{16}$$

which can be expressed in terms of the elevation-specific density of bed-load rate q_{Bz} as

$$c_B\left(\frac{\partial P}{\partial t} + p_z\frac{\partial \bar{z}_B}{\partial t}\right) = -\frac{\partial q_{Bz}}{\partial x} \tag{17}$$

Eq. (17) can be transformed into the following form using the density functions described by Eq. (9) and the bed-load function proposed by Elhakeem and Imran (2005)

$$c_B\left((0.557F^{-1} + 0.578z_Bc_2^{-1}F^{-0.756})\frac{\partial F}{\partial t} - c_2^{-1}F^{-0.578}\frac{\partial z_B}{\partial t} + N\frac{\partial \bar{z}_B}{\partial t}\right)$$
$$= -(1.422c_1NF + 0.578c_1c_2^{-1}mNF^{0.422}z_B)\frac{\partial F}{\partial x}$$
$$+ c_1c_2^{-1}mNF^{1.422}\frac{\partial z_B}{\partial x} \tag{18}$$

where $c_1 = 17.5\sqrt{(S_s-1)gd_s^3}$, $c_2 = 103.83d_s$, $N = (1/\sigma\sqrt{2\pi}) \times e^{-(1/2)[(z_B-\mu)/\sigma]^2}$, $F = (kR'_BS_f - \tau_{*c})$, and $k = [(S_s-1)d_s]^{-1}$.

The derivation of Eq. (18) is given in the Appendix. In deriving Eq. (18), it is assumed that the density functions of E_z, D_z, and p_z continue to follow the normal distribution curve under the nonequilibrium flow condition. However, the deterministic and statistical parameters can change due to change in sediment supply or flow condition. Elhakeem (2004) has shown that the distribution patterns of E_z, D_z, and p_z are not affected as long as the deviation from the equilibrium flow condition is moderate.

Eq. (18) includes terms that describe the temporal and spatial changes of the departure of bed-elevation from the mean bed elevation of the bed forms. If net aggradation or degradation occurs, i.e., $\partial\bar{z}_B/\partial t \neq 0$, the following deterministic form of the Exner equation should be solved simultaneously with Eq. (18)

$$\frac{\partial \bar{z}_B}{\partial t} = -2c_1F\frac{\partial F}{\partial x} \tag{19}$$

Eq. (18) also includes in its formulation, terms such as F and its derivatives, that are related to changes in the flow field. Solution of the Saint Venant equations (Chaudhry 1993) can be used simultaneously with Eqs. (18) and (19) to account for these changes.

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Conclusions

In this paper, a model has been developed to predict the density functions of bed-elevation fluctuations and entrainment and deposition rates within the conceptual framework of Parker et al. (2000) for the modeling of the vertical exchange of sediment particles in bed forms. The model is based on both statistical and deterministic parameters. Experiments have been conducted under the equilibrium flow condition within the lower regime using three grain sizes of uniform sand to validate the model and to estimate its statistical and deterministic parameters. The experimental results have shown that the density functions of bed-elevation fluctuations and entrainment and deposition rates can be approximated quite satisfactorily with the normal distribution curve. Transformation of these density functions for each individual run to the standardized normal distribution curve provides a universal pattern for all the experiments, regardless of the hydraulic conditions of flow, sediment grain size, and shapes and dimensions of the bed forms. The developed density functions along with the bed-load model of Elhakeem and Imran (2005) have been incorporated in the probabilistic formulation of sediment continuity proposed by Blom and Parker (2004) to provide a closure for the probabilistic Exner equation. One limitation of the present approach is that it cannot be applied at very low values of shear stress at which the bed forms do not develop and at very high values of shear stress at which the bed forms disappear.

Appendix. Derivation of the Exner Equation of Sediment Continuity from the Present Approach

Elhakeem and Imran (2005) have shown that the average bed-load rate $\bar{q}_B$ can be expressed as

$$\bar{q}_B = AL \tag{20}$$

From Eqs. (10) and (13)–(15)

$$AL = c_1 F^2 \tag{21a}$$

$$\sigma = c_2 F^{0.578} \tag{21b}$$

where $c_1 = 17.5\sqrt{(S_s-1)gd_s^3}$, $c_2 = 103.83 d_s$, $F = (kR'_B S_f - \tau_{*c})$, and $k = [(S_s-1)d_s]^{-1}$.

The normal distribution function of Eq. (9) can be written as

$$N = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(z_B-\mu)/\sigma]^2} \tag{22}$$

From Eqs. (21) and (22)

$$\partial(AL) = 2c_1 F\, \partial F \tag{23a}$$

$$\partial\sigma = 0.578 c_2 F^{-0.422}\, \partial F \tag{23b}$$

$$\partial N = \frac{-N}{\sigma}\, \partial\sigma + \frac{mNz_B}{\sigma^2}\, \partial\sigma - \frac{mN}{\sigma}\, \partial z_B \tag{23c}$$

From Eqs. (21b), (23b), and (23c)

$$\partial N = -0.578NF^{-1}\, \partial F + 0.578c_2^{-1} mNF^{-1.578} z_B\, \partial F - c_2^{-1} mNF^{-0.578}\, \partial z_B \tag{24}$$

Considering the effect of bed level fluctuation, Eq. (20) can be written with the aid of Eq. (9) as

$$q_{Bz} = AL\frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(z_B-\mu)/\sigma]^2} \text{ or} \tag{25a}$$

$$q_{Bz} = c_1 F^2 N \tag{25b}$$

Differentiating Eq. (25b) with respect to distance x gives

$$\frac{\partial q_{Bz}}{\partial x} = 2c_1 FN\frac{\partial F}{\partial x} + c_1 F^2 \frac{\partial N}{\partial x} \text{ or} \tag{26a}$$

$$\frac{\partial q_{Bz}}{\partial x} = (1.422c_1 NF + 0.578c_1 c_2^{-1} mNF^{0.422} z_B)\frac{\partial F}{\partial x} - c_1 c_2^{-1} mNF^{1.422}\frac{\partial z_B}{\partial x} \tag{26b}$$

From Eq. (9) the term $(\partial P/\partial t)$ in Eq. (16) can be written as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left\{1 - \frac{1}{\sigma\sqrt{2\pi}}\int_{-B}^{B} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\right\} \tag{27}$$

Applying the Leibnitz's rule to Eq. (27) gives

$$-\int_{-B}^{B}\frac{\partial}{\partial t}\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\right\} - \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(B-\mu)/\sigma]^2}\frac{\partial B}{\partial t} + \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(-B-\mu)/\sigma]^2}\frac{\partial(-B)}{\partial t} \tag{28a}$$

which reduces to

$$-\int_{-B}^{B}\frac{\partial}{\partial t}\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\right\} - \frac{0.03}{\sigma\sqrt{2\pi}}\frac{\partial B}{\partial t} \tag{28b}$$

In Eq. (28b), the term

$$-\int_{-B}^{B}\frac{\partial}{\partial t}\left\{\frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\right\}$$

gives

$$\frac{1}{\sigma^2\sqrt{2\pi}}\int_{-B}^{B} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\frac{\partial\sigma}{\partial t} - \frac{1}{\sigma\sqrt{2\pi}}\int_{-B}^{B}\frac{-(z_B-\mu)}{\sigma} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\frac{\partial}{\partial t}\left(\frac{z_B-\mu}{\sigma}\right)$$

which reduces to

$$\frac{1}{\sigma}\frac{\partial\sigma}{\partial t} - \frac{1}{\sigma}\frac{\partial z_B}{\partial t} + \frac{z_B}{\sigma^2}\frac{\partial\sigma}{\partial t} \tag{29}$$

Substitution of Eq. (29) into (28b) gives

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial t}\left\{\frac{1}{\sigma\sqrt{2\pi}}\int_{-B}^{B} e^{-(1/2)[(z_B-\mu)/\sigma]^2} dz\right\} = \frac{0.964}{\sigma}\frac{\partial\sigma}{\partial t} + \frac{z_B}{\sigma^2}\frac{\partial\sigma}{\partial t} - \frac{1}{\sigma}\frac{\partial z_B}{\partial t} \tag{30}$$

Substitution of Eqs. (21b) and (23b) into (30) gives

$$\frac{\partial P}{\partial t} = (0.557F^{-1} + 0.578 z_B c_2^{-1} F^{-0.756})\frac{\partial F}{\partial t} - c_2^{-1} F^{-0.578}\frac{\partial z_B}{\partial t} \tag{31}$$

Substitution of Eqs. (22), (26b), and (31) into (17) gives

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$$c_B\left((0.557F^{-1}+0.578z_Bc_2^{-1}F^{-0.756})\frac{\partial F}{\partial t}-c_2^{-1}F^{-0.578}\frac{\partial z_B}{\partial t}+N\frac{\partial \bar{z}_B}{\partial t}\right)$$
$$=-(1.422c_1NF+0.578c_1c_2^{-1}mNF^{0.422}z_B)\frac{\partial F}{\partial x}$$
$$+c_1c_2^{-1}mNF^{1.422}\frac{\partial z_B}{\partial x} \qquad (32)$$

in which

$$\frac{\partial F}{\partial x}=k\left(S_f\frac{\partial R_B'}{\partial x}+R_B'\frac{\partial S_f}{\partial x}\right) \text{ and } \frac{\partial F}{\partial t}=k\left(S_f\frac{\partial R_B'}{\partial t}+R_B'\frac{\partial S_f}{\partial t}\right)$$

Notation

The following symbols are used in this paper:

A = scale factor adjusting the area under the normal distribution curve;
A_* = dimensionless scale factor adjusting the area under the normal distribution curve;
A_D = amount of net sediment deposited into the bed within distance Δx;
A_E = amount of net sediment entrained from the bed within distance Δx;
B = $\Delta z_{B\max}/2$;
c = constant;
c_B = volumetric fraction of sediment in bed;
c_BA_{Dei} = net amount of sediment deposited into the ith bed layer;
c_BA_{Eei} = net amount of sediment entrained from the ith bed layer;
D = total rate of deposition in volume per unit area per unit time of bed sediment;
D_{ei} = rate of deposition into the ith bed layer;
D_z = elevation-specific density of deposition rate;
d_s = mean grain size or particle diameter;
E = total rate of entrainment in volume per unit area per unit time of bed sediment;
E_{ei} = rate of entrainment from the ith bed layer;
E_z = elevation-specific density of entrainment rate;
g = acceleration due to gravity;
K = number of layers within $\Delta z_{B\max}$;
L = bed form's average wave length;
L_* = dimensionless bed form's wave length;
m = transformation variable;
P = probability function of bed level fluctuations over a number of bed forms;
p_z = probability density function of bed-elevation fluctuations;
q = flow rate per unit width;
$\bar{q}_B$ = average bed-load transport rate in volume per unit width per unit time;
R = correlation coefficient;
R_B' = hydraulic radius of the channel bed corresponding to grain resistance only;
S_f = slope of energy grade line;
S_s = relative density of sediment to flow$=\rho_s/\rho$;
t = time;
U = mean flow velocity;
x = distance;
y = flow depth;
z_B = departure of bed elevation from the mean bed elevation;
$\bar{z}_B$ = mean bed elevation;
$z_{B1}(x)$ = bed profile at time t;
$z_{B2}(x)$ = bed profile at time $t+\Delta t$;
Δt = time interval between two observations of bed profile within distance Δx which gives maximum deposited area A_D and entrained area A_E;
Δz_B = $z_{B2}-z_{B1}$;
Δz_* = dimensionless range of the bed surface layer;
$\Delta z_{B\max}$ = range of the bed surface layer over which the processes of entrainment, deposition, and sediment movement occur;
λ_p = porosity;
μ = mean of the density functions of bed-elevation fluctuations, entrainment, or deposition rate;
ρ = fluid density;
ρ_s = sediment density;
σ = standard deviation of the density functions of bed-elevation fluctuations, entrainment, or deposition rate;
τ_{*c} = dimensionless critical bed shear stress; and
τ_*' = dimensionless shear stress due to grain resistance only.

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