Int. J. Appl. Comput. Math (2019) 5:38
https://doi.org/10.1007/s40819-019-0624-0

ORIGINAL PAPER

The Modified Fractional Power Series Method for Solving Fractional Non-isothermal Reaction–Diffusion Model Equations in a Spherical Catalyst

Muhammed I. Syam[1] · Mohamed-Naim Yehia Anwar[2] · Ahmet Yildirim[3] · Mahmmoud M. Syam[4]

Abstract

In this paper, we study the fractional non-isothermal reaction–diffusion model equations in a spherical catalyst. The modified fractional power series method (MFPS) is employed to compute an approximation to the solution of this problem. The validity of the MFPS is ascertained by computing the solutions as α approaches to 1 and comparing our results with other methods in the literature when $\alpha = 1$. The numerical results show the efficiency of our method. The existences of the solution is proved. In addition, the convergent of the proposed method is investigated. The influences of the main parameters in the model on the solution are discussed.

Keywords Fractional non-isothermal reaction–diffusion model equations in a spherical catalyst · Nonlinear boundary value problem · Modified fractional power series method

Mathematics Subject Classification 76A05 · 76W05 · 76Z99 · 65L05

Introduction

The Lane–Emden equation has the form

$$u'' + \frac{k}{x}u'(x) + f(u(x)) = 0,\ k > 0. \tag{1}$$

✉ Muhammed I. Syam
m.syam@uaeu.ac.ae

1 Department of Mathematical Sciences, College of Science, UAE University, Al-Ain, United Arab Emirates

2 Basic Sciences Department, Faculty of Engineering, Pharos University in Alexandria, Alexandria, Egypt

3 Department of Mathematics, Faculty of Science, Ege University, 35100 Izmir, Bornova, Turkey

4 Department of Mechanical Engineering, College of Engineering, Al-Ain, United Arab Emirates

Published online: 08 March 2019

The singular behavior at $x = 0$ gives the main difficulty for solving this class of problems. Several numerical techniques used to solve this type of problems, see [1–3]. Many industrial reactors involve heterogeneous kinetics of packed catalytic pellets. The dimensionless concentration of chemical species within a spherical catalyst is modeled mathematically in the form

$$u''(x) + \frac{2}{x}u'(x) - \phi^2 u(x) \exp\left(\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}\right) = 0 \tag{2}$$

subject to

$$u'(0) = 0, \quad u(1) = 1 \tag{3}$$

where

$$x = \frac{r}{R}, \; u = \frac{C_A}{C_{As}}, \; \gamma = \frac{E}{R_g T_s}, \tag{4}$$

$$\beta = \frac{(-\Delta H)DC_{As}}{KT_s}, \; \phi^2 = \frac{K_{ref}\;R_2}{D}, \; \eta = \frac{3}{\phi^2}\frac{du}{dx}(1), \tag{5}$$

where $\phi, \alpha.$ and β are constants [4–6]. Several researchers used different numerical techniques to solve this problem such as variational iteration method [7], Adomian decomposition method [8], and modified decomposition method [9–15]. In this paper, we study the generalized problem in the fractional form [14–22]

$$D^{2\alpha}u(x) + \frac{2}{x^\alpha}D^\alpha u(x) - \phi^2 u(x) \exp\left(\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}\right) = 0, \; \frac{1}{2} < \alpha \le 1, \tag{6}$$

subject to

$$D^\alpha u(0) = 0, \quad u(1) = 1. \tag{7}$$

The derivative in Eqs. (1–2) is in the Caputo derivative sense. We write the definition and some preliminary results of the Caputo fractional derivatives, as well as, the definition of the fractional power series and one of its properties.

Definition 1 For $\xi > 0, \; r-1 < \xi < r, \; r$ is an integer, $v > 0$, the Caputo fractional derivative is defined by

$$D^\xi q(v) = \begin{cases} \frac{1}{\Gamma(r-\xi)} \int_0^t (v-w)^{r-1-\xi} q^{(r)}(w) ds, & \xi > 0, \\ q(v), & \xi = 0, \end{cases} \tag{8}$$

where Γ is the well-known Gamma function.

The Caputo fractional derivative satisfies the following properties for $\alpha > 0$, see [12].

1. $D^\alpha c = 0$, where c is constant,
2. $D^\alpha v^\gamma = \left\{ \begin{array}{cc} 0, & \gamma < \alpha, \gamma \in \{0, 1, 2, \ldots\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} v^{\gamma-\alpha}, & \text{otherwise} \end{array} \right\}.$

Next, we write the definition and one of the properties of the fractional power series which are used in this paper. More details can be found in [13].

Definition 2 A series

$$\sum_{s=0}^{\infty} b_m (v - v_0)^{sr} = v_0 + v_1(v - v_0)^r + b_2(v - v_0)^{2r} + \cdots \tag{9}$$

is called fractional power series FPS about $v = v_0$.

Suppose that g has a FPS representation at $v = v_0$ of the form

$$g(v) = \sum_{s=0}^{\infty} b_s (v - v_0)^{sr}, \quad v_0 \le v < v_0 + \beta. \tag{10}$$

If $D^{sr} g(v), \;\; s = 0, 1, 2, \ldots$ are continuous on $\mathbb{R}$, then $b_s = \frac{D^{sr} g(v_0)}{\Gamma(1+sr)}$.

MFPS Method for Solving the Fractional Non-isothermal Reaction–Diffusion Model Equations in a Spherical Catalyst

Consider the following problem

$$D^{2\alpha} u(x) + \frac{2}{x^{\alpha}} D^{\alpha} u(x) - \phi^2 u(x) \exp\left(\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}\right) = 0, \; \frac{1}{2} < \alpha \le 1, \tag{11}$$

subject to

$$D^{\alpha} u(0) = 0, \;\; u(1) = 1. \tag{12}$$

Expand $g(u) = \left(\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}\right)$ by

$$g(u) = c_0 + c_1 u(x) + c_2 u^2(x) + \bigcirc(u^3) \tag{13}$$

where $c_0 = e^{\frac{\beta\gamma}{1+\beta}}$, $c_1 = -\frac{\beta\gamma e^{\frac{\beta\gamma}{1+\beta}}}{(1+\beta)^2}$, and $c_2 = -\frac{\beta^2\gamma(2+2\beta-\gamma)e^{\frac{\beta\gamma}{1+\beta}}}{2(1+\beta)^4}$. The MFPS method proposes the solution of the problem in the form of fractional power series as

$$u(x) = \sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{14}$$

To obtain the approximate values of the above series (14), we consider its n-th truncated series $u_n(x)$ which has the form

$$u_n(x) = \sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{15}$$

Since $u(0) = f_0 = A$ and $D^{\alpha} u(0) = f_1 = 0$, we rewrite Eq. (15) as

$$u_n(x) = A + \sum_{k=2}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad n = 2, 3, \ldots \tag{16}$$

where $u_1(x) = f_0 + f_1 \frac{x^{\alpha}}{\Gamma(1+\alpha)} = A$ is considered to be the 1st RPS approximate solution of $u(x)$. To find the values of the RPS-coefficients $f_n, \; n = 2, 3, 4, \ldots$, we solve the fractional differential equation

$$D^{(n-2)\alpha} Res_n(0) = 0, \;\; n = 2, 3, 4, \ldots \tag{17}$$

where $Res_n(x)$ is the n-th residual function and it is defined by

$$Res_n(x) = D^{2\alpha} u_n(x) + \frac{2}{x^{\alpha}} D^{\alpha} u_n(x) - \phi^2 u_n(x)\left(c_0 + c_1 u_n(x) + c_2 u_n^2(x)\right). \tag{18}$$

To find f_n, for $n \geq 2$, we consider its n-th truncated series $u_n(x)$ of the form

$$u_n(x) = \sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{19}$$

Substitute $u_n(x)$ into $Res_n(x)$ to get

$$\begin{aligned} Res_n(x) = {} & D^{2\alpha}\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) + \frac{2}{x^{\alpha}} D^{\alpha}\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) \\ & -\phi^2 \left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) \\ & \left(c_0 + c_1\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) + c_2\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right)^2\right) \end{aligned}$$

or

$$\begin{aligned} Res_n(x) = {} & \left(\sum_{k=2}^{n} f_k \frac{x^{(k-2)\alpha}}{\Gamma(1+(k-2)\alpha)}\right) + 2\left(\sum_{k=2}^{n} f_k \frac{x^{(k-2)\alpha}}{\Gamma(1+(k-1)\alpha)}\right) \\ & -\phi^2 \left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) \\ & \left(c_0 + c_1\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) + c_2\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right)^2\right). \end{aligned}$$

Then, set $D^{(n-2)\alpha} Res_n(0) = 0$ to get

$$\begin{aligned} 0 = {} & f_n + 2\ f_n \frac{\Gamma(1+(n-2)\alpha)}{\Gamma(1+(n-1)\alpha)} - \phi^2\ c_0\ f_{n-2} \\ & -\phi^2\ c_1 \Gamma(1+(n-2)\alpha) \sum_{k=0}^{n-2} \frac{f_k\ \ f_{n-2-k}}{\Gamma(1+k\alpha)\Gamma(1+(n-2-k)\alpha)} \\ & -\phi^2\ c_2 D^{(n-2)\alpha} \left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right)^3_{|x=0}. \end{aligned}$$

It is worth mention that $D^{(n-2)\alpha}\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right)^3_{|x=0}$ is a function of $f_0,\ f_1,\ \ldots,\ f_{n-2}$. For simplicity, we assume that

$$D^{(n-2)\alpha}\left(\sum_{k=0}^{n} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right)^3_{|x=0} = h(f_0,\ f_1,\ \ldots,\ f_{n-2}). \tag{20}$$

Thus,

$$f_n = \frac{\Gamma(1+(n-1)\alpha)\left(\begin{array}{c}\phi^2\, c_0\, f_{n-2} + \phi^2\, c_1 \Gamma(1+(n-2)\alpha) \sum_{k=0}^{n-2} \frac{f_k \quad f_{n-2-k}}{\Gamma(1+k\alpha)\Gamma(1+(n-2-k)\alpha)} \\ +\phi^2\, c_2 h(f_0,\ f_1,\ \ldots,\ f_{n-2})\end{array}\right)}{\Gamma(1+(n-1)\alpha) + 2\Gamma(1+(n-2)\alpha)} \tag{21}$$

for $n \geq 2$.

In the next theorem, we study the convergence of the series (7) to the solution of problem (4)–(5).

Theorem 2.1 *Let* $u(x) = \sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$ *and* $\frac{1}{2} < \alpha \leq 1$. *Then, the sequence* $\{u_k(x)\}$ *converges to the solution of problem* (4)–(5).

Proof First, we want to prove that $\sum_{n=2}^{\infty} f_n \frac{x^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ converges to $D^{2\alpha}u(x)$ when $0 < x < 1$. For any $1 > x > 0$,

$$\begin{aligned} D^{2\alpha}u(x) &= \frac{1}{\Gamma(2-2\alpha)} \int_0^x (x-s)^{1-2\alpha} u''(s) ds \\ &= \frac{1}{\Gamma(2-2\alpha)} \int_0^x (x-s)^{1-2\alpha} \left(\sum_{n=0}^{\infty} f_n \frac{s^{n\alpha}}{\Gamma(1+n\alpha)}\right)'' ds \\ &= \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n\alpha)} \frac{1}{\Gamma(2-2\alpha)} \int_0^x (x-s)^{1-2\alpha} \left(s^{n\alpha}\right)'' ds \\ &= \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n\alpha)} D^{2\alpha}(x^{n\alpha}) = \sum_{n=2}^{\infty} \frac{f_n}{\Gamma(1+(n-2)\alpha)} x^{(n-2)\alpha}. \end{aligned}$$

Thus, $\sum_{n=2}^{\infty} f_n \frac{x^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$ converges to $D^{2\alpha}u(x)$ when $1 > x > 0$. Similarly, one can show that $\sum_{n=1}^{\infty} f_n \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$ converges to $D^{\alpha}u(x)$ when $0 < x < 1$. Next, we want to prove the sequence $\{u_k(x)\}$ converges to the solution of problem (4)–(5). Let

$$\begin{aligned} \sum_{n=0}^{\infty} \xi_n x^{n\alpha} &= \left(\sum_{k=2}^{\infty} f_k \frac{x^{(k-2)\alpha}}{\Gamma(1+(k-2)\alpha)}\right) + 2\left(\sum_{k=2}^{\infty} f_k \frac{x^{(k-2)\alpha}}{\Gamma(1+(k-1)\alpha)}\right) \\ &\quad -\phi^2 \left(\sum_{k=0}^{\infty} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) \\ &\quad \left(c_0 + c_1 \left(\sum_{k=0}^{\infty} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right) + c_2 \left(\sum_{k=0}^{\infty} f_k \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\right)^2\right). \end{aligned}$$

Since $D^{2\alpha}u(x) = \sum_{n=2}^{\infty} f_n \frac{x^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}$, $D^{\alpha}u(x) = \sum_{n=1}^{\infty} f_n \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$, and $u(x) = \sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$, then

$$\sum_{n=2}^{\infty} \xi_n x^{n\alpha} = 0. \tag{22}$$

Let

$$S_k = \sum_{n=k}^{\infty} \xi_n x^{n\alpha}. \tag{23}$$

Then, the sequence $\{S_k\}$ converges to zero. We see that

$$Res_k(x) = S_k. \tag{24}$$

Thus,

$$\lim_{k\to\infty} Res_k(x) = \lim_{k\to\infty} S_k = 0. \tag{25}$$

Hence, the sequence $\{u_k(t)\}$ converges to the solution. □

Numerical Results

The exact solution of the fractional non-isothermal reaction–diffusion model equations in a spherical catalyst of the form

$$D^{2\alpha}u(x) + \frac{2}{x^\alpha} D^\alpha u(x) - \phi^2 u(x) \exp\left(\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}\right) = 0, \ \frac{1}{2} < \alpha \le 1, \tag{26}$$

subject to

$$D^\alpha u(0) = 0, \ \ u(1) = 1 \tag{27}$$

is not known. Therefore, the numerical solution has been determined by the proposed method in the previous section. We compare our results with other results obtained by other researchers for the case $\alpha = 1$. In order to be able to compare our results with others, we first take $\alpha = 1$. Let $\phi = 1$ and $\beta = 1$. We study the influence of the parameter γ on the solution. For $\gamma = \frac{1}{2}$ and following the same procedure that is described in the previous section, the first few approximations of the solution are given by

$$\begin{aligned}
u_0(x) &= u_1(x) = a_0, \\
u_2(x) &= u_3(x) = a_0 + \frac{e^{1/4}}{768}(128a_0 - 16a_0^2 - 7a_0^3)x^2, \\
u_4(x) &= u_5(x) = a_0 + \frac{e^{1/4}}{768}(128a_0 - 16a_0^2 - 7a_0^3)x^2 \\
&\quad + \frac{e^{1/2}}{1966080}\left(16384a_0 - 6144a_0^2 - 3072a_0^3 + 560a_0^4 + 147a_0^5\right), \\
&\quad \vdots
\end{aligned}$$

To find a_0, we solve the equation $u_k(1) = 1$. For $k = 6$, the solution of $u_6(1) = 1$ implies that $a_0 = 0.8395084016966201$ and

$$u_6(x) = 0.839508 + 0.153881x^2 + 0.00666358x^4 - 0.0000527903x^6. \tag{28}$$

In Table 1, we report the values of a_0 and $u_6(x)$ for different values of γ.

Figure 1 shows the influence of the parameter γ on the solution of the fractional non-isothermal reaction–diffusion model equations in a spherical catalyst.

Table 1 The values of a_0 and $u_6(x)$ for different values of γ

γ	a_0	$u_6(x)$
0.5	0.8395084016966201	$0.839508 + 0.153881x^2 + 0.00666358x^4 - 0.0000527903x^6$
1	0.8291503112149827	$0.82915 + 0.165927x^2 + 0.00536284x^4 - 0.000439684x^6$
1.5	0.8206637567337322	$0.820664 + 0.177593x^2 + 0.00277704x^4 - 0.00103413x^6$

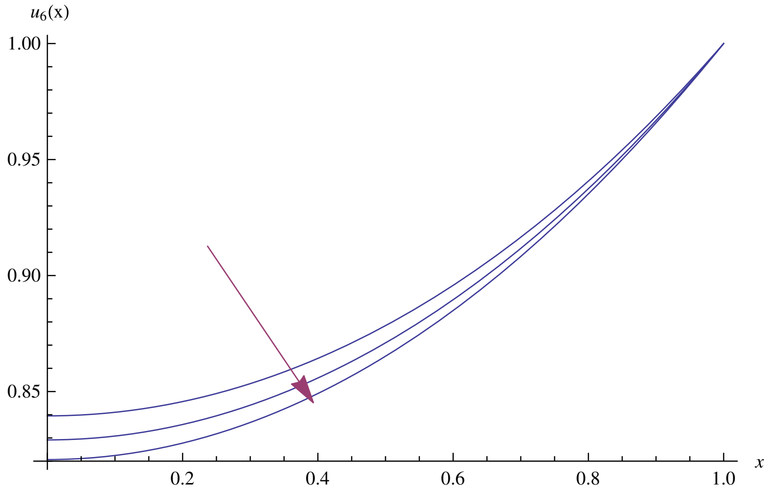

Fig. 1 Graphs of $u_6(x)$ for $\phi = 1,\ \beta = 1, \alpha = 1$, and $\gamma = \frac{1}{2}, 1, \frac{3}{2}$

Table 2 The values of η for different values of γ

γ	η
0.5	1.0022975640854817
1	1.0519989558224643
1.5	1.0802700990962877

Table 3 The values of a_0 and $u_6(x)$ for different values of γ

γ	a_0	$u_6(x)$
0.5	0.815341	$0.815341 + 0.174743x^{1.8} + 0.0099654x^{3.6} - 0.0000497697x^{5.4}$
1	0.802374	$0.802374 + 0.189372x^{1.8} + 0.0090584x^{3.6} - 0.0008046113x^{5.4}$
1.5	0.792034	$0.792034 + 0.204437x^{1.8} + 0.0055737x^{3.6} - 0.0020451675x^{5.4}$

Let $\eta = \frac{3}{\phi^2}\frac{du}{dx}(1)$. The corresponding values of η for different values of γ are reported in Table 2.

Next, let $\alpha = 0.9$, $\phi = 1$, and $\beta = 1$. We study the influence of the parameter γ on the solution. In Table 3, we report the values of a_0 and $u_6(x)$ for different values of γ.

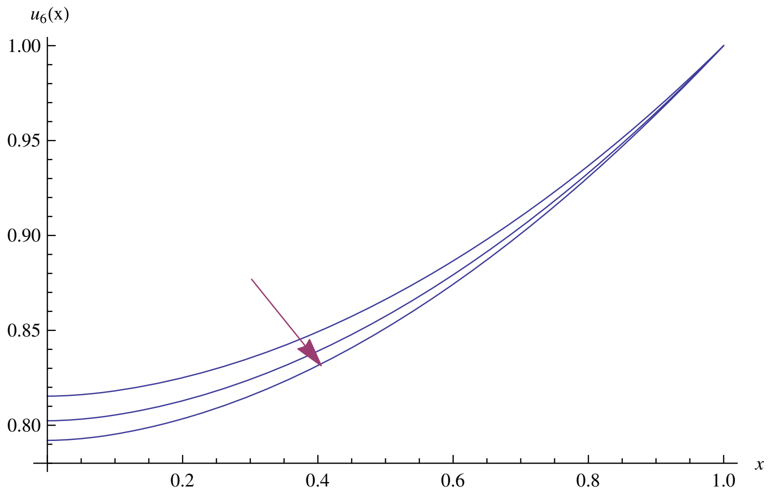

Fig. 2 Graphs of $u_6(x)$ for $\phi = 1,\ \beta = 1, \alpha = 0.9$, and $\gamma = \frac{1}{2}, 1, \frac{3}{2}$

Table 4 The values of η for different values of γ

γ	η
0.5	1.0504330114206435
1	1.107403897744165
1.5	1.131024682343269

Table 5 The values of a_0 and $u_6(x)$ for different values of γ

γ	a_0	$u_6(x)$
0.5	0.787925	$0.787925 + 0.195070x^{1.6} + 0.0165225x^{3.2} + 0.000482452x^{4.8}$
1	0.7738712	$0.773872 + 0.212782x^{1.6} + 0.0146748x^{3.2} - 0.001328940x^{4.8}$
1.5	0.7617318	$0.761732 + 0.231647x^{1.6} + 0.0103673x^{3.2} - 0.003746613x^{4.8}$

Figure 2 shows the influence of the parameter γ on the solution of the fractional non-isothermal reaction–diffusion model equations for $\alpha = 0.9$. The corresponding values of η for different values of γ are reported in Table 4.

Next, let $\alpha = 0.8$, $\phi = 1$, and $\beta = 1$. We study the influence of the parameter γ on the solution. In Table 5, we report the values of a_0 and $u_6(x)$ for different values of γ when $\alpha = 0.8$.

Figure 3 shows the influence of the parameter γ on the solution of the fractional non-isothermal reaction–diffusion model equations for $\alpha = 0.8$. The corresponding values of η for different values of γ are reported in Table 6.

Next, we want to study the influence of ϕ on the solution when $\beta = \gamma = 1$. In Table 6, we report the values of $a_0,\ \eta$, and $u_6(x)$ for different values of ϕ and α. Figure 4, 5 and 6 show the influence of the parameter ϕ on the solution of the fractional non-isothermal reaction–diffusion model.

Figure 7 shows the influence of the parameter α on the solution of the fractional non-isothermal reaction–diffusion model for $\gamma = \beta = 1, \phi = \frac{3}{2}$, and $\alpha = 1.0, 0.9, 0.8$.

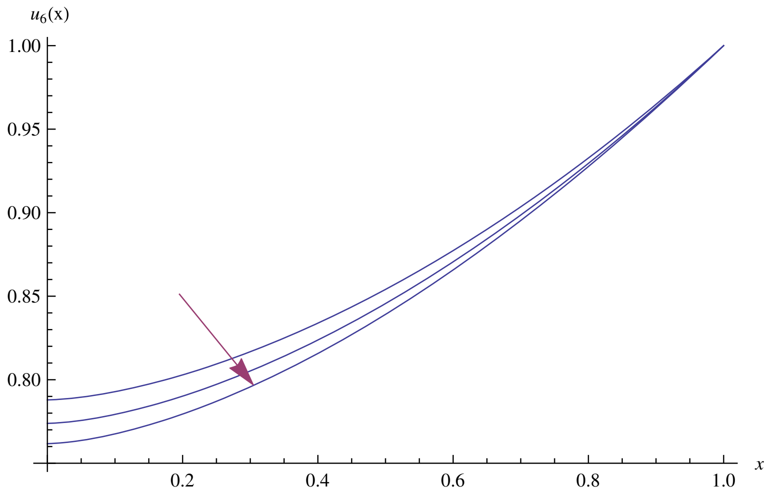

Fig. 3 Graphs of $u_6(x)$ for $\phi = 1,\ \beta = 1, \alpha = 0.8$, and $\gamma = \frac{1}{2}, 1, \frac{3}{2}$

Table 6 The values of η for different values of γ

γ	η
0.5	1.1019001482829287
1	1.1430969178284867
1.5	1.1574827132900705

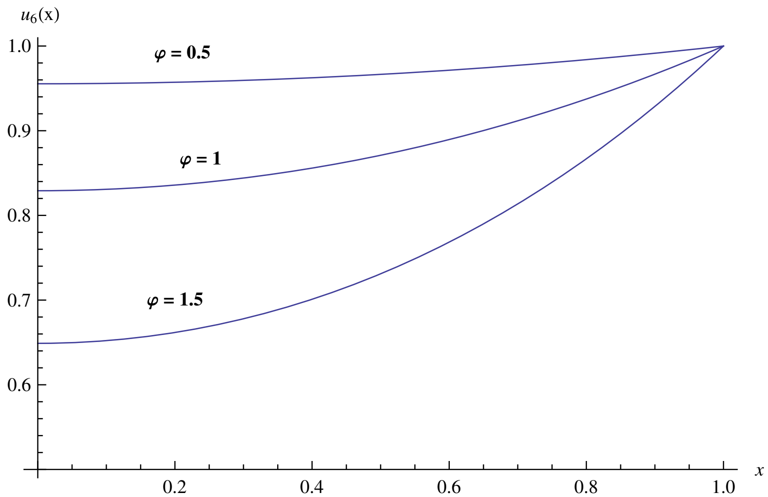

Fig. 4 Graphs of $u_6(x)$ for $\gamma = \ \beta = \alpha = 1$, and $\phi = \frac{1}{2}, 1, \frac{3}{2}$

Conclusions and Closing Remarks

In this paper, we study the fractional non-isothermal reaction–diffusion model equations in a spherical catalyst of the form

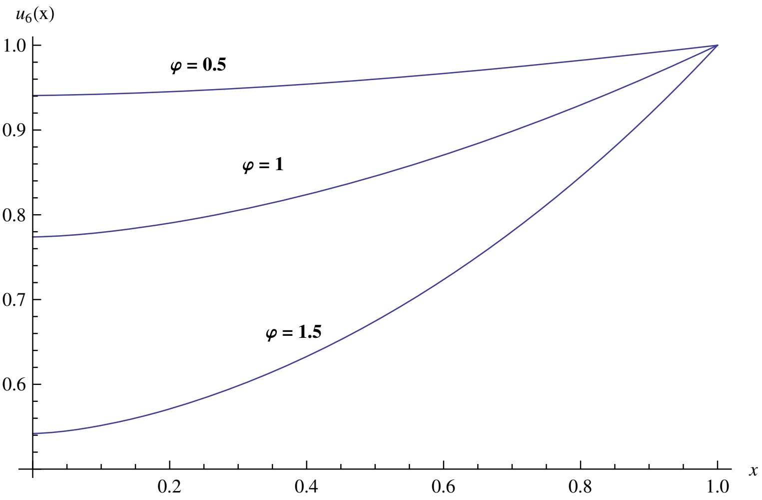

Fig. 5 Graphs of $u_6(x)$ for $\gamma = \ \beta = \alpha = 1$, and $\phi = \frac{1}{2}, 1, \frac{3}{2}$

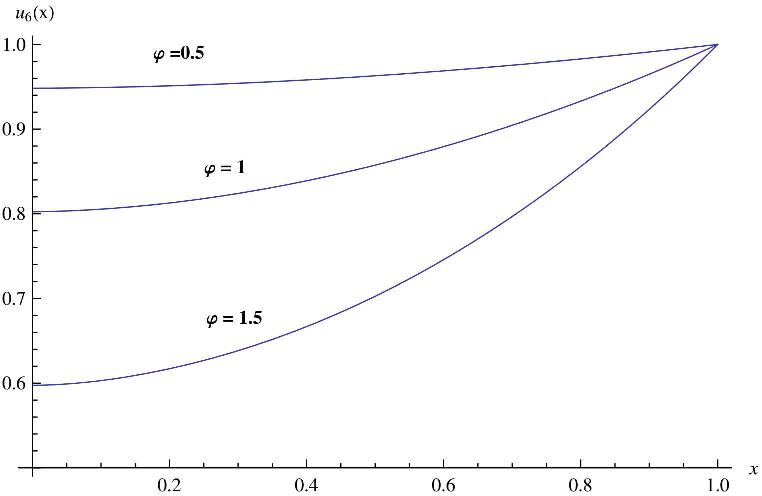

Fig. 6 Graphs of $u_6(x)$ for $\gamma = \ \beta = 1, \alpha = 0.9$, and $\phi = \frac{1}{2}, 1, \frac{3}{2}$

$$D^{2\alpha}u(x) + \frac{2}{x^\alpha}D^\alpha u(x) - \phi^2 u(x)\exp\left(\frac{\gamma\beta(1-u(x))}{1+\beta(1-u(x))}\right) = 0, \ \frac{1}{2} < \alpha \le 1, \tag{29}$$

subject to

$$D^\alpha u(0) = 0, \quad u(1) = 1. \tag{30}$$

The modified fractional power series method is employed to compute an approximation to the solution of this problem. The existence of the solution and the convergence of the approximate solution are discussed in Theorem 2.1. Our results are compared with already existing ones for the same problem when $\alpha = 1$ such as the variational iteration method [7] , the Adomian decomposition method [14], the homotopy perturbation method [15], and the

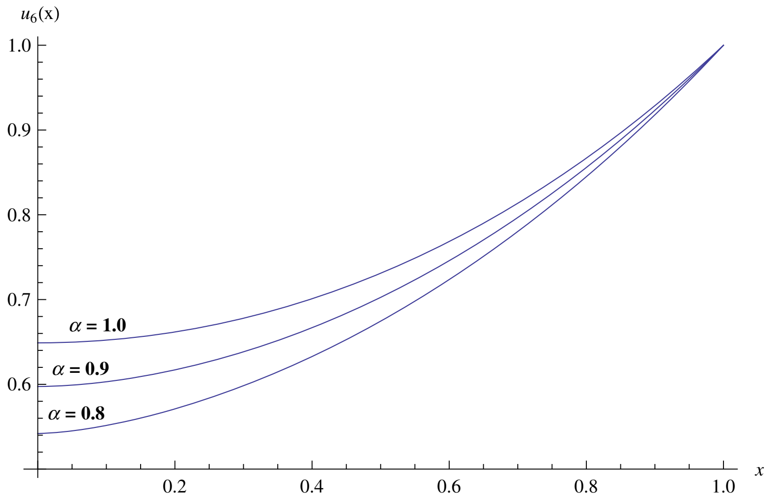

Fig. 7 Graphs of $u_6(x)$ for $\gamma = \beta = 1$, $\phi = \frac{3}{2}$, and $\alpha = 1.0, 0.9, 0.8$

Table 7 The values of a_0,η , and $u_6(x)$ for different values of ϕ where $\beta = \gamma = 1$

α	ϕ	η	a_0	$u_6(x)$
1	0.5	0.268786	0.955426	$0.955426 + 0.044340x^2 + 0.00024266x^4 - 9.376031 \times 10^{-6}x^6$
	1	1.051999	0.829150	$0.829150 + 0.165927x^2 + 0.00536284x^4 - 0.000439684015x^6$
	1.5	2.277603	0.648915	$0.648915 + 0.320280x^2 + 0.03309562x^4 - 0.002290094240x^6$
0.9	0.5	0.281462	0.948241	$0.948241 + 0.051377x^{1.8} + 0.0004015x^{3.6} - 0.00001900627x^{5.4}$
	1	1.107404	0.802374	$0.802374 + 0.189372x^{1.8} + 0.0090584x^{3.6} - 0.00080461130x^{5.4}$
	1.5	2.439305	0.597453	$0.597453 + 0.350804x^{1.8} + 0.0543079x^{3.6} - 0.00256567224x^{5.4}$
0.8	0.5	0.287654	0.940639	$0.940639 + 0.0587568x^{1.6} + 0.000639852x^{3.2} - 0.0000362144x^{4.8}$
	1	1.143097	0.773872	$0.773872 + 0.2127824x^{1.6} + 0.014674809x^{3.2} - 0.0013289403x^{4.8}$
	1.5	2.599940	0.541996	$0.541996 + 0.3739946x^{1.6} + 0.084367482x^{3.2} - 0.0003584310x^{4.8}$

modified Adomian decomposition method [11]. From the previous section, we can conclude the following:

- From Tables 1 and 2, we see that our results agree exceptionally well with the variational iteration method [7], the Adomian decomposition method [14], the homotopy perturbation method [15], and the modified Adomian decomposition method [11].
- MFPS method is an excellent tool due to its rapid convergent.
- The dimensionless activation energy has the same influence on both ordinary and fractional derivatives. From Figs. 1, 2 and 3, we see that by increasing the dimensionless activation energy, the solution profile decreases.
- The thiele modulus has the same influence on both ordinary and fractional derivatives. From Figs. 4, 5 and 6, we see that by increasing the thiele modulus, the solution profile decreases.
- From Fig. 7, we see that by increasing the dimensionless activation energy, the solution profile increases.

- The dimensionless activation energy has the same influence on both ordinary and fractional derivatives. From Tables 1, 3, and 5, we see that by increasing the dimensionless activation energy, the value of a_0 decreases.
- The dimensionless activation energy has the same influence on both ordinary and fractional derivatives. From Tables 2, 4, and 6, we see that by increasing the dimensionless activation energy, the value of dimensionless activation energy increases.
- From Table 7, we see that by increasing the fractional derivative, the value of a_0 and the dimensionless activation energy increase.
- From Fig. 7, we see that by increasing the fractional derivative to 1, the solution profile increases and it approaches to the solution produced by the variational iteration method [7], the Adomian decomposition method [14], the homotopy perturbation method [15], and the modified Adomian decomposition method [11].
- The proposed method convergent and easy to implement.

Compliance with Ethical Standards

Conflict of interest The author declares that there is no conflict of interests regarding the publication of the paper.

References

1. Wazwaz, A.M.: The variational iteration method for solving systems of equations of Emden–Fowler type. Int. J. Comput. Math. **82**(09), 1107–1115 (2011)
2. Wazwaz, A.: The variational iteration method for solving the Volterra integro-differntial forms of the Lane–Emden equations of the first and the second kind. J. Math. Chem. **52**(2), 613626 (2014)
3. Wazwaz, A.M., Rach, R.: Comparison of the Adomian decomposition method and the variational iteration method for solving the Lane–Emden equations of the first and second kinds. Kybernetes **40**(9/10), 1305–1318 (2011)
4. Muatjetjeja, B., Khalique, C.M.: Exact solutions of the generalized Lane–Emden equations of the first and second kind. Pramana **77**, 545–554 (2011)
5. Duan, J.-S., Rach, R., Wazwaz, A.M.: Steady-state concentrations of carbon dioxide absorbed into phenyl glycidyl ether solutions by the Adomian decomposition method. J. Math. Chem. **53**, 1054–1067 (2015)
6. Rach, R., Duan, J.S., Wazwaz, A.M.: Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. J. Math. Chem **52**, 255–267 (2014)
7. Wazwaz, A.M.: Solving the non-isothermal reaction–diffusion model equations in a spherical catalyst by the variational iteration method. Chem. Phys. Lett. **679**, 132–136 (2017)
8. Rach, R., Duan, J.S., Wazwaz, A.M.: Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. J. Math. Chem. **52**, 255–267 (2014)
9. Ananthaswamy, V., Shanthakumari, R., Subha, M.: Simple analytical expressions of the non-linear reaction–diffusion process in an immobilized biocatalyst particle using the new homotopy perturbation method. Rev. Bioinform. Biom. **3**, 23–28 (2014)
10. Ananthaswamy, V., Rajendran, L.: Approximate analytical solution of non-linear kinetic equation in a porous pellet. Glob. J. Pure Appl. Math. **8**(2), 101–111 (2012)
11. Sevukaperumal, S., Rajendran, L.: Analytical solution of the concentration of species using modified Adomian decomposition method. Int. J. Math. Arch. **4**(6), 107–117 (2013)
12. Oldham, K.B., Spanier, J.: The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Courier Corporation, Mineola (2002)
13. Abu Arqub, O., El-Ajou, A., Bataineh, A., Hashim, I.: A representation of the exact solution of generalized Lane-Emden equations using a new analytical method. Abstr. Appl. Anal. **2013**, 378593 (2013). https://doi.org/10.1155/2013/378593
14. Syam, M., Siyyam, H., Al-Subaihi, I.: Tau-path following method for solving the Riccati equation with fractional order. J. Comput. Methods Phys. **8**, 7 (2014)

15. Al-Mdallal, Q., Syam, M.: An efficient method for solving non-linear singularly perturbed two points boundary-value problems of fractional order. Commun. Nonlinear Sci. Numer. Simul. **17**(6), 2299–2308 (2012)
16. Alquran, M., Jaradat, H.M., Syam, M.: Analytical solution of the time-fractional Phi-4 equation by using modified residual power series method. Nonlinear Dyn. **90**(4), 2525–2529 (2017)
17. Jaradat, H., Jaradat, I., Alquran, M., Jaradat, M., Mustafa, Z., Abohassan, K., Abdelkarim, R.: Approximate solutions to the generalized time-fractional Ito system. Ital. J. Pure Appl. Math. **37**, 699–710 (2017)
18. Alquran, M., Al-Khaled, K., Sivasundaram, S., Jaradat, H.: Mathematical and numerical study of existence of bifurcations of the generalized fractional Burgers–Huxley equation. Nonlinear Stud. **24**(1), 235–244 (2017)
19. Jaradat, I., Alquran, M., Al-Khaled, K.: An analytical study of physical models with inherited temporal and spatial memory. Eur. Phys. J. Plus. **133**, 162 (2018)
20. Jaradat, I., Al-Dolat, M., Al-Zoubi, K., Alquran, M.: Theory and applications of a more general form for fractional power series expansion. Chaos Solitons Fract. **108**, 107–110 (2018)
21. Jaradat, I., Alquran, M., Al-Dolat, M.: Analytic solution of homogeneous time-invariant fractional IVP. Adv. Differ. Equ. **2018**, 143 (2018)
22. Jaradat, I., Alquran, M., Abdel-Muhsen, R.: An analytical framework of 2D diffusion, wave-like, telegraph, and Burgers' models with twofold Caputo derivatives ordering. Nonlinear Dyn. **93**(4), 1911–1922 (2018)