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Three-level atom–field in the context of time-dependent coupling and power-lower potentials

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A R T I C L E I N F O

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A B S T R A C T

In the framework of time-varying coupling and power-law potentials, we investigate quantum entanglement and quantum Fisher information in a system that consists of a three-level atom interacting with a quantized field. The results illustrate that the quantum entanglement and quantum Fisher information's temporal evolution depend critically on the physical properties of the field and its coupling to the atom. It is noteworthy to notice that the quantifiers are sensitive to the exponent parameter of the potential with and without time-varying coupling. Considerable nonlocal correlation with the accuracy of parameter estimation can be achieved through the proper control of physical parameters.

Introduction

When the Jaynes–Cummings model (J-CM) was first introduced in 1963 [1,2], its practical prominence was unclear. Due to technological advancement, the J-CM acquired significance in the 1980 s as its predictions of phenomena were experimentally confirmed [3,4]. It is observed that despite the ease of implementation of J-CM, it reproduces numerous remarkable effects, such as quantum squeezing [5], Rabi oscillations [6,7], collapses and revivals [8–11], field–atom entanglement [12,13], and nonclassical states [14–17]. The J-CM was designed to describe the field–atom interaction, and thus may be used to explain various physical phenomena in quantum systems like flux qubits [18], Josephson junctions [19,20], and Cooper-pair boxes [21]. Also, the model can be applied to solid-state systems to describe the qubits' coupling with resonator modes, in the context of superconducting quantum circuits [22–26] and dot systems [27,28]. The interaction of several two-level atoms with a quantized field represented by the Tavis-Cummings model, wherein N atoms adequately coupled to a quantized field have been examined, is an important J-CM generalization [29]. The exposure and understanding of nonclassical effects are the main theoretical and experimental tasks in different areas of quantum optics. In the past few decades, many of these quantum effects were intensively examined in the literature; e.g., quantum entanglement [30], quantum squeezing [31,32], and the antibunching phenomenon [33]. Moreover, there are other effects including quantum correlations due to the intensity-field correlation [34,35].

Entanglement phenomenon has gained a lot of attention and been regarded as a prominent aspect of quantum mechanical theory since the work of Einstein et al. [36]. Entanglement is used in detecting and testing a kind of quantum nonlocality [37] and plays a considerable role in topics of quantum information and computation, such as the quantum key distribution [38], teleportation [39], dense coding [40], etc. Numerous proposals [41–44] have been presented for the formation of entanglement through the interaction of fields and atoms in the framework of cavity QED. The entanglement features of models of the interaction between light and matter have recently received a lot of attention. Systems formed by multiple two-level atoms that interact with a one-mode field and present dipole–dipole and Ising-like interactions

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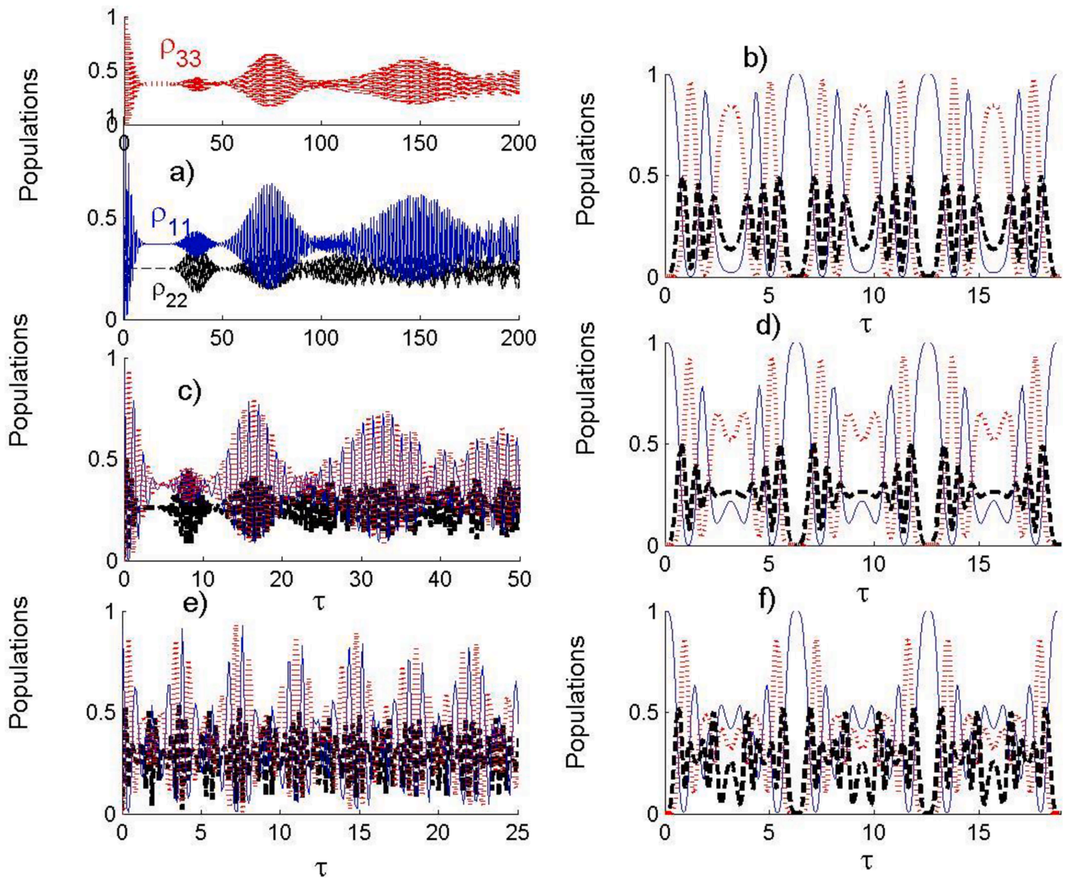

Fig. 1. Evolution of the population probabilities ρ_{11} (solid blue line), ρ_{22} (dashed black line), and ρ_{33} (dotted red line) for $\xi = \sqrt{10}$. Panels (a), (c), and (e) present results in the absence of T-VC ($\lambda(t) = \varepsilon$) and panels (b), (d) and (f) those for the T-VC ($\lambda(t) = \varepsilon\sin(t)$). Panels (a) and (b) present results for a triangular potential ($\ell = 1$, $\eta = 3$), panels (c) and (d) those for a harmonic potential ($\ell = 2$), and panels (e) and (f) those for an infinite potential ($\ell \to \infty$, $\eta = 4$). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

[45,46], or include the spin systems [47,48] or coupled molecules [49,50] have been explored.

Studies have recently investigated the precision of parameter estimation [51,52] by investigating the effects of the particle loss and environmental noises [53–57]. The detection of the sensitivity of the quantum states with regard to a change in the parameter estimation has been made possible in large part by quantum Fisher information (QFI). QFI was introduced for the first time to study the statistical estimation [58]. The QFI is employed to extract the maximum information from the parameter estimation for a particular measurement. By improving and developing new techniques for detecting the sensitivity of parameters, the estimation of an unknown parameter promotes the development of numerous fields of quantum technologies [52,59–61]. Obtaining with great accuracy the rate of an unknown parameter arising from physical effects is an important task in estimation theory. In this regard, the lower bound provided by the Cramér–Rao inequality is described through the Fisher information [62]. Subsequently, QFI can finally be considered as the main problem to be identified. Lately, the dynamics of QFI were well investigated considering decoherence and fluctuation phenomena due to the interaction between the system and its own environment [63–66].

Coherent states (CSs) have attracted a lot of interest in latest decades [67,68] and have been crucial in a variety of quantum physics applications. These states are described as a superposition of Fock states [69,70]. The CSs associated with power-law potentials (P-LPs) depending on the Hamiltonian eigenvalues with one-dimensional P-LP have introduced [71]. Lately, promising applications of P-LPs in experimental and theoretical physics have been introduced, presenting the possibility of describing a wide set of quantum systems [72,73]. The CSs associated with this type of potential are useful and provide insight for different topics of quantum physics [74–76]. The present paper thus investigates quantum entanglement and QFI in a system comprising a three-level atom that interacts with a quantized field in the context of time-varying coupling (T-VC) and P-LPs. We show that the physical nature of the field and the atom–field interaction play important roles in the time evolution of quantum entanglement and QFI. Considerable nonlocal correlation with the accuracy of parameter estimation is achieved through the proper control of physical parameters.

The rest of the article is structured as following: The physical model and system dynamics are described in Section "Physical Model and Dynamics". The quantumness measures are introduced in Section "Measures of Quantumness". Section "Numerical Results and Discussion" discusses the obtained numerical results. Section "Conclusions" includes a succinct conclusion.

Physical model and dynamics

We consider here a model of a three-level atom having the transitions 1→2 and 2→3. The photon transition between 1 and 3 is not allowed. The system Hamiltonian under the rotating wave approximation, in the interaction picture, can be written as

$$\widehat{H}_I = \lambda_1(t)(\widehat{A}\widehat{S}_{12} + \widehat{A}^{\dagger}\widehat{S}_{21}) + \lambda_2(t)(\widehat{A}\widehat{S}_{23} + \widehat{A}^{\dagger}\widehat{S}_{32}), \tag{1}$$

where $\widehat{A}$ and$\widehat{A}^{\dagger}$ *represent*, respectively, the annihilation and creation operators of the generalized Heisenberg algebra [79], $\widehat{S}_{ij} = |i\rangle\langle j|$ denotes the atomic operators, and $\lambda_1(t)$ and $\lambda_2(t)$ are coefficients of the T-VC of

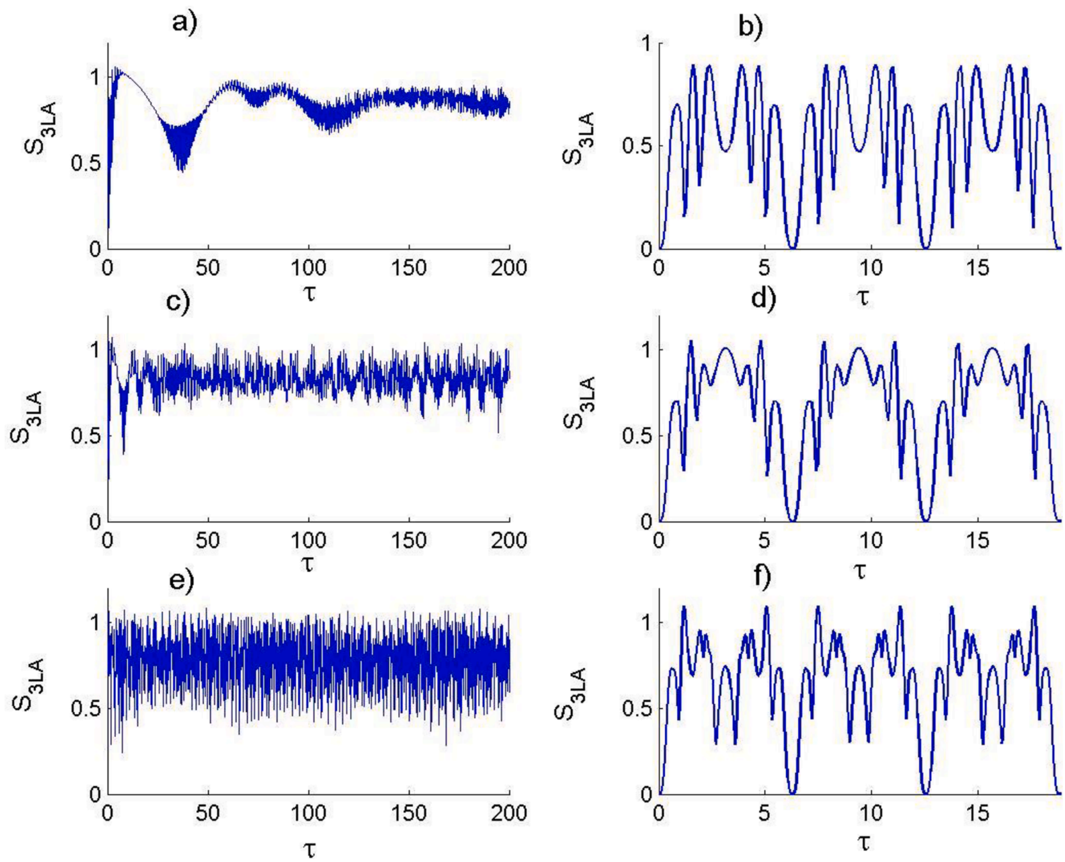

Fig. 2. Evolution of the entanglement between the field and three-level atom as measured by von Neumann entropy S_{3LA} for $\xi = \sqrt{10}$. Panels (a),(c), and (e) present results in the absence of T-VC ($\lambda(t) = \varepsilon$) and panels (b), (d), and (f) those for the T-VC ($\lambda(t) = \varepsilon\sin(t)$). Panels (a) and (b) present results for the triangular well potential ($\ell = 1, \eta = 3$), panels (c) and (d) those for the harmonic well potential ($\ell = 2$), and panels (e) and (f) those for the infinite well potential ($\ell \to \infty, \eta = 4$).

the field and three-level atom, where $\lambda(t) = \lambda_1(t) = \lambda_1(t) = \varepsilon\sin t$.

In this case, the Schrödinger equation

$$\frac{d|\varpi(\tau)\rangle}{d\tau} = -i\widehat{H}_I|\varpi(\tau)\rangle, \ with\, h = 1 \tag{2}$$

can be solved. Here, we assume that time τ is scaled. One unit of time is the inverse of the parameter ε; i.e., $\tau = \varepsilon t$.

For $\tau > 0$, the system's state vector can be expressed as

$$|\omega(\tau)\rangle = \sum_{n=0}^{\infty}\sum_{j=1}^{3}\left(R_j(n,\tau)|n+j-1,j\right\rangle \tag{3}$$

Using the interaction Hamiltonian (1) and state vector (3), the Schrödinger equation (2) is reduced to a set of coupled equations:

$$\frac{dR_1(n,\tau)}{dt} = -i\lambda_1(\tau)\sqrt{n}R_2(n-2,\tau), \tag{4}$$

$$\frac{dR_2(n,\tau)}{dt} = -i\lambda_1(\tau)\sqrt{n+2}R_1(n+2,\tau) - i\lambda_2(\tau)\sqrt{n+1}R_3(n-2,\tau), \tag{5}$$

$$\frac{dR_3(n,\tau)}{d\tau} = -i\lambda_2(\tau)\sqrt{n+2}R_2(n+2,\tau). \tag{6}$$

We suppose that the three-level atom is initially set in the upper state $|1\rangle$, and the quantized field is in the superposition CS of the P-LP as $N(|\xi,\ell\rangle + |-\xi,\ell\rangle)$. Thus, $|\varpi(0)\rangle = N_c(|\xi,\ell,1\rangle + |-\xi,\ell,1\rangle)$, N_c is the normalization constant, and $\rho(0) = |\varpi(0)\rangle\langle\varpi(0)|$. The CS of P-LPs is given by

$$|\xi,\ell\rangle = \left(\sum_{m=0}^{\infty}\frac{|\xi|^{2m}}{\upsilon(m,n)}\right)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\left(\frac{\xi}{\sqrt{\upsilon(n,\eta)}}\right)^{n}|n\rangle \tag{7}$$

where.

$\upsilon(m,n) = \prod_{j=1}^{m}\left(\left(j+\frac{\eta}{4}\right)^{\frac{2\ell}{\ell+2}} - \left(\frac{\eta}{4}\right)^{\frac{2\ell}{\ell+2}}\right)\upsilon(0,\ell) = 1.$ (8).

The CS of P-LPs involves three different potential wells, each determined by the parameters ℓ and η. In particular, $\ell = 2$ for the harmonic oscillator potential, $\ell \to \infty$ for infinite square potential and $\ell \to 1$ for the triangular potential.

By using the state (3), the atomic density matrix is obtained as

$$\widehat{\rho}^{3-\mathrm{LA}}(\tau) = Tr_{\{field\}}\{|\varpi(\tau)\rangle\langle\varpi(\tau)|\} = \begin{pmatrix} \widehat{\rho}_{11} & \widehat{\rho}_{12} & \widehat{\rho}_{13} \\ \widehat{\rho}_{21} & \widehat{\rho}_{22} & \widehat{\rho}_{23} \\ \widehat{\rho}_{31} & \widehat{\rho}_{32} & \widehat{\rho}_{33} \end{pmatrix}, \tag{9}$$

where $\widehat{\rho}_{11}$, $\widehat{\rho}_{22}$, and $\widehat{\rho}_{33}$ are, respectively, the atomic occupation probabilities of levels 1, 2, and 3.

Measures of quantumness

The significance of quantum correlations, particularly entanglement, in several aspects of quantum optics and quantum information technology is widely known. For a closed quantum system, the atom–field entanglement can be measured using quantum entropy. The von Neumann is defined as

$$S = -\mathrm{Tr}(\widehat{\rho}\ln\widehat{\rho}), \tag{10}$$

For pure state the function S takes zero value, where $\widehat{\rho}$ is the density matrix that describes a given quantum system and it is considered as a quantifier of the purity of states.

In the present model, the entanglement of the atom–field state is detected through the von Neumann entropy of reduced density matrices as:

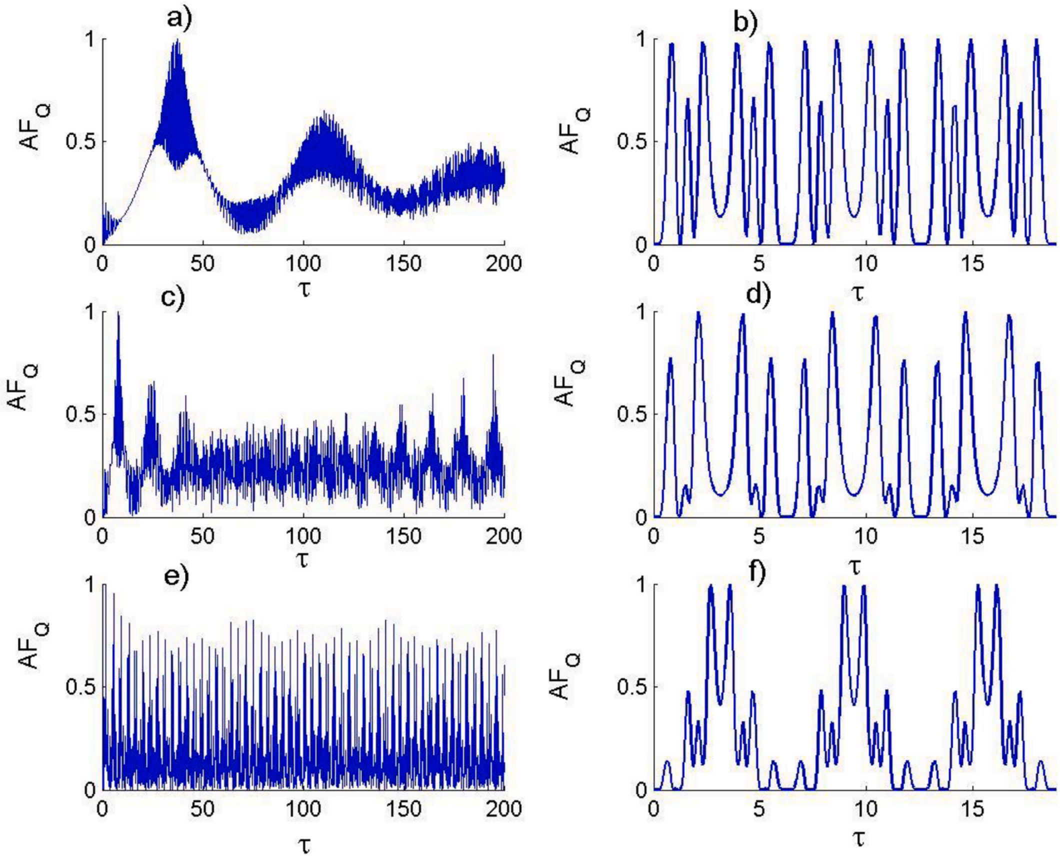

Fig. 3. Evolution of the atomic QFI AF_Q for $\xi = \sqrt{10}$. Panels (a),(c), and (e) present results in the absence of *T*-VC ($\lambda(t) = \varepsilon$) and panels (b), (d), and (f) those for the *T*-VC ($\lambda(t) = \varepsilon\sin(t)$). Panels (a) and (b) present results for the triangular well potential ($\ell = 1, \eta = 3$), panels (c) and (d) those for the harmonic well potential ($\ell = 2$), and panels (e) and (f) those for the infinite well potential ($\ell \to \infty$, $\eta = 4$).

$S_{Field} = S_{3-LA} = -\sum_{j=1}^{3}\mu_j \ln \mu_j$, (11).

where μ_j denotes the eigenvalues of the state $\hat{\rho}^{3-\mathrm{LA}}(t)$.

For a given process characterized by estimator parameter ϕ, which is related to the initial state density of the three-level atom, $u(\phi) = |1\rangle\langle 1| + \exp(i\phi)|2\rangle\langle 2|$, and hence the QFI of the three-level atom is defined by [77]

$$AF_Q(\tau) = Tr\{\hat{\rho}^{3-\mathrm{LA}}(\phi,\tau)\Theta(\phi,\tau)\Theta(\phi,\tau)\}, \tag{12}$$

where $\Theta(\phi,\tau)$ is the symmetric logarithmic derivative operator satisfying

$$2\frac{\partial\hat{\rho}^{3-\mathrm{LA}}(\phi,\tau)}{\partial\phi} = \hat{\rho}^{3-\mathrm{LA}}(\phi,\tau)\Theta(\phi,\tau) + \Theta(\phi,\tau)\hat{\rho}^{3-\mathrm{LA}}(\phi,\tau). \tag{13}$$

The next section discusses the obtained numerical results of the dynamical behavior for atomic populations, von Neumann entropy, and QFI.

Numerical results and discussion

Using equations (9), (11), and (12), we introduced the main results for the effects of physical parameters on the time evolution of the atomic populations, von Neumann entropy, and QFI.

Fig. 1 presents the numerical simulations of the populations ρ_{11}, ρ_{22}, and ρ_{33} considering different cases of the quantized field with $\xi = \sqrt{10}$ with and without the time dependence of the coupling parameter. The blue line shows the results for ρ_{11}, the black line those for ρ_{22}, and the red line those for ρ_{33}. For $\lambda(t) = \varepsilon$ and the three-level atom starting in the upper state, we observe that the populations exhibit quasi-periodic oscillations with less intensity in the case of harmonic and infinite potentials. For $\lambda(t) = \varepsilon\sin t$, we note that the populations experience a periodic behavior with large amplitude and low frequency of oscillations.

Fig. 2 presents the entanglement between the three-level atom and the single-mode field according to equation (11), considering $\xi = \sqrt{10}$ with $\lambda(t) = \varepsilon$ and $\lambda(t) = \varepsilon\sin t$. In general, the entanglement of the system state is sensitive to the physical nature of the quantized field. In the first case of the triangular potential with $\ell = 1$ and $\eta = 3$, the von Neumann entropy gradually increases to a maximum and then exhibits considerable oscillatory behavior. For the harmonic ($\ell = 2$ and $\eta = 2$) and infinite ($\ell \to \infty$ and $\eta = 4$) potentials, the function $S_{3-\mathrm{LA}}$ oscillates randomly from the beginning of the interaction, with the intensity of the oscillations being lower than that in the previous case. There is thus strong atom–field entanglement and the von Neumann entropy does not vanish, which corresponds to a separable state, in the dynamics. When $\lambda(t) = \varepsilon\sin t$, the function $S_{3-\mathrm{LA}}$ has a periodic behavior for different values of the parameters ℓ and η with the same period of time and exhibits sudden death and reveals sudden birth and death entanglement as time evolves.

Fig. 3 illustrates the dynamics of the QFI in scaled time τ for different values of the system parameters. Generally, we see that the QFI depends on the nature of the field and the coupling of the atom and the field. The parameters act in a similar way on the QFI as on $S_{3-\mathrm{LA}}$, where the amount of QFI initially increases and tends to oscillate with fluctuations and amplitudes that depend on the field parameter $\lambda(t) = \varepsilon$. The previous irregular behaviour becomes regular for $\lambda(t) = \varepsilon\sin t$, for various values of the field parameter, the QFI periodically achieves minimum and maximum values. Furthermore, there is a unilateral relationship between the QFI and von Neumann entropy in the dynamics. The results demonstrate that the accuracy of the parameter estimation can be the controlled and maintained through appropriate choice of the atom–field interaction and field parameters.

Conclusions

We have analyzed the quantum entanglement and QFI in a system comprising a three-level atom interacting with a quantized field in the context of *T*-VC and P-LPs. We have shown that the dynamics of quantum entanglement and QFI is strongly influenced by the physical properties of the field and the coupling between the field and the atom. We have discussed the sensitivity of the quantifiers to the parameter of the potential with and without *T*-VC, as well as how a significant nonlocal correlation with the parameter estimation accuracy can be achieved by carefully controlling the physical parameters. These results provide potential uses for quantum optics and information under optimal conditions.

CRediT authorship contribution statement

Mariam Algarni: Writing – review & editing. **Kamal Berrada:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Sayed Abdel-Khalek:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Hichem Eleuch:** Validation, Funding acquisition, Investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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