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OSCILLATION CRITERION FOR CERTAIN SECOND ORDER DIFFERENTIAL EQUATIONS

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1. INTRODUCTION.

The purpose of this paper is to introduce a new oscillation criteria for second order nonlinear differential equations with retarded argument. Consider the delay differential equation

$$\ddot{x}(t) + q(t)f(x(t))g(\dot{x}(t)) = 0 \tag{1}$$

where $q : [t_0, +\infty) \to (0, +\infty)$, $f, g : R \to R$ are continuous and not identically zero on any ray of the form $[T, +\infty), T > t_0 > 0$ and

$$\lim_{t\to\infty} \sup \frac{\dot{x}(t)}{t^2} = c \tag{2}$$

where c is a constant.

Some qualitative behavior of differential equations of the form (1) has been studied by many authors. Grace and Lalli [2] prove a theorem with strong assumptios that equation (1) is oscillatory. The aim of this paper is using the ideas of Hamedani [3] to find a new oscillation criterion with weaker (or different) set of assumptions for Equation(1).

Solution $x(t)$ of equation (1) which exist on some ray $[T, +\infty), T \geq t_0$ and which are nontrivial in any neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory. It is clear that with a solution $x(t)$ of (1) also $-x(t)$ is its solution. This enables us to consider e.g. only positive nonoscillatory solutions of (1).

2. RESULTS. The following results are basic for all later discussions.

Lemma 1. Let $f(t) \in C^2(-\infty, +\infty)$ and for each $0 < k_1 < 1$ there exists a $T_{k_1} \geq T \geq t_0$ such that

$$f(x(t)) \geq k_1 \frac{x(t)}{t} f(t) \tag{3}$$

Lemma 2. Let $g(t) \in C^2(-\infty, +\infty)$ and for each $0 < k_2 < 1$ there exists a $T_{k_2} \geq T \geq t_0$ such that

$$g(\dot{x}(t)) \geq k_2 \frac{\dot{x}(t)}{t} g(t) \tag{4}$$

Lemma 3. Let $x(t) \in C^2[t_0, +\infty)$ and $x(t) > 0$, $\dot{x}(t) > 0$, $\ddot{x}(t) \leq 0$ for $t \geq T \geq t_0$ then for each $k_3 \in (0,1)$ there is a $T_{k_3} \geq T \geq t_0$ such that

$$x(t) \geq k_3 t \dot{x}(t) \tag{5}$$

Proofs of Lemma 1 and Lemma 2 analogous to that of lemma given by Erbe [1] and the proof of Lemma 3 may be found in Ohriska [4].

Theorem 1. Let $q(t) \in C[t_0, +\infty)$, $q(t) \geq 0$ for $t \geq t_0$ and with the condition (2) if

$$\int^{\infty} q(s)f(s)g(s)ds = \infty \tag{6}$$

holds, then Equation (1) has oscillatory solution.

Proof. If the conclusion is not true, then assume that x(t) be a positive unbounded nonoscillotory solution on an interval $[T, +\infty), T \geq t_0$. Integrating the equation (1) from t to $+\infty, t \geq T$ it follows that

$$\dot{x}(t) \geq \int_t^{\infty} q(s)f(x(s))g(\dot{x}(s))ds \tag{7}$$

Using Lemma 3 in (7), it yields

$$x(t) \geq k_3 t \int_t^{\infty} q(s)f(x(s))g(\dot{x}(s))ds \tag{8}$$

Applying the results of Lemma 1 and Lemma 2 in (8) for the functions f and g respectively it follows

$$x(t) \geq k_1k_2k_3 t \int_t^{\infty} q(s)f(s)g(s)\frac{\dot{x}(s)}{s}\frac{x(s)}{s}ds \tag{9}$$

Letting $k = \min k_1, k_2, k_3$ and since the solution x(t) is positive increasing function therfore it is easily obtain that

$$1 \geq k^3 t \int_t^{\infty} q(s) f(s) g(s) ds \frac{\dot{x}(s)}{s^2} ds \tag{10}$$

Now using the condition (2) in (10) it follows that

$$1 \geq k^3 t \int_t^{\infty} q(s) f(s) g(s) ds \tag{11}$$

This contradiction proves the theorem and completes the proof.

Consider the second order differential equation

$$\ddot{x}(t) + q(t) f(x(t)) = 0 \tag{12}$$

where $q(t) : [t_0, +\infty) \to (0, +\infty)$ and f(t) satisfies the Lemma 1. Then we have the following theorem. Theorem 2. Let $q(t) \in C[t_0, +\infty), q(t) \geq 0, t \geq t_0$ and

$$\lim_{x \to \infty} \frac{f(x)}{x} = c, c > 0 \tag{13}$$

then the solutions of equations (12) are oscillatory if

$$\int_{t_0}^{\infty} q(s) ds = \infty \tag{14}$$

Proof of the theorem is quite similar to that of the Theorem 1 and hence is omitted. The condition (14) which involve the behavior of the integral of q. This is a well known classical result for nonlinear form of the equation (12) as well as for a linear equation of the form

$$\ddot{x}(t) + q(t) x(t) = 0 \tag{15}$$

Theorem 3. Consider the equation

$$\ddot{x}(t) + q(t) f(x(h(t))) g(\dot{x}(t)) = 0 \tag{16}$$

where $q(t) : [t_0, +\infty) \to (0, +\infty)$ and $f(t)$ satisfies Lemma 1 and g(t) satisfies Lemma 2. Assume that for sufficently large t

$$h(t) \leq t \tag{17}$$

and for each $0 < k_4 < 1$ there exists a $T_{k_4} \geq T \geq t_0$ such that x(t) satisfies Lemma 1. Then the equation (16) has oscillatory solutions if the condition (6) and (2) hold. Proof of the theorem is analogous to that of the Theorem 1. It would require obviously some trivial modifications.

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