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ABSTRACT

This paper presents the fractional generalized q-deformed sinh-Gordon equation. The fractional effects of the temporal derivative of the proposed model are studied using a conformable derivative. The analytical solutions of the governing model depend on the specified parameters. The resulting equation is studied with two integration architectures: the sine-Gordon expansion method and the modified auxiliary equation method. These strategies extract hyperbolic, trigonometric, and rational form solutions. For appropriate parametric values and different values of fractional parameter α, the acquired findings are displayed via 3D graphics, 2D line plots, and contour plots. The graphical simulations of the constricted solutions depict the existence of bright soliton, dark soliton, and periodic waves. The considered model is useful in describing physical mechanisms that possess broken symmetry and incorporate effects such as amplification or dissipation.

I. INTRODUCTION

Recently, many researchers have been interested in finding accurate solutions to nonlinear partial differential equations (NPDs) and they have employed a number of techniques. A few illustrations are shown below. Ghanbari and Baleanu presented unique methods for constructing exact solutions for NPDs. Hosseini *et al.* investigated solitary wave profiles for Hirota–Satsuma–Ito issue using the linear superposition approach. The bilinear neural network approach for obtaining accurate analytical solutions to NPDs. The p-gBKP equation was developed by Zhang and Bilige. Batool *et al.* used a pair of integration norms to extract solitons from a nonlinear model. Inc *et al.* obtained analytic solutions for the Ito problem using the simplest equation technique. Sun *et al.* employed the Hirota bilinear technique to solve a nonlinear problem for M-lump solutions. Faheem *et al.* developed a strategy based on the Hermite polynomials to solve fractional differential equations. Raza *et al.* solved a nonlinear Kudryashov problem by using the Painlevé approach.[1–9]

Nonlinear wave phenomena have been seen in a variety of fields, including tsunami waves, plasma physics, and control theory.[10,11] These phenomena can be clearly defined by analytical and computational soliton solutions. Solitons are of great interest due to their genuine capability for unique applications in a variety of scientific fields. They are formed by a variety of prominent NPDs such as in Refs. 12 and 13. Many researchers have partnered to introduce an array of approaches for investigating such nonlinear phenomena including inverse scattering, Wronskian formulation, invariant subspaces, Hirota bilinear, symmetries, and auxiliary equation strategy.[14–17] A variety of new analytical techniques are being employed by researchers to extract new soliton solutions for different nonlinear evolution models.[28–39]

Fractional derivatives are the generalizations of usual derivatives to arbitrary real or complex orders. Many physical phenomena are well described by fractional derivatives, such as viscoelasticity, anomalous diffusion, and chaotic systems. Fractional derivatives can capture the memory and nonlocal effects of some physical processes. The fundamental advantages of fractional derivatives are flexibility and non-locality. Fractional differential equations are the central area of research because fractional-order mathematical models are more compliant, offer generalization from the classical models, and reveal abundant interesting hereditary and memory properties of the underlying physical phenomena. Only fractional derivatives (FDs) can correctly explain models because they account for memory loss and crossover effects in a range of physical models. The fractional operator system makes it possible to comprehend dynamics in great detail. This is the reason many new fractional definitions have been introduced to present improved and new versions of FDs in recent decades. The most commonly used FD is the conformable derivative,[27] which has many useful mathematical properties.

Several researchers have investigated a wide range of ordinary or PD equations with important advances in a number of domains.[18,19] The q-deformed nonlinear equations are very interesting in characterizing the behavior of several physical systems when their symmetry is broken, such as the perturbation in quantum optics, deformed nuclei in nuclear processes, and elasticity changes. Several researchers have examined the q-deformed equation and produced varied conclusions. Song obtained traveling solutions to the q-deformed double sinh-Gordon equations.[20] Alrebdi *et al.* derived a number of novel soliton solutions of q-deformed sinh-Gordon (qdsG) model.[21] Raza *et al.*[22] completed lie symmetry analysis and qualitative analysis on the qdsG equation. Ali *et al.* conducted a thorough investigation of the qdsG equation.[23]

In this paper, a novel form of generalized qdsG equation, introduced by Eleuch,[24] is considered, which is of the following form:

$$\frac{\partial^2 u}{\partial \tilde{z} \partial \zeta} = e^{\tilde{\alpha} u}\left[\sinh_p\left(u^{\Lambda}\right)\right]^q - \Omega, \tag{1}$$

where $\Lambda, q, \tilde{\alpha} \in R^*$ and $0 < p < 1$. The $\sinh_p(t)$ is defined as

$$\sinh_p(t) = \frac{e^t - pe^{-t}}{2}. \tag{2}$$

The soliton solutions for Eq. (1) have been constructed in this paper adopting two versatile analytical approaches.[25,26] By applying these techniques, hyperbolic, trigonometric, and periodic solutions are constructed in Sec. III. The proposed methods are the most efficient approaches employed on the model that is proposed in this paper for the first time for extracting new and novel solutions in the form of hyperbolic, trigonometric, and rational solutions. The obtained hyperbolic solutions can further be classified as bright soliton, dark soliton, Kink and anti-Kink solutions, and singular solitons. The trigonometric solutions can be referred to as periodic solutions. All analytical methods have some limitations. The analytical techniques that are applied in this paper also have some limitations. The proposed approaches extracted dark solitons, periodic waves, and bright solitons. The approaches are strong and play an efficient role in finding solutions to a variety of NLPDEs. The proposed techniques are widely employed in nonlinear dynamics and soliton theory to create solitonic shapes. These techniques provide the full spectrum of soliton solutions. This one-of-a-kind combination of problem and methodology has led to the discovery of a plethora of novel soliton solutions and their accompanying behaviors.

The paper is designed as follows: The governing model is discussed in Sec. II. Section III provides all the details to find soliton solutions using the underlined methods. The obtained results are summarized in Sec. IV.

II. MATHEMATICAL ANALYSIS OF THE GOVERNING MODEL

The following transformation is applied:

$$x = l\tilde{z} + \frac{\zeta}{l}, \quad t = l\tilde{z} - \frac{\zeta}{l}, \tag{3}$$

where l is the arbitrary constant. The fractional form of Eq. (1) is considered as follows:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = e^{\tilde{\alpha} u}\left[\sinh_p\left(u^{\Gamma}\right)\right]^q - \Omega. \tag{4}$$

The following traveling wave transformation is applied to find the traveling wave solutions of Eq. (4):

$$u(x,t) = \breve{\Pi}(\check{\chi}), \quad \check{\chi} = Kx - \Theta\frac{t^{\alpha}}{\alpha}, \tag{5}$$

where Θ represents the speed of the wave of propagation.

The conformal derivative[27] of order α is defined as follows:

$$D_t^{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad \text{for all} \quad t > 0, \quad \alpha \in (0,1). \tag{6}$$

The conformable derivative possesses the following properties:

$$\begin{aligned}
&\mathbf{1}: \quad D_t^{\alpha}(\alpha_1\, g(t) + \beta_1\, h(t)) = D_t^{\alpha} g(t) + D_t^{\alpha} h(t),\\
&\mathbf{2}: \quad D_t^{\alpha}(c) = 0,\\
&\mathbf{3}: \quad D_t^{\alpha}(g(t) * h(t)) = h(t) D_t^{\alpha} g(t) + g(t) D_t^{\alpha} h(t),\\
&\mathbf{4}: \quad D_t^{\alpha}\left(\frac{g(t)}{h(t)}\right) = \frac{h(t) D_t^{\alpha} g(t) - g(t) D_t^{\alpha} h(t)}{h^2(t)},
\end{aligned}$$

where g and h are differentiable of order α at $t > 0$.

By using Eq.(5), Eq.(4) becomes

$$\left(K^2 - \Theta^2\right)\check{\Pi}''(\check{\chi}) + \Omega - e^{\tilde{\alpha}\check{\Pi}(\check{\chi})}\left[\sinh_p(\check{\Pi}(\check{\chi})^{\Gamma})\right]^q = 0. \tag{7}$$

The following three cases for Eq. (7) have been studied in this paper.

CASE 1: Take $q = \Gamma = \tilde{\alpha} = 1$ and $\Omega = -\frac{p}{2}$.

Equation (7) becomes

$$\left(K^2 - \Theta^2\right)\check{\Pi}''(\check{\chi}) - \frac{p}{2} - e^{\check{\Pi}(\check{\chi})}\left[\sinh_p(\check{\Pi}(\check{\chi}))\right] = 0. \tag{8}$$

By integrating the above equation, we obtain

$$\frac{1}{2}\left(K^2 - \Theta^2\right)\check{\Pi}'(\check{\chi})^2 - \frac{1}{2}e^{2\check{\Pi}(\check{\chi})} - C_1 = 0, \tag{9}$$

where C_1 is the nonzero integrating constant.

Let

$$\check{\Pi}(\check{\chi}) = \frac{1}{2}\ln(\Lambda(\check{\chi})). \tag{10}$$

Now, Eq. (9) takes the following form:

$$-8C_1\Lambda(\check{\chi})^2 + \left(K^2 - \Theta^2\right)\Lambda'(\check{\chi})^2 - 2\Lambda(\check{\chi})^3 = 0. \tag{11}$$

Equation (11) is solved in the following section to find the soliton solutions for Eq. (4).

CASE 2: Take $q = \Gamma = 1$, $\tilde{\alpha} = -1$, and $\Omega = \frac{1}{2}$.

Equation (7) becomes

$$\left(K^2 - \Theta^2\right)\check{\Pi}''(\check{\chi}) + \frac{1}{2} - e^{-\check{\Pi}(\check{\chi})}\left[\sinh_p(\check{\Pi}(\check{\chi}))\right] = 0. \tag{12}$$

By integrating the above equation, we obtain

$$\frac{1}{2}\left(K^2 - \Theta^2\right)\check{\Pi}'(\check{\chi})^2 - \frac{1}{2}e^{-2\check{\Pi}(\check{\chi})} - C_2 = 0, \tag{13}$$

where C_2 is the nonzero integrating constant.

Let

$$\check{\Pi}(\check{\chi}) = \frac{1}{2}\ln(\Lambda(\check{\chi})). \tag{14}$$

Now, Eq. (13) takes the following form:

$$-8C_2\Lambda(\check{\chi})^2 + \left(K^2 - \Theta^2\right)\Lambda'(\check{\chi})^2 - 2p\Lambda(\check{\chi}) = 0. \tag{15}$$

Equation (15) is solved in Sec. III to find the soliton solutions for Eq. (4).

CASE 3: Take $q = 2$, $\Gamma = 1$, $\tilde{\alpha} = 2$, and $\Omega = (\frac{p}{2})^2$.

Equation (7) becomes

$$\left(K^2 - \Theta^2\right)\check{\Pi}''(\check{\chi}) - e^{-2\check{\Pi}(\check{\chi})}\left[\sinh_p(\check{\Pi}(\check{\chi}))\right]^2 + \left(\frac{p}{2}\right)^2 = 0. \tag{16}$$

By integrating the above equation, we obtain

$$\frac{1}{4}\left(pe^{2\check{\Pi}(\check{\chi})} - 2\left(\Theta^2 - K^2\right)\check{\Pi}'(\check{\chi})^2 - \frac{1}{4}e^{4\check{\Pi}(\check{\chi})}\right) - C_3 = 0, \tag{17}$$

where C_3 is the nonzero integrating constant.

Let

$$\check{\Pi}(\check{\chi}) = \frac{1}{2}\ln(\Lambda(\check{\chi})). \tag{18}$$

Now, Eq. (17) takes the following form:

$$-16C_3\Lambda(\check{\chi})^2 + 4p\Lambda(\check{\chi})^4 + 8\left(K^2 - \Theta^2\right)\Lambda'(\check{\chi})^2 - \Lambda(\check{\chi})^6 = 0. \tag{19}$$

Equation (19) is solved in Sec. III to find the soliton solutions for Eq. (4).

III. SOLUTION EXTRACTION USING THE GIVEN METHODOLOGIES

The suggested model has been integrated using two methodologies: sGe and mAe. By applying these techniques, the solutions are constructed in Secs. III A and III B.

A. Method 1: Sine-Gordon expansion method

The method 1 is offered on Eqs. (11), (15), and (19).

CASE 1: According to the method,[25] for balance number $S = 2$, the assumed solution for Eq. (11) is

$$\Lambda(\check{\chi}) = A_0 + A_1 \cos(\check{\chi}) + B_1 \sin(\check{\chi}) + A_2 \cos(\check{\chi})^2 + B_2 \sin(\check{\chi})\cos(\check{\chi}), \tag{20}$$

where A_0, A_1, B_1, A_2, and B_2 are unknowns. The implementation of the sGe method yields the two sets of solutions as follows:

Set 1:

$$A_0 = -4C_1, B_1 = 0, \quad A_1 = 0, A_2 = 4C_1, \quad B_2 = 0,$$
$$\Theta = \sqrt{K^2 - 2C_1}.$$

Set 2:

$$A_0 = -8C_1, \quad B_1 = 0, \quad A_1 = 0, \quad A_2 = 8C_1, \quad B_2 = 8\imath C_1$$
$$\Theta = \sqrt{K^2 - 8C_1}.$$

The bright solitons extracted from **Set 1** are as follows:

$$u(x,t) = \ln\left(-4C_1 \operatorname{sech}^2(\check{\chi})\right)^{\frac{1}{2}}. \tag{21}$$

The complexiton solutions for **Set 2** are as follows:

$$u(x,t) = \frac{1}{2}\ln\left(8C_1\left(-1 + \imath\operatorname{sech}(\check{\chi})\tanh(\check{\chi}) + \tanh^2(\check{\chi})\right)\right). \tag{22}$$

CASE 2: According to this method,[25] for $N = 2$, the assumed solution for Eq. (15) is

$$\check{\vartheta}(\tau) = A_0 + A_1 \cos(\check{\chi}) + B_1 \sin(\check{\chi}) + A_2 \cos(\check{\chi})^2 + B_2 \sin(\check{\chi})\cos(\check{\chi}), \tag{23}$$

where A_0, A_1, B_1, A_2, and B_2 are unknowns.

Set 3:

$$A_0 = -\frac{4C_2}{p}, \quad A_2 = \frac{4C_2}{p}, \quad B_1 = A_1 = B_2 = 0, \quad \Theta = \sqrt{K^2 - 2C_2}.$$

Set 4:

$$A_0 = -\frac{8C_2}{p}, \quad A_2 = \frac{8C_2}{p}, \quad B_1 = A_1 = 0, \quad B_2 = -\frac{8\imath C_2}{p},$$
$$\Theta = \sqrt{K^2 - 8C_2}.$$

The bright solitons extracted from **Set 3** are as follows:

$$u(x,t) = \ln\left(-\frac{4C_2}{p}\operatorname{sech}^2(\check{\chi})\right)^{\frac{1}{2}}. \tag{24}$$

The complexiton solutions from **Set 4** are as follows:

$$u(x,t) = \frac{1}{2}\ln\left(\frac{8C_1}{p}\left(-1 - \imath\operatorname{sech}(\check{\chi})\tanh(\check{\chi}) + \tanh^2(\check{\chi})\right)\right). \tag{25}$$

CASE 3: According to this method,[25] for $S = 1$, the assumed solution for Eq. (19) is

$$\Lambda(\check{\chi}) = A_0 + A_1 \cos(\check{\chi}) + B_1 \sin(\check{\chi}), \tag{26}$$

where A_0, A_1, and B_1 are unknowns. The application of the sGe method gives the following sets of solutions.

Set 5:

$$A_0 = p, \quad A_1 = p, \quad B_1 = 0, \quad \Theta = \sqrt{K^2 - \frac{p^2}{2}}, \quad C_3 = \frac{p^2}{4}.$$

Set 6:

$$A_0 = p, \quad A_1 = p, \quad B_1 = -\sqrt{3}p, \quad \Theta = \sqrt{K^2 - 2p^2}, \quad C_3 = \frac{p^2}{4}.$$

The solutions extracted from **Set 5** are as follows:

$$u(x,t) = \frac{1}{2}\ln\left(p(1 + \tanh(\check{\chi}))\right). \tag{27}$$

The solutions extracted from **Set 6** are as follows:

$$u(x,t) = \frac{1}{2}\ln\left(p\left(1 - \sqrt{3}\operatorname{sech}(\check{\chi}) + \tanh(\check{\chi})\right)\right). \tag{28}$$

B. Method 2: Modified auxiliary equation method

According to this technique,[26] for N = 2, the assumed solution for Eq. (11) is of the following form:

$$v(\check{\chi}) = a_0 + a_1(z^g) + b_1(z^g)^{-1} + a_2(z^g)^2 + b_2(z^g)^{-2}, \tag{29}$$

where a_0, a_1, b_1, a_2, and b_2 are unknowns to be evaluated. The following sets of solutions are obtained by utilizing the mAe method.

Set 7:

$$a_0 = \frac{16C_1\tilde{\alpha}\check{\sigma}}{\beta^2 - 4\tilde{\alpha}\check{\sigma}}, \quad a_1 = \frac{16C_1\beta\check{\sigma}}{\beta^2 - 4\tilde{\alpha}\check{\sigma}},$$
$$a_2 = \frac{16C_1\check{\sigma}^2}{\beta^2 - 4\tilde{\alpha}\check{\sigma}}, \quad b_1 = 0, \quad b_2 = 0,$$
$$k = \frac{\sqrt{8C_1 + \Theta^2(\beta^2 - 4\tilde{\alpha}\check{\sigma})}}{\sqrt{\beta^2 - 4\tilde{\alpha}\check{\sigma}}}.$$

Set 8:

$$a_0 = \frac{16C_1\tilde{\alpha}\check{\sigma}}{\beta^2 - 4\tilde{\alpha}\check{\sigma}}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = \frac{16C_1\tilde{\alpha}\beta}{\beta^2 - 4\tilde{\alpha}\check{\sigma}},$$
$$b_2 = \frac{16C_1\tilde{\alpha}^2}{\beta^2 - 4\tilde{\alpha}\check{\sigma}}, \quad k = \frac{\sqrt{8C_1 + \Theta^2(\beta^2 - 4\tilde{\alpha}\check{\sigma})}}{\sqrt{\beta^2 - 4\tilde{\alpha}\check{\sigma}}},$$

provided that $\beta^2 - 4\tilde{\alpha}\check{\sigma} \neq 0$.

The solutions corresponding to **Set 7** are expressed in the following.

When $\beta^2 - 4\tilde{\alpha}\check{\sigma} > 0$ and $\check{\sigma} \neq 0$, then the following bright and singular solitons are obtained:

$$u(x,t) = \frac{1}{2}\ln\left(-4C_1\operatorname{sech}\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)^2\right), \tag{30}$$

or

$$u(x,t) = \frac{1}{2}\ln\left(-4C_1\operatorname{csch}\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)^2\right). \tag{31}$$

The obtained soliton solutions for **Set 8** are given in the following.

When $\beta^2 - 4\tilde{\alpha}\check{\sigma} > 0$ and $\check{\sigma} \neq 0$, then then following hyperbolic function solutions are obtained:

$$u(x,t) = \frac{1}{2}\ln\left(-\frac{16C_1\tilde{\alpha}\check{\sigma}}{\left(\beta\cosh\left(\frac{1}{2}(\beta^2 - 4\tilde{\alpha}\check{\sigma})^{\frac{1}{2}}\check{\chi}\right) + \sqrt{\beta^2 - 4\tilde{\alpha}\check{\sigma}}\sinh\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)\right)^2}\right), \tag{32}$$

or

$$u(x,t) = \frac{1}{2}\ln\left(\frac{16C_1\tilde{\alpha}\check{\sigma}}{\left(\beta\sinh\left(\frac{1}{2}\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)+\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\cosh\left(\frac{1}{2}\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)\right)^2}\right). \tag{33}$$

According to this method,[26] for $N = 2$, the assumed solution for Eq. (15) is

$$v(\check{\chi}) = a_0 + a_1(z^g) + b_1(z^g)^{-1} + a_2(z^g)^2 + b_2(z^g)^{-2}, \tag{34}$$

where a_0, a_1, b_1, a_2, and b_2 are unknowns to be evaluated. The following sets of solutions have been obtained.

Set 9:

$$a_0 = \frac{16C_2\tilde{\alpha}\check{\sigma}}{q(\beta^2-4\tilde{\alpha}\check{\sigma})},\quad a_1 = \frac{16C_2\beta\check{\sigma}}{q(\beta^2-4\tilde{\alpha}\check{\sigma})},\quad a_2 = \frac{16C_2\check{\sigma}^2}{q(\beta^2-4\tilde{\alpha}\check{\sigma})},$$
$$b_1 = 0,\quad b_2 = 0,\quad k = \frac{\sqrt{8C_2+\Theta^2(\beta^2-4\tilde{\alpha}\check{\sigma})}}{\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}}.$$

Set 10:

$$a_0 = \frac{16C_2\tilde{\alpha}\check{\sigma}}{q(\beta^2-4\tilde{\alpha}\check{\sigma})},\quad a_1 = 0,\quad a_2 = 0,\quad b_1 = \frac{16C_2\tilde{\alpha}\beta}{q(\beta^2-4\tilde{\alpha}\check{\sigma})},$$

$$b_2 = \frac{16C_2\tilde{\alpha}^2}{q(\beta^2-4\tilde{\alpha}\check{\sigma})},\quad k = \frac{\sqrt{8C_2+\Theta^2(\beta^2-4\tilde{\alpha}\check{\sigma})}}{\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}},$$

provided that $\beta^2 - 4\tilde{\alpha}\check{\sigma} \neq 0$.

The solutions for **Set 9** are expressed in the following.

When $\beta^2 - 4\tilde{\alpha}\check{\sigma} > 0$ and $\check{\sigma} \neq 0$, then hyperbolic solutions are

$$u(x,t) = \frac{1}{2}\ln\left(-\frac{8C_2}{q+q\cosh\left(\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)}\right), \tag{35}$$

or

$$u(x,t) = \frac{1}{2}\ln\left(\frac{4C_2}{q}\operatorname{csch}\left(\frac{1}{2}\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)^2\right). \tag{36}$$

The soliton solutions obtained for **Set 10** are given in the following.

When $\beta^2 - 4\tilde{\alpha}\check{\sigma} > 0$ and $\check{\sigma} \neq 0$, then hyperbolic solutions are evaluated as

$$u(x,t) = \frac{1}{2}\ln\left(-\frac{16C_2\tilde{\alpha}\check{\sigma}}{q\left(\beta\cosh\left(\frac{1}{2}\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)+\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\sinh\left(\frac{1}{2}\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}\check{\chi}\right)\right)^2}\right), \tag{37}$$

or

$$u(x,t) = \frac{1}{2}\ln\left(\frac{16C_2\tilde{\alpha}\check{\sigma}}{q\left(\beta\sinh\left(\frac{1}{2}(\beta^2-4\tilde{\alpha}\check{\sigma})^{\frac{1}{2}}\check{\chi}\right)+(\beta^2-4\tilde{\alpha}\check{\sigma})^{\frac{1}{2}}\cosh\left(\frac{1}{2}(\beta^2-4\tilde{\alpha}\check{\sigma})^{\frac{1}{2}}\check{\chi}\right)\right)^2}\right). \tag{38}$$

According to this method,[26] for S = 1, the assumed solution for Eq. (19) is

$$v(\check{\chi}) = a_0 + a_1(z^g) + b_1(z^g)^{-1}, \tag{39}$$

where a_0, a_1, and b_1 are unknowns to be evaluated. The following sets of solutions have been obtained.

Set 11:

$$a_0 = q + \frac{q\beta}{\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}},\quad a_1 = 0,\quad b_1 = \frac{2q\tilde{\alpha}}{\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}},$$
$$k = \frac{\sqrt{2q^2+\Theta^2(\beta^2-4\tilde{\alpha}\check{\sigma})}}{\sqrt{\beta^2-4\tilde{\alpha}\check{\sigma}}},\quad C_3 = \frac{q^2}{4}.$$

Set 12:

$$a_0 = q + \frac{q\beta}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}}, \quad a_1 = \frac{2q\breve{\sigma}}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}}, \quad b_1 = 0,$$

$$k = \frac{\sqrt{2q^2 + \Theta^2(\beta^2 - 4\tilde{\alpha}\breve{\sigma})}}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}}, \quad C_3 = \frac{q^2}{4},$$

provided that $\beta^2 - 4\tilde{\alpha}\breve{\sigma} \neq 0$.

The solutions for **Set 11** are expressed in the following. When $\beta^2 - 4\tilde{\alpha}\breve{\sigma} > 0$ and $\check{\breve{\sigma}} \neq 0$, then hyperbolic solutions are

$$u(x,t) = \frac{1}{2}\ln\left(q + \frac{q\beta}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}} - \frac{4q\tilde{\alpha}\breve{\sigma}}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\left(\beta + \sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\check{\chi}\right)\right)}\right), \tag{40}$$

or

$$u(x,t) = \frac{1}{2}\ln\left(q + \frac{q\beta}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}} - \frac{4q\tilde{\alpha}\breve{\sigma}}{\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\left(\beta + \sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\check{\chi}\right)\right)}\right). \tag{41}$$

The obtained soliton solutions for **Set 12** are given in the following.

When $\beta^2 - 4\tilde{\alpha}\breve{\sigma} > 0$ and $\check{\breve{\sigma}} \neq 0$, then hyperbolic solutions are evaluated as

$$u(x,t) = \frac{1}{2}\ln\left(q - q\tanh\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\check{\chi}\right)\right), \tag{42}$$

or

$$u(x,t) = \frac{1}{2}\ln\left(q - q\coth\left(\frac{1}{2}\sqrt{\beta^2 - 4\tilde{\alpha}\breve{\sigma}}\check{\chi}\right)\right). \tag{43}$$

C. Results and discussion

Few of the constructed results are depicted graphically using Maple or Mathematica. The suggested schemes provide explicit and reliable solutions for the proposed model and the fractional effects are visualized by selecting different values of fractional parameters α for the first time in this paper. The graphical simulations of the obtained solutions depict the construction of trigonometric function solutions, hyperbolic solutions, and periodic waves. The construction of the graphs is presented in Figs. 1–8, upon taking suitable values of parameters. Graphs of analytical solutions include dark, periodic waves, bright solitons, and singular solitons.

Figure 1 illustrates the graphical representation of Set 7 [Eq. (29)], by choosing the values of parameters as $\beta = 4$, $\Theta = 1.1$, $C_1 = -0.01$, $\tilde{\alpha} = 2$, $\tilde{\sigma} = 1.4$, and $q = 0.4$. In Figs. 1(a)–1(c), three surface plots of Eq. (29) are plotted using $\alpha = 0.45$, $\alpha = 0.75$, and $\alpha = 0.85$, respectively. Figure 2(a) depicts the contour plot of Eq. (29), whereas Fig. 2(b) displays the line plot for $\alpha = 0.85$. It has been observed that

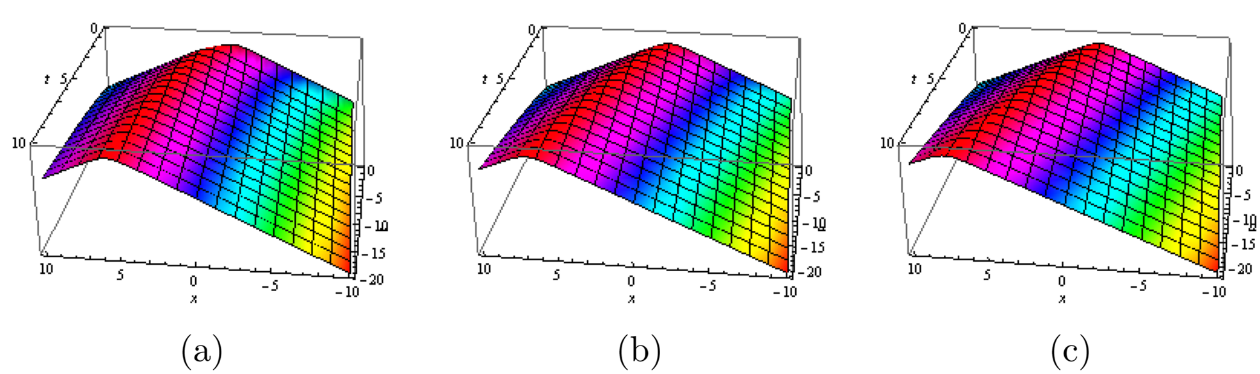
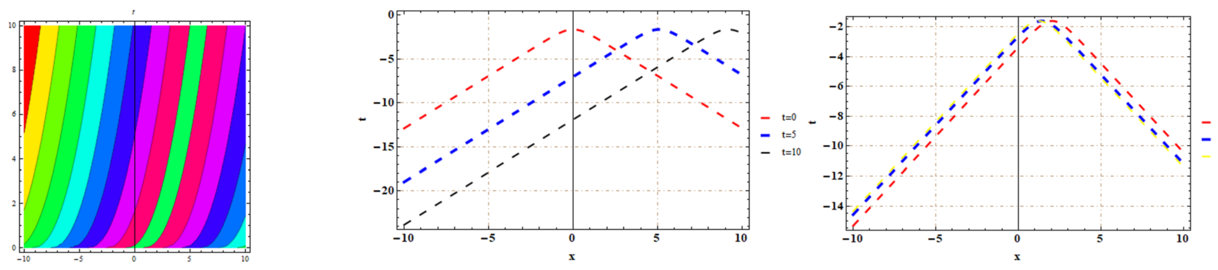
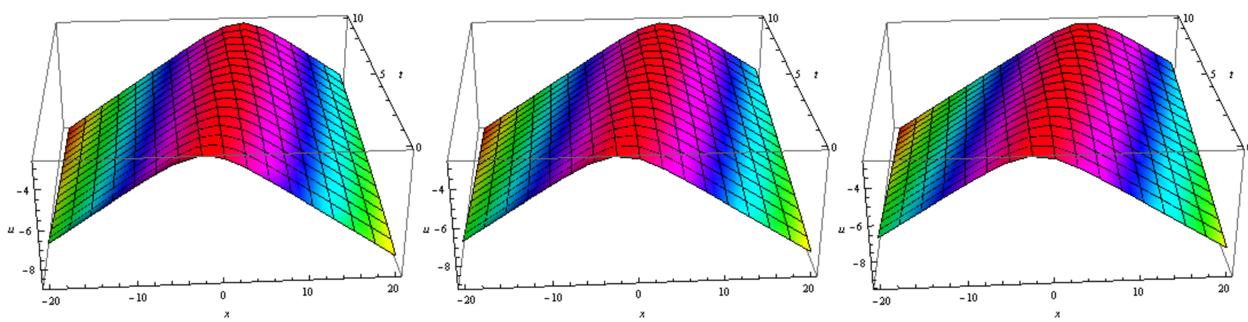


FIG. 1. Surface plots of Eq. (29) for (a) $\alpha = 0.45$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.85$, respectively.



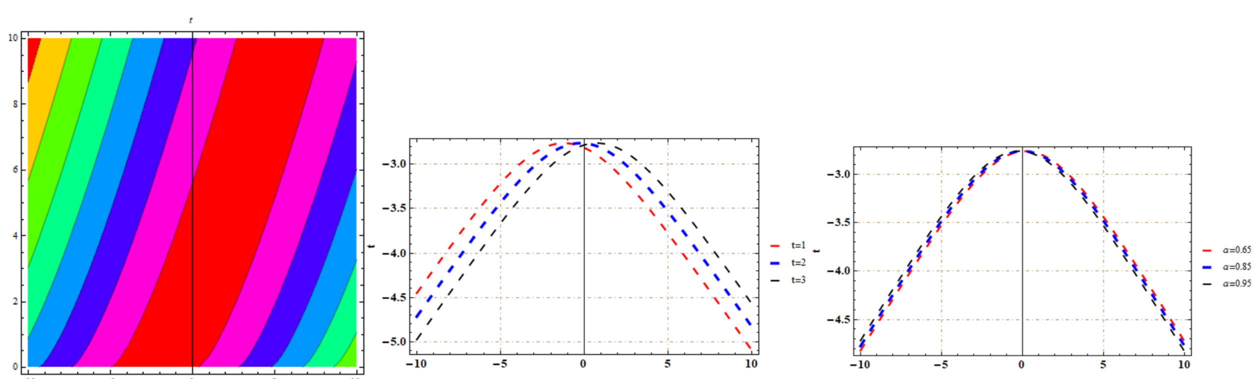
(a) (b) (c)

FIG. 2. (a) Contour plot of Eq. (29). (b) Line plot for $\alpha = 0.85$. (c) Line plot for $t = 1$.



(a) (b) (c)

FIG. 3. Surface plots of Eq. (31) for (a) $\alpha = 0.45$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.85$, respectively.

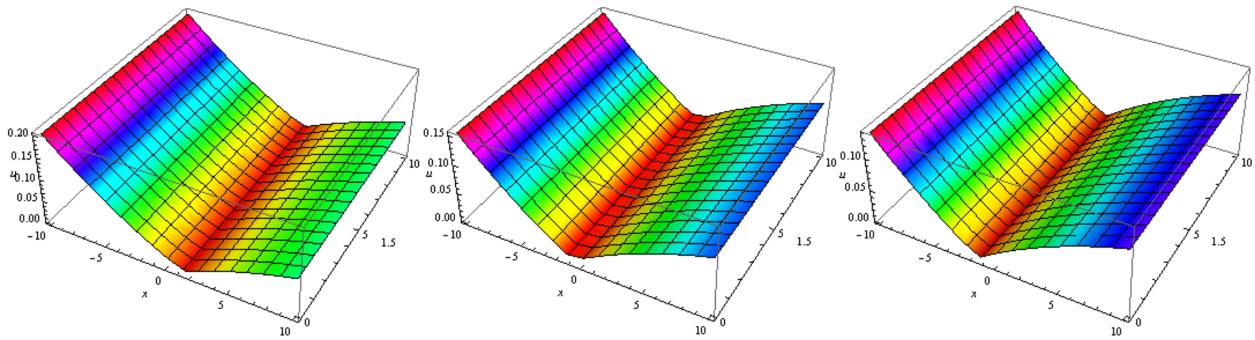


(a) (b) (c)

FIG. 4. (a) Contour plot of Eq. (31). (b) Line plot for $\alpha = 0.85$. (c) Line plot for $t = 1$.

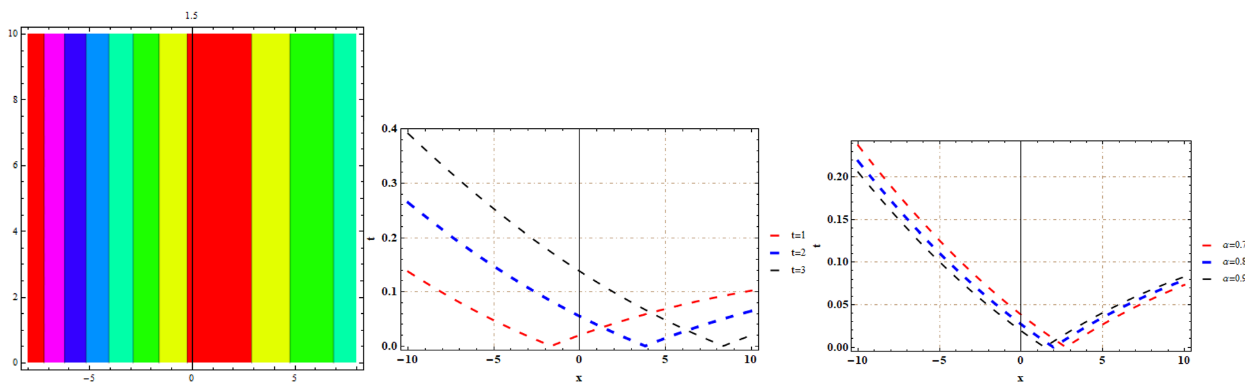
as the wave advances, it retains its structure and features for $t = 0$, $t = 5$, and $t = 10$, and with time, it shifts down. In Fig. 2(c), as time is fixed as $t = 1$ and the fractional parameter α changes its values, the wave shifts in the upward direction.

Figure 3 illustrates the graphical representation of Set 8 [Eq. (31)] by choosing the values of parameters as $\beta = 4$, $\Theta = 0.25$, $C_1 = -0.001$, $\tilde{\alpha} = 2$, $\tilde{\sigma} = 1.4$, and $q = 0.4$. In Figs. 3(a)–3(c), three surface plots of Eq. (31) are plotted using $\alpha = 0.45$, $\alpha = 0.75$, and $\alpha = 0.85$,



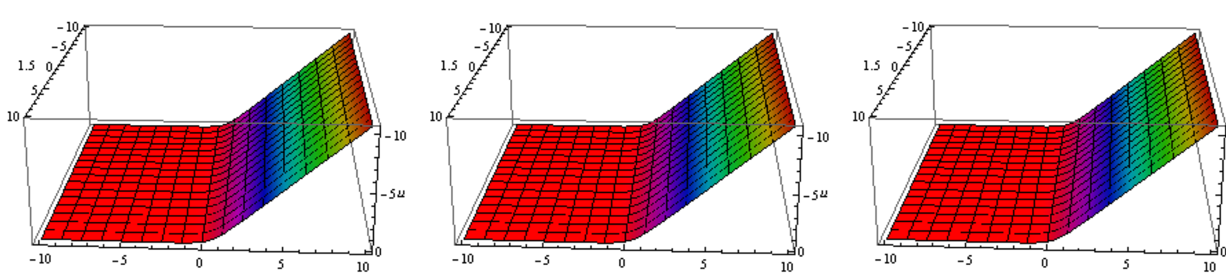
(a) (b) (c)

FIG. 5. Surface plots of Eq. (39) for (a) α = 0.45, (b) α = 0.75, and (c) α = 0.85, respectively.



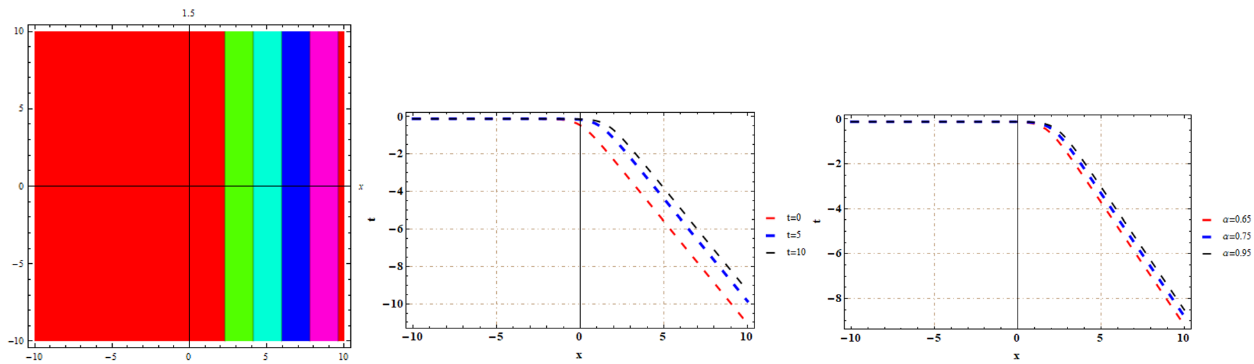
(a) (b) (c)

FIG. 6. (a) Contour plot of Eq. (39). (b) Line plot for α = 0.85. (c) Line plot for t = 1.



(a) (b) (c)

FIG. 7. Surface plots of Eq. (41) for (a) α = 0.45, (b) α = 0.75, and (c) α = 0.85, respectively.



(a) (b) (c)

FIG. 8. (a) Contour plot of Eq. (41). (b) Line plot for α = 0.85. (c) Line plot for t = 1.

respectively. Figure 4(a) depicts the contour plot of Eq. (31), whereas Fig. 4(b) displays the line plot for $\alpha = 0.85$. It has been observed that as the wave advances, it retains its structure and features for $t = 1, t = 2$, and $t = 3$ and it shifts down as time progress. In Fig. 4(c), as time is fixed as $t = 1$ and the fractional parameter α changes its values, the wave shifts upward.

Figure 5 illustrates the graphical representation of Set 11 [Eq. (39)] by choosing the values of parameters as $\beta = 3$, $\Theta = 0.25$, $\tilde{\alpha} = 1$, $\tilde{\sigma} = 1$, and $q = 0.7$. In Figs. 5(a)–5(c), three surface plots of Eq. (39) are plotted using $\alpha = 0.65$, $\alpha = 0.85$, and $\alpha = 0.95$, respectively. Figure 6(a) depicts the contour plot of Eq. (39), whereas Fig. 6(b) displays the line plot for $\alpha = 0.85$. It has been observed that the wave moves, retaining its shape and properties for $t = 1, t = 2$, and $t = 3$ and it shifts upward as time evolves. In Fig. 6(c), as time is fixed as $t = 1$ and the fractional parameter α changes its values, the wave shifts downward.

Figure 7 illustrates the graphical representation of Set 12 [Eq. (41)] by choosing the values of parameters as $\beta = 4$, $\Theta = 0.25$, $\tilde{\alpha} = 2$, $\tilde{\sigma} = 1.4$, and $q = 0.4$. In Figs. 7(a)–7(c), three surface plots of Eq. (41) are plotted using $\alpha = 0.65$, $\alpha = 0.85$, and $\alpha = 0.95$, respectively. Figure 8(a) depicts the contour plot of Eq. (41), whereas Fig. 8(b) displays the line plot for $\alpha = 0.85$. It has been observed that the wave moves, retaining its shape and properties for $t = 0, t = 5$, and $t = 10$, and it shifts up as time evolves. In Fig. 8(c), as time is fixed as $t = 1$ and the fractional parameter α changes its values, the wave shifts upward.

IV. CONCLUSION

In this paper, the fractional generalized q-deformed sinh-Gordon model has been studied successfully by employing sine-Gordon and modified auxiliary methods. The fractional effects of the proposed model have been studied using conformable derivatives. The novel traveling wave solutions to the governing model have been extracted, especially dark solitons, periodic waves, and bright solitons. On comparing our findings with Refs. 23 and 24, it has been cleared that our obtained solutions are new and have been not found in the literature. Our obtained soliton solutions exhibit some interesting properties: They are of permanent form and can interact with other solitons without losing their shape. They can increase the data transfer capabilities and the distance between repeater stations in fiber optics. They can be switched and guided by other light beams in nonlinear optics and communication electronics. The retrieved solutions' behaviors are presented using 3D surface graphs and related 2D line graphs along with the contour plots. The achieved findings show that the offered approaches are vital tools for investigating nonlinear equations. The offered work contains useful information regarding the q-deformed model, and other accurate solutions to this equation will be considered to be an important task in the future. The observed solitary wave structures corresponding to the obtained solutions are important for further explorations and applications in describing physical systems that have broken symmetry and incorporate effects such as amplification or dissipation.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Abdel-Haleem Abdel-Aty: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal). **Saima Arshed**: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing –

original draft (equal). **Nauman Raza**: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal). **Tahani A. Alrebdi**: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal). **K. S. Nisar**: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal). **Hichem Eleuch**: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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