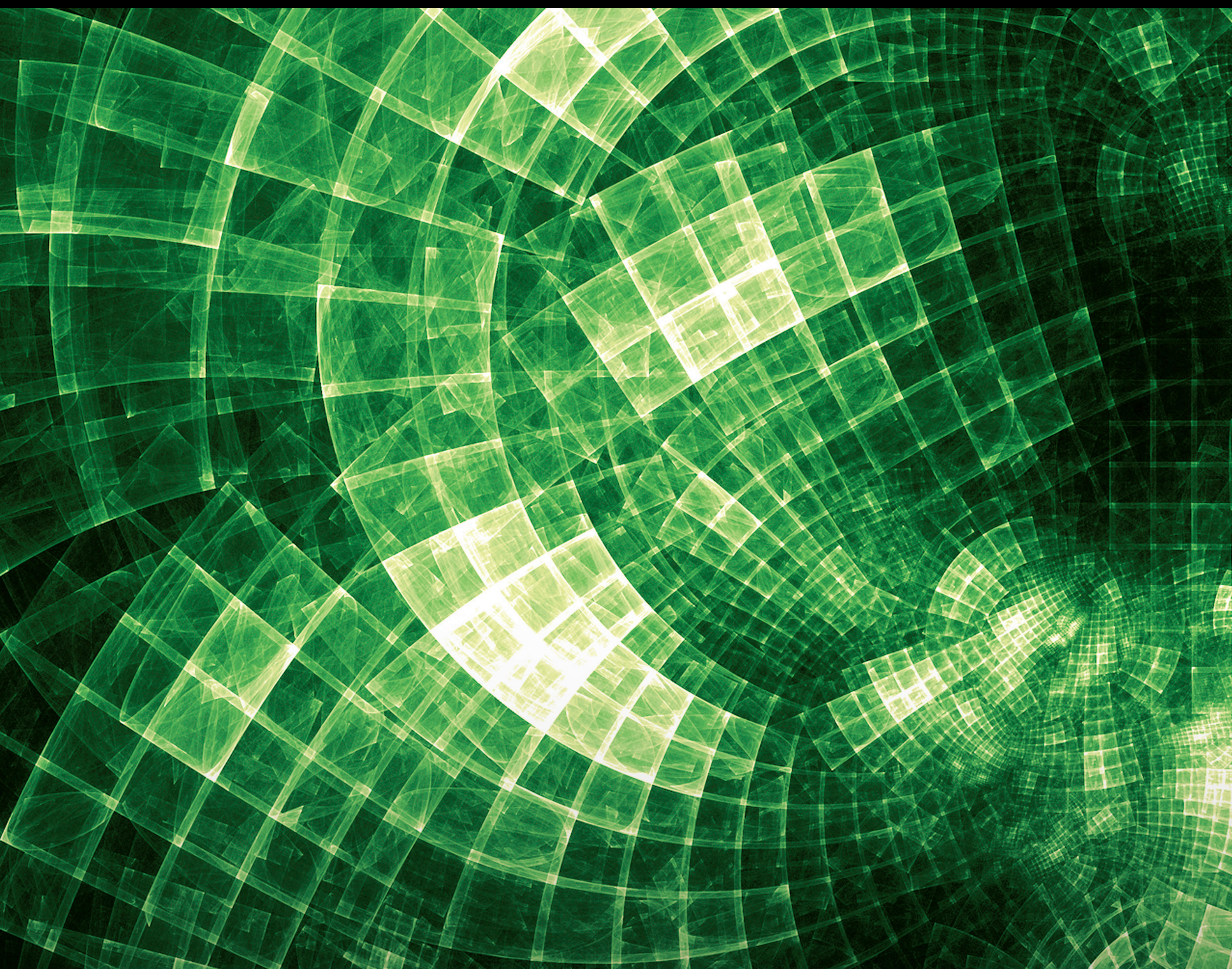
Journal of Mathematics

Recent Trends in Computational and Theoretical Aspects in Differential and Difference Equations

Lead Guest Editor: Ram Jiwari
Guest Editors: Mehmet Emir Köksal and Haydar Akca



Recent Trends in Computational and Theoretical Aspects in Differential and Difference Equations

Journal of Mathematics

Recent Trends in Computational and Theoretical Aspects in Differential and Difference Equations

Lead Guest Editor: Ram Jiwari
Guest Editors: Mehmet Emir Köksal and Haydar Akca

Editorial Board

Contents

Hindawi
Journal of Mathematics
Volume 2017, Article ID 3406716, 1 page
https://doi.org/10.1155/2017/3406716

Editorial

Recent Trends in Computational and Theoretical Aspects in Differential and Difference Equations

Ram Jiwari,[1] Mehmet Emir Koksal,[2] and Haydar Akca[3]

[1]*Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India*
[2]*Department of Mathematics, Ondokuz Mayis University, 55139 Samsun, Turkey*
[3]*Department of Applied Sciences and Mathematics, College of Arts and Sciences, Abu Dhabi University, P.O. Box 59911, Abu Dhabi, UAE*

Correspondence should be addressed to Ram Jiwari; ram1maths@gmail.com

Received 6 July 2017; Accepted 9 July 2017; Published 11 September 2017

Differential and difference equations are used for modeling, solving, and discussing many problems arising in engineering and natural sciences. Therefore, analysis of analytical and numerical solutions to such equations is crucial for applications. It is important to develop new theories and methods, as well as modify the well-known techniques.

The main propose of this special issue was to attract high quality contributions and highlight significant recent developments on the subject of differential and difference equations. The focus of this special issue was also originated with numerical methods for solving differential equations and modelling and/or considering engineering problems involving differential and difference equations. Below we give a brief survey of the content of this special issue.

The special issue succeeded in bringing together a number of papers on various branches of the theory and applications of both differential and difference equations. One of the papers in the issue is on the convergence of the solutions of the bistable reaction-diffusion equations in two-dimensional spaces to a pair of diverging traveling curved fronts. Another one focuses on developing a mixed discontinuous Galerkin scheme for the following control constrained optimal control problem governed by a transient convection-diffusion equation. One paper studies the stability analysis of the periodic solutions of some kind of predator-prey dynamical systems. There is also an important paper on finding exact solution of nonlinear sixth-order solitary wave equation using the improved generalized tanh-coth method. Finally, as a novel application of differential equations in system and control theory, the investigation of the inverse commutativity conditions for second-order linear continuous time-varying systems is studied in another paper.

Acknowledgments

Finally, the guest editors of the present special issue wish to take this opportunity to thank all authors for their contributions, which have provided a range of in-depth overviews of the most relevant topics in differential and difference equations. They are also grateful to distinguished scientists who served as referees for the submitted manuscripts. It is hoped that the papers published in the special issue will enhance and stimulate future studies and developments in the field.

Ram Jiwari
Mehmet Emir Koksal
Haydar Akca

Hindawi
Journal of Mathematics
Volume 2017, Article ID 3070236, 6 pages
https://doi.org/10.1155/2017/3070236

Research Article

Inverse Commutativity Conditions for Second-Order Linear Time-Varying Systems

Mehmet Emir Koksal

Department of Mathematics, Ondokuz Mayis University, Atakum, 55139 Samsun, Turkey

Correspondence should be addressed to Mehmet Emir Koksal; emir_koksal@hotmail.com

Received 14 May 2017; Revised 24 June 2017; Accepted 27 June 2017; Published 27 August 2017

Academic Editor: Willi Freeden

The necessary and sufficient conditions where a second-order linear time-varying system A is commutative with another system B of the same type have been given in the literature for both zero initial states and nonzero initial states. These conditions are mainly expressed in terms of the coefficients of the differential equation describing system A. In this contribution, the inverse conditions expressed in terms of the coefficients of the differential equation describing system B have been derived and shown to be of the same form of the original equations appearing in the literature.

1. Introduction

As one of the main fields of applied mathematics, differential equations arise in acoustics, electromagnetics, electrodynamics, fluid dynamics, wave motion, wave distribution, and many other sciences and branches of engineering. There is a tremendous amount of work on the theory and techniques for solving differential equations and on their applications [1–4]. Particularly, they are used as a powerful tool for modelling, analyzing, and solving real engineering problems and for discussing the results turned up at the end of analyzing for resolution of naturel problems. For example, they are used in system and control theory, which is an interdisciplinary branch of electric-electronics engineering and applied mathematics that deal with the behavior of dynamical systems with inputs and how their behavior is modified by different combinations such as cascade and feedback connections [5–8]. When the cascade connection in system design is considered, the commutativity concept places an important role to improve different system performances [9–11].

As shown in Figure 1, when two linear time-varying systems A and B described by linear time-varying differential equations are connected in cascade so that the output of one appears as the input of the other, it is said that systems A and B are connected in cascade. If the order of connection does not affect the input-output relation of the combined system AB or BA, it is said that systems A and B are commutative.

Although the first paper about the commutativity appeared in 1977 [12] which had introduced the commutativity concept for the first time and studied the commutativity of the first-order continuous-time linear time-varying systems, this paper is important for proving that a time-varying system can be commutative with another time-varying system only; further very few classes of systems can be commutative. In particular, commutativity conditions for relaxed second-order systems first appeared in 1982 [13]. In 1984 [14] and 1985 [15], commutativity conditions for third- and fourth-order continuous-time linear time-varying systems were studied, respectively. The content of the published but undistributed work [15] can be found in journal paper [16] which presents an exhaustive study on the commutativity of continuous-time linear time-varying systems. That paper is the first tutorial paper in the literature.

During the period from 1989 to 2011, no publication about commutativity had appeared in the literature. In 2011, the second basic journal publication [17] appeared. In this paper, commutativity in case of nonzero initial conditions, commutativity of Euler's systems, new results about effects of commutativity, reduction of disturbance by change of connection order in a chain structure of subsystems, and the

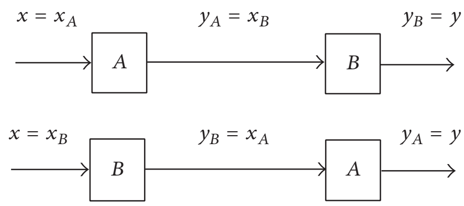


Figure 1: Cascade connection of the differential systems A and B.

most important explicit commutativity conditions for fifth-order systems were studied.

This work is directly focused on the commutativity of second-order linear time-varying systems with zero initial conditions. Section 2 summarizes the necessary and sufficient conditions of commutativity of such systems as appearing in the literature and then presents the inverse conditions. Section 3 covers an example and its simulation results. Finally, Section 4 includes conclusions.

2. Inverse Conditions of Commutativity

Let A be a second-order linear time-varying system described by

$$\begin{aligned} &A: \\ &a_2(t)\,\ddot{y}_A(t) + a_1(t)\,\dot{y}_A(t) + a_0(t)\,y_A(t) = x_A(t); \\ &\qquad \forall t \geq t_0, \\ &y_A(t_0) = 0, \\ &\dot{y}_A(t_0) = 0, \end{aligned} \tag{1a}$$

where $x_A(\cdot)$ is the input; $y_A(\cdot)$ is the output functions of the system; $a_i(t)$'s, $i = 2, 1, 0$, are the time-varying coefficients. They are all defined for $[t_0, \infty) \to R$. The (double) dot on the top represents the (second) first-order derivative with respect to time $t \in R$, where $t \geq t_0$, t_0 being the initial time. Since A is of second order, $a_2(t) \not\equiv 0$; further, for the unique solvability of (1a) for the output $y_A(\cdot)$, it is sufficient that $x_A(\cdot), a_i(\cdot) \in P[t_0, \infty)$, which is the set of piecewise continuous mappings $[t_0, \infty) \to R$ [2].

Let B be another second-order linear time-varying system defined by

$$\begin{aligned} &B: \\ &b_2(t)\,\ddot{y}_B(t) + b_1(t)\,\dot{y}_B(t) + b_0(t)\,y_B(t) = x_B(t); \\ &\qquad \forall t \geq t_0, \\ &y_B(t_0) = 0, \\ &\dot{y}_B(t_0) = 0, \end{aligned} \tag{1b}$$

where $x_B(\cdot)$, $y_B(\cdot)$, $b_i(t)$'s are defined in a similar manner as for system A and $b_2(t) \not\equiv 0$.

For the commutativity of systems A and B, it is necessary and sufficient that the coefficients of B must be expressible in terms of those of A by the matrix equation

$$\begin{bmatrix} b_2(t) \\ b_1(t) \\ b_0(t) \end{bmatrix} = \begin{bmatrix} a_2(t) & 0 & 0 \\ a_1(t) & a_2^{0,5}(t) & 0 \\ a_0(t) & a_2^{-0,5}(t)\,\dfrac{2a_1(t) - \dot{a}_2(t)}{4} & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}, \tag{2a}$$

where $c_2 \neq 0, c_1, c_0$ are arbitrary constants; further, the coefficients of A must satisfy the following equation for the general values of $t \geq t_0$:

$$-a_2^{0.5}\frac{d}{dt}\left[a_0 - \frac{4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1 a_2 - 4a_2\ddot{a}_2}{16a_2}\right]c_1 \equiv 0. \tag{2b}$$

Defining

$$f_A(t) = a_2^{-0.5}(t)\,\frac{2a_1(t) - \dot{a}_2(t)}{4}, \tag{3}$$

it can be shown by routine mathematical operations that the necessary and sufficient conditions in (2a) and (2b) can be rewritten as

$$\begin{bmatrix} b_2(t) \\ b_1(t) \\ b_0(t) \end{bmatrix} = \begin{bmatrix} a_2(t) & 0 & 0 \\ a_1(t) & a_2^{0.5}(t) & 0 \\ a_0(t) & f_A(t) & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}, \tag{4a}$$

$$-a_2^{0.5}\frac{d}{dt}\left[A_0(t)\right]c_1 \equiv 0, \tag{4b}$$

where

$$A_0(t) = a_0(t) - f_A^2(t) - a_2^{0.5}(t)\,\dot{f}_A(t). \tag{4c}$$

If $c_1 = 0$, (2b) and (4b) are automatically satisfied and hence they are redundant. But if $c_1 \neq 0$, these equations, together with the information $a_2(t) \not\equiv 0$, are replaced by (4c) with $A_0(t)$ being constant, that is, time invariant.

It is naturally expected that if A and B are commutative, similar equations to (4a), (4b), and (4c) are valid when the coefficients of B are used. In fact, the first equation in (4a) is equivalent to

$$a_2(t) = \left(\frac{1}{c_2}\right)b_2(t). \tag{5a}$$

Using this equation in the second line of equation in (4a) and solving it for $a_1(t)$, we obtain

$$a_1(t) = \left(\frac{1}{c_2}\right)b_1(t) - \left(\frac{c_1}{c_2^{1.5}}\right)b_2^{0.5}(t). \tag{5b}$$

Using (5a) and (5b) in (3), we express $f_A(t)$ as

$$
\begin{aligned}
f_A(t) &= \left(\frac{b_2}{c_2}\right)^{-0,5} b\frac{(2/c_2)\, b_1 - \left(2c_1/c_2^{1.5}\right) b_2^{0.5} - (1/c_2)\, \dot{b}_2}{4} \\
&= \left(\frac{b_2}{c_2}\right)^{-0,5} \frac{\left(2b_1 - \dot{b}_2\right)/c_2 - \left(2c_1/c_2^{1.5}\right) b_2^{0.5}}{4} \\
&= \frac{1}{c_2^{0.5}} \frac{b_2^{-0.5}\left(2b_1 - \dot{b}_2\right)}{4} - \frac{c_1}{2c_2} = \frac{1}{c_2^{0.5}} f_B(t) - \frac{c_1}{2c_2},
\end{aligned}
\tag{5c}
$$

where

$$
f_B(t) = b_2^{-0.5}(t)\, \frac{2b_1(t) - \dot{b}_2(t)}{4}. \tag{6}
$$

Finally, substituting (5c) in the third line of equation of (4a) and solving it for $a_0(t)$, we obtain

$$
\begin{aligned}
a_0(t) &= \frac{1}{c_2}\left[b_0(t) - \frac{c_1}{c_2^{0.5}} f_B(t) + \frac{c_1^2}{2c_2} - c_0\right] \\
&= \left(\frac{1}{c_2}\right) b_0(t) + \left(-\frac{c_1}{c_2^{1.5}}\right) f_B(t) + \frac{c_1^2 - 2c_2c_0}{2c_2^2}.
\end{aligned}
\tag{7}
$$

Hence, writing (5a), (5b), and (7) in matrix form and letting

$$
\begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{c_2} \\ -\dfrac{c_1}{c_2^{1.5}} \\ \dfrac{c_1^2 - 2c_2c_0}{2c_2^2} \end{bmatrix} \tag{8a}
$$

or equivalently (note $c_2 \neq 0$ so $k_2 \neq 0$)

$$
\begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{k_2} \\ -\dfrac{k_1}{k_2^{1.5}} \\ \dfrac{k_1^2 - 2k_2k_0}{2k_2^2} \end{bmatrix}, \tag{8b}
$$

we obtain the first set of the inverse equations derived as

$$
\begin{bmatrix} a_2(t) \\ a_1(t) \\ a_0(t) \end{bmatrix} = \begin{bmatrix} b_2(t) & 0 & 0 \\ b_1(t) & b_2^{0.5}(t) & 0 \\ b_0(t) & f_B(t) & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}. \tag{9a}
$$

To find the second inverse equation, we substitute $a_0(t)$ from (7), $f_A(t)$ from (5c), and $a_2(t)$ from (5a), all into (4c); and rearranging, we obtain

$$
\begin{aligned}
A_0(t) &= \frac{b_0}{c_2} - \frac{c_1}{c_2^{1.5}} f_B + \frac{c_1^2 - 2c_2c_0}{2c_2^2} - \left(\frac{f_B}{c_2^{0.5}} - \frac{c_1}{2c_2}\right)^2 \\
&\quad - \frac{b_2^{0.5}}{c_2^{0.5}}\left(\frac{\dot{f}_B}{c_2^{0.5}}\right) \\
&= \frac{1}{c_2}\left[b_0(t) - f_B^2(t) - b_2^{0.5}(t)\, \dot{f}_B(t)\right] \\
&\quad + \frac{c_1^2 - 4c_2c_0}{4c_2^2}.
\end{aligned}
\tag{$*$}
$$

Using this in (4b), eliminating a_2 by (5a) and c_2, c_1 by (8b), and finally multiplying by -1, we proceed with

$$
\begin{aligned}
-a_2^{0.5}\frac{d}{dt}\left[\frac{1}{c_2}\left(b_0 - f_B^2 - b_2^{0.5}\dot{f}_B\right) + \frac{c_1^2 - 4c_2c_0}{4c_2^2}\right] c_1 &= 0, \\
-\frac{b_2^{0.5}}{c_2^{0.5}}\frac{d}{dt}\left[\frac{1}{c_2}\left(b_0 - f_B^2 - b_2^{0.5}\dot{f}_B\right)\right] c_1 &= 0, \\
b_2^{0.5}\frac{d}{dt}\left[\left(b_0 - f_B^2 - b_2^{0.5}\dot{f}_B\right)\right]\left(\frac{-c_1}{c_2^{1.5}}\right) &= 0, \\
-b_2^{0.5}\frac{d}{dt}\left[B_0(t)\right] k_1 &\equiv 0,
\end{aligned}
\tag{9b}
$$

where

$$
B_0(t) = b_0(t) - f_B^2(t) - b_2^{0.5}(t)\, \dot{f}_B(t). \tag{9c}
$$

Hence, (9a) and (9b) constitute the inverse necessary and sufficient conditions in terms of the coefficients of subsystem B. These equations are the duals of (4a) and (4b), respectively; similarly, (9c) is the dual of (4c).

Finally, using (9c) in the above obtained intermediate equation for $A_0(t)$, we write

$$
A_0(t) = \frac{1}{c_2} B_0(t) + \frac{c_1^2 - 4c_2c_0}{4c_2^2}. \tag{10a}
$$

Solving it for $B_0(t)$, we obtain

$$
B_0(t) = c_2 A_0(t) + c_0 - \frac{c_1^2}{4c_2}. \tag{10b}
$$

Using (8a) and (8b), we obtain the duals of (10a) and (10b) as

$$
B_0(t) = \frac{1}{k_2} A_0(t) + \frac{k_1^2 - 4k_2k_0}{4k_2^2}, \tag{11a}
$$

$$
A_0(t) = k_2 B_0(t) + k_0 - \frac{k_1^2}{4k_2}, \tag{11b}
$$

respectively.

We remark that $f_A(t)$ in (3) and $f_B(t)$ in (6) are similarly defined so they constitute also dual equations. We express the results we obtained by a theorem.

Theorem 1. *Let A and B be two second-order linear time-varying systems described by (1a) and (1b), respectively.*

The necessary and sufficient conditions where subsystem B is commutative with subsystem A are as follows:

(i) *The coefficients of B are expressed in terms of the coefficients of A as in (4a) where $c_2 \neq 0$, c_1, c_0 are some constants.*

(ii) *Further, the coefficients of A and c_1 satisfy (4b).*

Conversely, the necessary and sufficient conditions where subsystem A is commutative with subsystem B are as follows:

(iii) *The coefficients of A are expressed in terms of the coefficients of B as in (9a) where $k_2 \neq 0$, k_1, k_0 are some constants.*

(iv) *Further, the coefficients of B and k_1 satisfy (9b).*

Conditions (i) and (ii) together are equivalent to conditions (iii) and (iv) together. There exists a unique relation between the constants c_2, c_1, c_0 and k_2, k_1, k_0 which are expressed by the dual equations (8a) and (8b).

If $c_1 \neq 0$ which is equivalent to $k_1 \neq 0$ due to (8a) and (8b), the second and fourth conditions above are replaced, respectively, by the following:

(v) *$A_0(t)$ defined in (4c) is independent of time and equal to a constant.*

(vi) *$B_0(t)$ defined in (9c) is independent of time and equal to a constant.*

Conditions (i) and (v) together are equivalent to conditions (iii) and (vi) together. Further (v) and (vi) are equivalent conditions due to (10a), (10b), (11a), and (11b).

3. Example

Let the coefficients of A be $a_2 = t^4$ and $a_1 = t^3$. Then, by (3), $f_A(t) = -t/2$; hence $\dot{f}_A(t) = -1/2$. Substituting these in (4c) and choosing $A_0 = -1$, we find $a_0(t) = -1 - t^2/4$. Hence, A is described by

$$A\colon\ t^4 \ddot{y}_A(t) + t^3 \dot{y}_A(t) - \left(1 + \frac{t^2}{4}\right) y_A(t) = x_A(t). \tag{12a}$$

Since $a_0(t)$ is chosen so that (4c) is satisfied with a constant A_0, the above system has second-order pairs computed by (4a) where c_1 can be chosen different from zero due to condition (v) of Theorem 1. In fact, choosing $c_2 = 4$, $c_1 = -3$, and $c_0 = 1$, the following commutative pair B of A is obtained:

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} t^4 & 0 & 0 \\ t^3 & t^2 & 0 \\ -\left(1 + \dfrac{t^2}{4}\right) & -\dfrac{t}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4t^4 \\ 4t^3 - 3t^2 \\ -t^2 + 1.5t - 3 \end{bmatrix},$$

$$B\colon\ 4t^4 \ddot{y}_B(t) + \left(4t^3 - 3t^2\right) \dot{y}_B(t) - \left(t^2 - 1.5t + 3\right) y_B(t) = x_B(t). \tag{12b}$$

Using (6)

$$f_B(t) = \frac{1}{2t^2} \frac{2\left(4t^3 - 3t^2\right) - 16t^3}{4} = -t - 0.75, \tag{13}$$

where $\dot{f}_B(t) = -1$. Evaluating (9c), we find

$$B_0 = -t^2 + 1{,}5t - 3 - (-t - 0{,}75)^2 - 2t^2(-1) = -\frac{57}{16}. \tag{14}$$

Or directly from (10b)

$$B_0 = -1(4) + 1 - \frac{9}{16} = -3 - \frac{9}{16} = -\frac{57}{16}, \tag{15}$$

which is the same result in (14). The constants k_2, k_1, k_0 are found from (8a) as

$$\begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{4} \\ -\dfrac{(-3)}{8} \\ \dfrac{[9 - 2(4)]}{2(16)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{4} \\ \dfrac{3}{8} \\ \dfrac{1}{32} \end{bmatrix}. \tag{16}$$

So that the coefficients of A can be inversely computed by using (9a):

$$\begin{aligned} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} &= \begin{bmatrix} 4t^4 & 0 & 0 \\ 4t^3 - 3t^2 & 2t^2 & 0 \\ -t^2 + 15t - 3 & -t - 0.75 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{4} \\ \dfrac{3}{8} \\ \dfrac{1}{32} \end{bmatrix} \\ &= \begin{bmatrix} t^4 \\ t^3 - \dfrac{3}{4}t^2 + \dfrac{3}{4}t^2 \\ \dfrac{t^2}{4} + \dfrac{1{,}5t}{4} - \dfrac{3}{4} - \dfrac{3t}{8} - \dfrac{9}{32} + \dfrac{1}{32} \end{bmatrix} \\ &= \begin{bmatrix} t^4 \\ t^3 \\ -\dfrac{t^2}{4} - 1 \end{bmatrix}. \end{aligned} \tag{17}$$

Simulation results for AB and BA for inputs $x_1(t) = -6.5 + 10\sin(8\pi t)$ and $x_2(t) = 2.5e^{-t}$ are shown for $0.5 \le t \le 4$ in Figure 2. For both inputs, AB and BA give the same response ($AB1 = BA1$ and $AB2 = BA2$), respectively. If the coefficient $a_0(t)$ of A is changed to $-1 - 1.25t^2$, then (4c) is not constant any more since $A_0(t) = -1 - t^2$; so, A will not have a second-order time-varying commutative pair unless $c_1 = 0$.

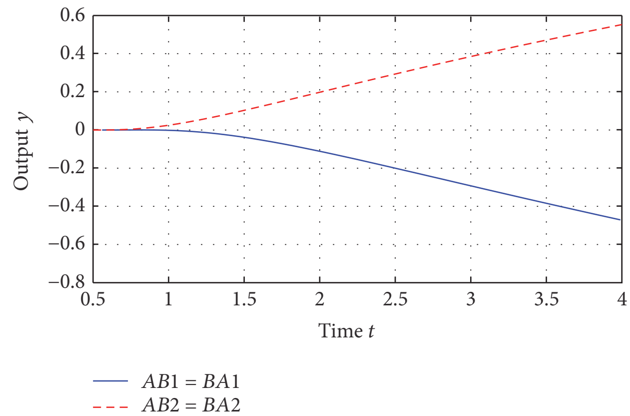


Figure 2: Responses of cascade connections AB and BA for sinusoidal ($AB1 = BA1$) and exponential ($AB2 = BA2$) inputs.

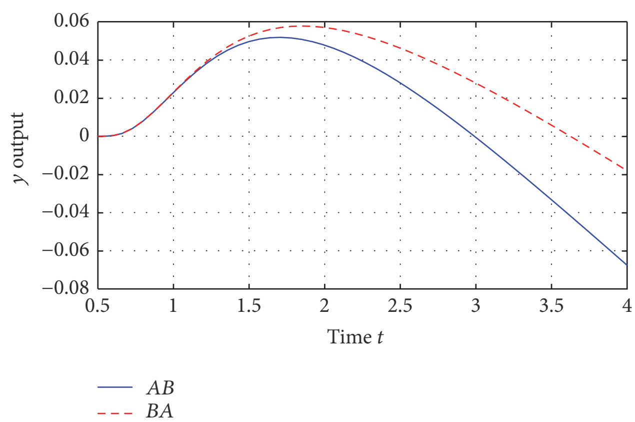


Figure 3: Responses of AB and BA for sinusoidal input when A and B do not satisfy commutativity conditions.

For $c_1 = -3 \neq 0$ and (2a) is spoiled between the coefficients of A and B both, A will not commute with B any more. This is verified by Figure 3 which is obtained by input $x_1(t) = -6.5 + 10\sin(8\pi t)$; it is seen obviously that the responses of AB and BA are not coincident at all. All simulations are performed with the initial and final times $t_0 = 0.5$ and $t = 4$, respectively, by Simulink program of MATLAB 2010a.

4. Conclusions

The commutativity conditions for a second-order linear time-varying system B commutative with another second-order linear time-varying system A is well known in the literature. In this paper, the inverse commutativity conditions are obtained. It is shown that the commutativity conditions for any two second-order linear time-varying systems A and B are in the same form whether they are expressed in terms of the coefficients of systems A or B. The results are illustrated by an example. Inverse commutativity conditions obtained are used in transitivity property of commutativity for time-varying systems, which is the subject of future work. Further, the problem of commutativity of switched systems [18], which are also linear time-varying systems, can be an interesting research subject which has not been studied before. Additionally, commutativity conditions could be studied for fractional order linear differential systems and even linear systems of some fractional difference equations [19–21].

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

This study is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the Project no. 115E952.

References

[1] G. M. Murphy, *Ordinary Differential Equations and Their Solutions*, D. Van Nostrand Co., Inc., London, UK, 1960.

[2] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, London, UK, 2nd edition, 2003.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, NY, USA, 7th edition, 2007.

[4] W. G. Kelley and A. C. Peterson, *The Theory of Differential Equations: Classical and Qualitative*, Springer, 2nd edition, 2010.

[5] K. Ogata, *Modern Control Engineering*, Prentice Hall, New York, NY, USA, 5th edition, 2010.

[6] A. G. J. Holt and K. M. Reineck, "The transfer function synthesis for a cascade connection network," *IEEE Transactions on Circuit Theory*, vol. 15, no. 2, pp. 162–165, 1968.

[7] M. R. Ainbund and I. P. Maslenkov, "Improving the characteristics of microchannel plates in cascade connection," *Instrumens and Experimental Techniques*, vol. 26, no. 3, pp. 650–652, 1983.

[8] I. Gohberg, M. A. Kaashoek, and A. C. Ran, "Partial pole and zero displacement by cascade connection," *SIAM Journal on Matrix Analysis and Applications*, vol. 10, no. 3, pp. 316–325, 1989.

[9] M. Koksal, "Effects of nonzero initial conditions on commutativity and those of commutativity on system sensitivity," in *Proceeding of the 2nd National Congress of Electrical Engineers*, vol. 2/2, pp. 570–573, Ankara, Turkey, 1987.

[10] M. Koksal, "Effects of nonzero initial conditions on the commutativity of linear time-varying systems," in *Proceedings of the International Conference on Modeling and Simulation*, vol. 1A, pp. 49–55, Istanbul, Turkey, June 1988.

[11] M. Koksal, "Effects of commutativity on system's sensitivity," in *Proceeding of the 6th International Symposium on Networks, Systems and Signal Processing*, pp. 61-62, Zagreb, Yugoslavia, June 1989.

[12] J. E. Marshall, "Commutativity of time-varying systems," *Electronics Letters*, vol. 13, no. 18, pp. 539-540, 1977.

[13] M. Koksal, "Commutativity of second order time varying systems," *International Journal of Control*, vol. 36, no. 3, pp. 541–544, 1982.

[14] M. Koksal, "General conditions for commutativity of time-varying systems," in *Proceedings of the International Conference*

on Telecommunication and Control, pp. 223–225, Halkidiki, Greece, 1984.

[15] M. Koksal, "A Survey on the Commutativity of Time-Varying Systems," METU, GEEE CAS-85/1, Gaziantep Engineering Faculty, 1985.

[16] M. Koksal, "Exhaustive study on the commutativity of time-varying systems," *International Journal of Control*, vol. 47, no. 5, pp. 1521–1537, 1988.

[17] M. Koksal and M. E. Koksal, "Commutativity of linear time-varying differential systems with nonzero initial conditions: a review and some new extensions," *Mathematical Problems in Engineering*, vol. 2011, Article ID 678575, 25 pages, 2011.

[18] X. Zhao, S. Yin, H. Li, and B. Niu, "Switching stabilization for a class of slowly switched systems," *IEEE Transactions on Automatic Control*, vol. 60, pp. 221–226, 2015.

[19] I. K. Dassios, D. I. Baleanu, and G. I. Kalogeropoulos, "On non-homogeneous singular systems of fractional nabla difference equations," *Applied Mathematics and Computation*, vol. 227, pp. 112–131, 2014.

[20] I. K. Dassios, "Optimal solutions for mon-consistent singular linear systems of fractional nabla difference equations," *Circuits, Systems, and Signal Processing*, vol. 34, no. 6, pp. 1769–1797, 2015.

[21] I. K. Dassios, "Stability and robustness of singular systems of fractional nabla difference equations," *Circuits, Systems, and Signal Processing*, vol. 36, no. 1, pp. 49–64, 2017.

Hindawi
Journal of Mathematics
Volume 2017, Article ID 8751097, 11 pages
https://doi.org/10.1155/2017/8751097

Research Article

The Improved Generalized tanh-coth Method Applied to Sixth-Order Solitary Wave Equation

M. Torvattanabun, J. Simmapim, D. Saennuad, and T. Somaumchan

Department of Mathematics, Faculty of Science and Technology, Loei Rajabhat University, Loei 42000, Thailand

Correspondence should be addressed to M. Torvattanabun; montri_kai@windowslive.com

Received 10 February 2017; Revised 1 May 2017; Accepted 29 May 2017; Published 17 July 2017

Academic Editor: Emir Köksal

The improved generalized tanh-coth method is used in nonlinear sixth-order solitary wave equation. This method is a powerful and advantageous mathematical tool for establishing abundant new traveling wave solutions of nonlinear partial differential equations. The new exact solutions consisted of trigonometric functions solutions, hyperbolic functions solutions, exponential functions solutions, and rational functions solutions. The numerical results were obtained with the aid of Maple.

1. Introduction

Nonlinear evolution equations (NLEEs) play an important role in various branches of scientific disciplines, such as fluid mechanics, optical fibers, plasma physics, chemical physics, biology, solid state physics, oceans engineering, and many other scientific applications. The solitary wave was introduced by Russell more than a century ago [1]. In the past years, many powerful methods for finding exact solutions of NLEEs have been proposed, such as the generalized (G'/G)-expansion method [2], the tanh-coth method [3], the modified sine-cosine method [4], the generalized unified method [5], the improved F-expansion method [6], the generalized Kudryashov method [7], the generalized Riccati equation mapping method [8], the modified Kudryashov method [9], the $\exp(-\phi(\xi))$ method [10], the lie symmetry analysis method [11], the first integral method [12], and the consistent Riccati expansion [13].

Another powerful method has been presented by Malfliet [14], who had customized the tanh technique and called the tanh method. In 2002, Fan and Hona [15] extended the tanh method which is called the extended tanh method, by using $U(\xi) = \sum_{k=0}^{M} a_k Y^k$ as traveling wave solutions. In 2007, Wazwaz [3] extended and improved this method which is called the tanh-coth method. In this method $U(\xi) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}$ is used as traveling wave solutions. In 2008 Gómez and Salas [16] improved and generalized this method which is called the improved generalized tanh-coth method, by using $U(\xi) = \sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}$, where ϕ is the solution of the generalized Riccati equation. Afterwards, several researchers applied this method to obtain new exact solutions for nonlinear PDEs [17–20].

In 2017, Christou [21] studies solitons occurring in electrical nonlinear transmission lines; there are called electrical solitons. The problem is applied to Ohm's law of solid state physics by using Taylor-series expansions.

In this paper, we focus on using the improved generalized tanh-coth method for finding exact solutions of the sixth-order solitary wave equation:

$$V_{tt} = \left[aV + bV^2 - cV^3 + gV_{xx} + hVxxxx\right]_{xx}, \qquad (1)$$

which was proposed by Christou [21] and

$$a = \frac{h^2}{C_0 L}$$

$$b = \frac{1}{2F_0}$$

$$c = \frac{1}{3F_0^2}$$

$$g = \frac{(12\delta + 1) h^4}{12C_0 L}$$

$$h = \frac{(60\delta + 1) h^6}{360C_0 L}. \tag{2}$$

In Section 2, we briefly describe the improved generalized tanh-coth method; in Section 3, the improved generalized tanh-coth method is applied to the sixth-order solitary wave equations. The last section is short summary and discussion.

2. The Improved Generalized tanh-coth Method

Consider the nonlinear partial differential equation in the variables x and t

$$P_1 (u, u_x, u_t, u_{x,t}, u_{x,x}, u_{t,t}, \dots) = 0. \tag{3}$$

The traveling wave transformation is given by

$$u(x,t) = U(\xi), \quad \xi = x - \lambda t + \xi_0, \tag{4}$$

where λ is the wave speed. We can reduce (3) to the ordinary differential equation

$$P_2 \left(U, U', U'', \dots\right) = 0. \tag{5}$$

According to the improved generalized tanh-coth method, we seek the exact solution of (3) that can be expressed in the following form:

$$U(\xi) = \sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}, \tag{6}$$

where M is a positive integer that will be determined by balancing the highest order derivative term with the highest order nonlinear term. The coefficients a_i are constants ($a_M \neq 0$ and $a_{-M} \neq 0$) that are determined later while the new variable $\phi(\xi)$ is the solution to the generalized Riccati equation

$$\phi'(\xi) = \alpha + \beta\phi(\xi) + \gamma(\phi(\xi))^2, \tag{7}$$

where α, β, and γ are constants. The solutions of generalized Riccati equation are given by [18].

Case 1 (exponential function solutions). When $\alpha = 0$

$$\phi(\xi) = \frac{\beta}{-\gamma + \beta e^{-\beta\xi}}. \tag{8}$$

Case 2 (trigonometric and hyperbolic function solution). When $\beta = 0$,

$$\phi(\xi) = \begin{cases} \dfrac{\sqrt{\alpha\gamma}}{\gamma} \tan\left(\sqrt{\alpha\gamma}\xi\right), & \alpha > 0,\ \gamma > 0, \\ \dfrac{\sqrt{\alpha\gamma}}{\gamma} \tanh\left(\sqrt{\alpha\gamma}\xi\right), & \alpha > 0,\ \gamma < 0, \\ \dfrac{\sqrt{-\alpha\gamma}}{\gamma} \tanh\left(-\sqrt{-\alpha\gamma}\xi\right), & \alpha < 0,\ \gamma > 0, \\ \dfrac{\sqrt{\alpha\gamma}}{\gamma} \tan\left(-\sqrt{\alpha\gamma}\xi\right), & \alpha < 0,\ \gamma < 0. \end{cases} \tag{9}$$

Case 3 (exponential function solutions). When $\gamma = 0$,

$$\phi(\xi) = \frac{-\alpha + \beta e^{\beta\xi}}{\beta}. \tag{10}$$

Case 4 (rational function solution). When $\alpha = \beta = 0$,

$$\phi(\xi) = -\frac{1}{\gamma\xi}. \tag{11}$$

Case 5 (rational function solution). When $\beta^2 \neq 0$ and $\beta^2 = 4\alpha\gamma$,

$$\phi(\xi) = -\frac{2\alpha(\beta\xi + 2)}{\beta^2\xi}. \tag{12}$$

Case 6 (trigonometric function solution). When $\beta^2 < 4\alpha\gamma$ and $\gamma \neq 0$,

$$\phi(\xi) = \frac{\sqrt{4\alpha\gamma - \beta^2} \tan\left((1/2)\sqrt{4\alpha\gamma - \beta^2}\xi\right) - \beta}{2\gamma}. \tag{13}$$

Case 7 (hyperbolic function solution). When $\beta^2 > 4\alpha\gamma$ and $\gamma \neq 0$,

$$\phi(\xi) = \frac{\sqrt{\beta^2 - 4\alpha\gamma} \tanh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma}\xi\right) - \beta}{2\gamma}. \tag{14}$$

We substitute (6) into (5) and collect all terms with the same order of $\phi^j(\xi)$; we get a polynomial in $\phi(\xi)$. Equating each coefficient of the polynomial to zero, we will give a system of algebraic equations involving the parameters a_i, α, γ, and β. Solving the system, we can construct a variety of exact solutions of (5).

3. The Improved Generalized tanh-coth Method Applied to Sixth-Order Solitary Wave Equation

We use the wave transformations $V(x,t) = V(\xi)$, $\xi = x - \lambda t + \xi_0$, to reduce (1) to the following ODE:

$$\begin{gathered} \left(\lambda^2 - a\right) V''(\xi) - b\left(V^2(\xi)\right)'' + c\left(V^3(\xi)\right)'' \\ + gV''''(\xi) - hV''''''(\xi) = 0. \end{gathered} \tag{15}$$

Balancing the highest order term V'''''' with the highest order nonlinear term $(V^3)'''$ in (13), we have

$$M + 6 = 3M + 2; \quad (16)$$

then $M = 2$. Consequently, we set

$$V(\xi) = a_0 + a_1\phi(\xi) + a_2(\phi(\xi))^2 + a_3(\phi(\xi))^{-1} + a_4(\phi(\xi))^{-2}. \quad (17)$$

Using (6) and (14) in (13) and equating all the coefficients of power of $\phi(\xi)$ to be zero, we obtain a system of algebraic equations in the unknowns a_0, a_1, a_2, a_3, a_4, λ, α, β, and γ.

$$\phi(\xi)^{-8}: -5040\alpha^6 za_4 + 42\alpha^2 ca_4^3 = 0,$$

$$\phi(\xi)^{-7}: -720\alpha^6 za_3 - 19440\alpha^5\beta za_4 + 90\alpha^2 ca_3a_4^2 + 8\alpha\beta ca_4^3 = 0,$$

$$\phi(\xi)^{-6}: 165ca_3a_4^2\alpha\beta + 72ca_4^3\alpha\gamma + 60ca_0a_4^2\alpha^2 + 60ca_3^2\alpha^2a_4 - 29400za_4\alpha^4\beta^2 - 13440za_4\alpha^5\gamma - 2520za_3\alpha^5\beta + 36ca_4^3\beta^2 - 20ba_4^2\alpha^2 - 120ga_4\alpha^4 = 0,$$

$$\phi(\xi)^{-5}: 75ca_3a_4^2\beta^2 + 66ca_4^3\beta\gamma + 72ca_0a_3a_4\alpha^2 + 108ca_0a_4^2\alpha\beta + 150ca_3a_4^2\alpha\gamma + 108ca_3^2a_4\alpha\beta - 38640za_4\alpha^4\beta\gamma - 24ba_3a_4\alpha^2 - 36ba_4^2\alpha\beta + 36ca_1a_4^2\alpha^2 - 21840za_4\alpha^3\beta^3 - 3360za_3\alpha^4\beta^2 - 1680za_3\alpha^5\gamma - 336ga_4\alpha^3\beta + 12ca_3^3\alpha^2 - 24ga_3\alpha^4 = 0,$$

$$\phi(\xi)^{-4}: 96ca_0a_4^2\alpha\gamma + 36ca_1a_3a_4\alpha^2 + 135ca_3a_4^2\beta\gamma - 4200za_3\alpha^4\beta\gamma - 40152za_4\alpha^3\beta^2\gamma - 42ba_3a_4\alpha\beta + 96ca_3^2a_4\alpha\gamma + 63ca_1a_4^2\alpha\beta - 12096za_4\alpha^4\gamma^2 - 8106za_4\alpha^2\beta^4 - 2100za_3\alpha^3\beta^3 - 240ga_4\alpha^3\gamma - 330ga_4\alpha^2\beta^2 - 60ga_3\alpha^3\beta + 21ca_3^3\alpha\beta + 48ca_3^2a_4 - 12ba_0a_4\alpha^2 - 32ba_4^2\alpha\gamma + 18ca_0^2a_4\alpha^2 + 18ca_0a_3^2\alpha^2 + 48ca_0a_4^2\beta^2 + 18ca_2a_4^2\alpha^2 - 6aa_4\alpha^2 + 6\lambda^2a_4\alpha^2 + 30ca_4^3\gamma^2 - 6ba_3^2\alpha^2 - 16ba_4^2\beta^2 + 126ca_0a_3a_4\alpha\beta = 0,$$

$$\phi(\xi)^{-3}: 54ca_0a_3a_4\beta^2 + 84ca_0a_4^2\beta\gamma + 12ca_2a_3a_4\alpha^2 + 30ca_2a_4^2\alpha\beta - 20ba_0a_4\alpha\beta - 440ga_4\alpha^2\beta\gamma - 3584za_3\alpha^3\beta^2\gamma - 17920za_4\alpha^2\beta^3\gamma - 22960za_4\alpha^3\beta\gamma^2 - 36ba_3a_4\alpha\gamma + 30ca_0^2a_4\alpha\beta + 54ca_1a_4^2\alpha\gamma + 84ca_3^2a_4\beta\gamma + 12ca_0a_1a_4\alpha^2 + 30ca_0a_3^2\alpha\beta - 1330za_4\beta^5\alpha - 1232za_3\alpha^4\gamma^2 - 602za_3\alpha^2\beta^4 - 130ga_4\alpha\beta^3 - 40ga_3\alpha^3\gamma - 50ga_3\alpha^2\beta^2 + 18ca_3^3\alpha\gamma + 60ca_3a_4^2\gamma^2 + 10a_4\lambda^2\alpha\beta - 4ba_0a_3\alpha^2 - 10aa_4\alpha\beta - 4ba_1a_4\alpha^2 - 10ba_3^2\alpha\beta - 18ba_3a_4\beta^2 - 28ba_4^2\beta\gamma + 6ca_0^2a_3\alpha^2 + 2ca_1a_4^2\beta^2 + 6ca_1a_3^2\alpha^2 - 2aa_3\alpha^2 + 2a_3\lambda^2\alpha^2 + 9ca_3^3\beta^2 + 108ca_0a_3a_4\alpha\gamma + 60ca_1a_3a_4\alpha\beta = 0,$$

$$\phi(\xi)^{-2}: 24ca_0a_3^2\alpha\gamma + 9ca_1a_3^2\alpha\beta + 24ca_1a_3a_4\beta^2 + 24ca_2a_4^2\alpha\gamma - 6ba_0a_3\alpha\beta - 16ba_0a_4\alpha\gamma - 232ga_4\alpha\gamma\beta^2 - 60ga_3\alpha^2\beta\gamma - 116za_3\alpha^2\beta\gamma - 1848za_3\alpha^3\beta\gamma^2 - 13320za_4\alpha^2\beta^2\gamma^2 - 3096za_4\alpha\beta^2\gamma - 30ba_3a_4\beta\gamma - 6ba_1a_4\alpha\beta + 9ca_0^2a_3\alpha\beta + 24ca_0^2a_4\alpha\gamma + 45ca_1a_4^2\beta\gamma - 3968za_4\alpha^3\gamma^3 - 63za_3\beta^5\alpha - 136ga_4\alpha^2\gamma^2 - 15ga_3\alpha\beta^3 + 15ca_3^3\beta\gamma + 36ca_3^2a_4\gamma^2 + 3a_3\lambda^2\alpha\beta + 8a_4\lambda^2\alpha\gamma - 8ba_0a_4\beta^2 - 3aa_3\alpha\beta - 8aa_4\alpha\gamma - 8ba_3^2\alpha\gamma + 12ca_0^2a_4\beta^2 + 12ca_0a_3^2\beta^2 + 36ca_0a_4^2\gamma^2 + 12ca_2a_4^2\beta^2 - 4aa_4\beta^2 + 4a_4\lambda^2\beta^2 - 4a_3^2b\beta^2 - 12ba_4^2\gamma^2 - 64za_4\beta^6 - 16ga_4\beta^4 + 90ca_0a_3a_4\beta\gamma + 18ca_0a_1a_4\alpha\beta + 48ca_1a_3a_4\alpha\gamma + 18ca_2a_3a_4\alpha\beta = 0,$$

$$\phi(\xi)^{-1}: 18ca_0a_3^2\beta\gamma + 36ca_0a_3a_4\gamma^2 + 6ca_1a_3^2\alpha\gamma + 6ca_2a_3a_4\beta^2 + 18ca_2a_4^2\beta\gamma - 4ba_0a_3\alpha\gamma - 12ba_0a_4\beta\gamma - 120ga_4\alpha\beta\gamma^2 - 22ga_3\alpha\beta^2\gamma - 720za_3\alpha^2\beta^2\gamma^2 - 114za_3\alpha\beta^4\gamma - 3696za_4\alpha^2\beta\gamma^3 - 2352za_4\alpha\beta^3\gamma^2 - 4ba_1a_4\alpha\gamma + 6ca_0^2a_3\alpha\gamma$$

$$+ 18ca_0^2a_4\beta\gamma + 6ca_0a_1a_4\beta^2 - 126za_4\beta^5\gamma$$
$$- 22za_3\alpha^3\gamma^3 - 30ga_4\beta^3\gamma - 16ga_3\alpha^2\gamma^2 + 2a_3\lambda^2\alpha\gamma$$
$$+ 6a_4\lambda^2\beta\gamma - 2ba_0a_3\beta^2 - 2aa_3\alpha\gamma - 6aa_4\beta\gamma$$
$$- 2ba_1a_4\beta^2 - 6ba_3^2\beta\gamma - 12ba_3a_4\gamma^2 + 3ca_0^2a_3\beta^2$$
$$+ 18ca_1a_4^2\gamma^2 + 3ca_1a_3^2\beta^2 - aa_3\beta^2 + a_3\lambda^2\beta^2$$
$$+ 6ca_3^3\gamma^2 - za_3\beta^6 - ga_3\beta^4 + 12ca_0a_1a_4\alpha\gamma$$
$$+ 36ca_1a_3a_4\beta\gamma + 12ca_2a_3a_4\alpha\gamma = 0,$$

$$\phi(\xi)^0 : a_3\lambda^2\beta\gamma + a_1\lambda^2\beta\alpha + 6ca_2^2a_4\alpha^2 + 6ca_2a_4^2\gamma^2$$
$$+ 6ca_0a_1^2\alpha^2 + 6ca_0a_3^2\gamma^2 + 6ca_0^2a_4\gamma^2 + 6ca_0^2a_2\alpha^2$$
$$- 62za_2\beta^4\alpha^2 - 272za_2\gamma^2\alpha^4 - za_1\beta^5\alpha - 62za_4\beta^4\gamma^2$$
$$- 22za_4\alpha^2\gamma^4 - za_3\beta^5\gamma - 16ga_4\alpha\gamma^3 - 14ga_4\beta^2\gamma^2$$
$$- ga_3\beta^3\gamma - ga_1\beta^3\alpha - 14ga_2\beta^2\alpha^2 - 16ga_2\gamma\alpha^3$$
$$- 4ba_0a_4\gamma^2 - 4ba_0a_2\alpha^2 - aa_3\beta\gamma - aa_1\beta\alpha$$
$$- 584za_2\beta^2\alpha^3\gamma - 52za_1\beta^3\gamma\alpha^2 - 136za_1\beta\gamma^2\alpha^3$$
$$- 136za_3\alpha^2\beta\gamma^3 - 52za_3\alpha\beta^3\gamma^2 - 584za_4\alpha\beta^2\gamma^3$$
$$- 8ga_1\beta\gamma\alpha^2 - 8ga_3\alpha\beta\gamma^2 - 2ba_0a_3\beta\gamma - 2ba_0a_1\beta\alpha$$
$$- 2ba_1a_4\beta\gamma - 2ba_2a_3\alpha\beta + 3ca_0^2a_3\beta\gamma + 3ca_0^2a_1\beta\alpha$$
$$+ 12ca_1a_3a_4\gamma^2 + 3ca_1^2a_3\alpha\beta + 12ca_1a_2a_3\alpha^2$$
$$+ 3ca_1a_3^2\beta\gamma - 2ba_1^2\alpha^2 - 2ba_3^2\gamma^2 - 2aa_4\gamma^2 - 2aa_2\alpha^2$$
$$+ 2a_4\lambda^2\gamma^2 + 2a_2\lambda^2\alpha^2 + 6ca_0a_1a_4\beta\gamma + 6ca_0a_2a_3\alpha\beta$$
$$+ 6ca_1a_2a_4\alpha\beta + 6ca_2a_3a_4\beta\gamma = 0,$$

$$\phi(\xi) : 18ca_2^2a_4\alpha\beta - 4ba_2a_3\alpha\gamma - 4ba_0a_1\gamma\alpha + 6ca_0^2a_1\gamma\alpha$$
$$+ 18ca_0^2a_2\alpha\beta + 6ca_0a_2a_3\beta^2 + 18ca_0a_1^2\alpha\beta$$
$$+ 36ca_0a_1a_2\alpha^2 + 6ca_1^2a_3\alpha\gamma + 6ca_1a_2a_4\beta^2$$
$$- 12ba_0a_2\alpha\beta - 22ga_1\beta^2\gamma\alpha - 120ga_2\beta\alpha^2\gamma$$
$$- 114za_1\beta^4\gamma\alpha - 720za_1\beta^2\gamma^2\alpha^2 - 2352za_2\beta^3\gamma\alpha^2$$
$$- 3696za_2\beta\gamma^2\alpha^3 - 6ba_1^2\alpha\beta - 12ba_1a_2\alpha^2$$
$$- 2ba_2a_3\beta^2 + 3ca_0^2a_1\beta^2 + 18ca_2^2a_3\alpha^2 + 3ca_1^2a_3\beta^2$$
$$+ 2a_1\lambda^2\gamma\alpha + 6a_2\lambda^2\alpha\beta - 126za_2\beta^5\alpha - 272za_1\gamma^3\alpha^3$$
$$- 16ga_1\gamma^2\alpha^2 - 30ga_2\beta^3\alpha - 2ba_0a_1\beta^2 - 2aa_1\gamma\alpha$$
$$- 6aa_2\alpha\beta + 6ca_1^3\alpha^2 - za_1\beta^6 - ga_1\beta^4 - aa_1\beta^2$$
$$+ a_1\lambda^2\beta^2 + 12ca_0a_2a_3\alpha\gamma + 36ca_1a_2a_3\alpha\beta$$
$$+ 12ca_1a_2a_4\alpha\gamma = 0,$$

$$\phi(\xi)^2 : 24ca_1a_2a_3\beta^2 + 9ca_1^2a_3\beta\gamma + 24ca_2^2a_4\alpha\gamma$$
$$- 16ba_0a_2\alpha\gamma - 60ga_1\beta\alpha\gamma^2 - 232ga_2\beta^2\alpha\gamma$$
$$- 1848za_1\beta\gamma^3\alpha^2 - 1176za_1\beta^3\alpha\gamma^2 - 3096za_2\beta^4\gamma\alpha$$
$$- 13320za_2\beta^2\gamma^2\alpha^2 - 30ba_1a_2\alpha\beta - 6ba_2a_3\beta\gamma$$
$$- 6ba_0a_1\beta\gamma + 9ca_0^2a_1\beta\gamma + 24ca_0^2a_2\alpha\gamma + 45ca_2^2a_3\alpha\beta$$
$$+ 24ca_0a_1^2\alpha\gamma - 3968za_2\gamma^3\alpha^3 - 63za_1\beta^5\gamma$$
$$- 15ga_1\beta^3\gamma - 136ga_2\gamma^2\alpha^2 + 3a_1\lambda^2\beta\gamma + 8a_2\lambda^2\alpha\gamma$$
$$- 8ba_0a_2\beta^2 - 3aa_1\beta\gamma - 8aa_2\alpha\gamma - 8ba_1^2\alpha\gamma$$
$$+ 12ca_0^2a_2\beta^2 + 36ca_0a_2^2\alpha^2 + 12ca_0a_1^2\beta^2 + 15ca_1^3\alpha\beta$$
$$+ 36ca_1^2a_2\alpha^2 + 12ca_2^2a_4\beta^2 - 4aa_2\beta^2 + 4a_2\lambda^2\beta^2$$
$$- 12ba_2^2\alpha^2 - 4ba_1^2\beta^2 - 64za_2\beta^6 - 16ga_2\beta^4$$
$$+ 90ca_0a_1a_2\alpha\beta + 18ca_0a_2a_3\beta\gamma + 48ca_1a_2a_3\alpha\gamma$$
$$+ 18ca_1a_2a_4\beta\gamma = 0,$$

$$\phi(\xi)^3 : 6ca_1^2a_3\gamma^2 + 60ca_1a_2^2\alpha^2 + 10a_2\lambda^2\beta\gamma$$
$$- 1330za_2\beta^5\gamma - 602za_1\beta^4\gamma^2 - 1232za_1\gamma^4\alpha^2$$
$$- 50ga_1\beta^2\gamma^2 - 40ga_1\alpha\gamma^3 - 130ga_2\beta^3\gamma - 4ba_0a_1\gamma$$
$$- 10aa_2\beta\gamma - 10ba_1^2\beta\gamma - 18ba_1a_2\beta^2 - 28ba_2^2\alpha\beta$$
$$- 4ba_2a_3\gamma^2 + 6ca_0^2a_1\gamma^2 + 27ca_2^2a_3\beta^2 + 18ca_1^3\alpha\gamma$$
$$+ 108ca_0a_1a_2\alpha\gamma + 60ca_1a_2a_3\beta\gamma - 20ba_0a_2\beta\gamma$$
$$- 440ga_2\beta\alpha\gamma^2 - 3584za_1\beta^2\alpha\gamma^3 - 22960za_2\beta\gamma^3\alpha^2$$
$$+ 12ca_1a_2a_4\gamma^2 + 84ca_1^2a_2\alpha\beta + 30ca_2^2a_4\beta\gamma$$
$$- 17920za_2\beta^3\alpha\gamma^2 + 54ca_2^2a_3\alpha\gamma - 36ba_1a_2\alpha\gamma$$
$$+ 30ca_0^2a_2\beta\gamma + 30ca_0a_1^2\beta\gamma + 54ca_0a_1a_2\beta^2$$
$$+ 84ca_0a_2^2\alpha\beta + 12ca_0a_2a_3\gamma^2 - 2aa_1\gamma^2 + 2a_1\lambda^2\gamma^2$$
$$+ 9ca_1^3\beta^2 = 0,$$

$$\phi(\xi)^4 : 135ca_1a_2^2\alpha\beta + 36ca_1a_2a_3\gamma^2 + 96ca_1^2a_2\alpha\gamma$$
$$- 4200za_1\beta\gamma^4\alpha - 40152za_2\beta^2\alpha\gamma^3 + 63ca_2^2a_3\beta\gamma$$

$$
\begin{aligned}
&- 42ba_1a_2\beta\gamma + 96ca_0a_2^2\alpha\gamma - 8106za_2\beta^4\gamma^2 \\
&- 12096za_2\gamma^4\alpha^2 - 2100za_1\beta^3\gamma^3 - 60ga_1\beta\gamma^3 \\
&- 330ga_2\beta^2\gamma^2 - 240ga_2\gamma^3\alpha - 12ba_0a_2\gamma^2 \\
&- 32ba_2^2\alpha\gamma + 18ca_0^2a_2\gamma^2 + 18ca_0a_1^2\gamma^2 + 48ca_0a_2^2\beta^2 \\
&+ 21ca_1^3\beta\gamma + 48ca_1^2a_2\beta^2 + 18ca_2^2a_4\gamma^2 - 6aa_2\gamma^2 \\
&- 6aa_2\gamma^2 + 6a_2\lambda^2\gamma^2 + 30ca_2^3\alpha^2 - 6ba_1^2\gamma^2 \\
&- 16ba_2^2\beta^2 + 126ca_0a_1a_2\beta\gamma = 0, \\
\phi(\xi)^5 :\ &- 336ga_2\beta\gamma^3 - 24ba_1a_2\gamma^2 - 36ba_2^2\beta\gamma \\
&+ 36ca_2^2a_3\gamma^2 + 75ca_1a_2^2\beta^2 + 66ca_2^3\alpha\beta \\
&- 21840za_2\beta^3\gamma^3 - 3360za_1\beta^2\gamma^4 - 1680za_1\gamma^5\alpha \\
&+ 72ca_0a_1a_2\gamma^2 + 108ca_0a_2^2\beta\gamma + 150ca_1a_2^2\alpha\gamma \\
&+ 108ca_1^2a_2\beta\gamma - 38640za_2\beta\gamma^4\alpha + 12ca_1^3\gamma^2 \\
&- 24ga_1\gamma^4 = 0, \\
\phi(\xi)^6 :\ &60ca_0a_2^2\gamma^2 + 60ca_1^2a_2\gamma^2 + 72ca_2^3\alpha\gamma \\
&- 29400za_2\beta^2\gamma^2 - 13440za_2\gamma^5\alpha - 2520za_1\beta\gamma^5 \\
&+ 165ca_1a_2^2\beta\gamma + 36ca_2^3\beta^2 - 20ba_2^2\gamma^2 \\
&- 120ga_2\gamma^4 = 0, \\
\phi(\xi)^7 :\ &- 19440\beta\gamma^5za_2 - 720a_1z\gamma^6 + 8ca_2^3\beta\gamma \\
&+ 90ca_1a_2^2\gamma^2 = 0, \\
\phi(\xi)^8 :\ &- 5040a_2z\gamma^6 + 42ca_2^3\gamma^2 = 0.
\end{aligned}
\tag{18}
$$

Solving the system of algebraic equations with the aid of Maple, using (18), we obtain the following results.

First Set

$$
\begin{aligned}
a_0 &= \pm\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right), \\
a_1 &= 0, \\
a_2 &= 0, \\
a_3 &= \pm\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}, \\
a_4 &= \pm\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}, \\
\lambda &= \pm\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 360\alpha\gamma\beta^2ch^2 + 45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}.
\end{aligned}
\tag{19}
$$

Case 1. When $\beta = 0$, $\alpha > 0$, and $\gamma > 0$, $\phi(\xi) = (\sqrt{\alpha\gamma}/\gamma)\tan(\sqrt{\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$
\begin{aligned}
&V_1(x,t) \\
&= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) \\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\gamma\alpha^2}{c}\cot^2\left(\sqrt{\alpha\gamma}\left(x + \lambda t + \xi_0\right)\right), \\
&V_2(x,t) \\
&= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) \\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\gamma\alpha^2}{c}\cot^2\left(\sqrt{\alpha\gamma}\left(x - \lambda t + \xi_0\right)\right),
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\lambda &= \frac{1}{30} \\
&\cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}.
\end{aligned}
\tag{21}
$$

Case 2. When $\beta = 0$, $\alpha < 0$, and $\gamma > 0$, $\phi(\xi) = (\sqrt{-\alpha\gamma}/\gamma)\tanh(-\sqrt{-\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$
\begin{aligned}
&V_3(x,t) \\
&= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) \\
&\quad - \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\coth^2\left(-\sqrt{-\alpha\gamma}\left(x + \lambda t + \xi_0\right)\right),
\end{aligned}
$$

$$V_4(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\coth^2\left(-\sqrt{-\alpha\gamma}\left(x - \lambda t + \xi_0\right)\right), \quad (22)$$

where

$$\lambda = \frac{1}{30} \cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \quad (23)$$

Case 3. When $\gamma = 0$, $\phi(\xi) = (-\alpha + \beta e^{\beta\xi})/\beta$, the combined formal single kink solutions of (1) are

$$V_5(x,t) = \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha}{c\left(-\alpha + \beta e^{\beta(x+\lambda t+\xi_0)}\right)} + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha^2}{c\left(-\alpha + \beta e^{\beta(x+\lambda t+\xi_0)}\right)^2},$$

$$V_6(x,t) = \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha}{c\left(-\alpha + \beta e^{\beta(x-\lambda t+\xi_0)}\right)} + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha^2}{c\left(-\alpha + \beta e^{\beta(x-\lambda t+\xi_0)}\right)^2}, \quad (24)$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \quad (25)$$

Case 4. When $\beta^2 \neq 0$ and $\beta^2 = 4\alpha\gamma$, $\phi(\xi) = -2\alpha(\beta\xi + 2)/\beta^2\xi$, the rational solutions of (1) are

$$V_7(x,t) = \frac{1}{30}\left(\frac{15\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{ch}\right) - \frac{\beta^3\sqrt{30}\sqrt{hc}\left(x - \lambda t + \xi_0\right)}{c\left(\beta\left(x - \lambda t + \xi_0\right) + 2\right)} + \frac{\beta^5\sqrt{30}\sqrt{hc}\left(x - \lambda t + \xi_0\right)^2}{2c\left(\beta\left(x - \lambda t + \xi_0\right) + 2\right)^2},$$

$$V_8(x,t) = \frac{1}{30}\left(\frac{15\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{ch}\right) - \frac{\beta^3\sqrt{30}\sqrt{hc}\left(x + \lambda t + \xi_0\right)}{c\left(\beta\left(x + \lambda t + \xi_0\right) + 2\right)} + \frac{\beta^5\sqrt{30}\sqrt{hc}\left(x + \lambda t + \xi_0\right)^2}{2c\left(\beta\left(x + \lambda t + \xi_0\right) + 2\right)^2}, \quad (26)$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(-30ach - 10b^2h + 3cg^2\right)}}{ch}. \quad (27)$$

Case 5. When $\beta^2 < 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{4\alpha\gamma - \beta^2}\tan((1/2)\sqrt{4\alpha\gamma - \beta^2}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$V_9(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{4\alpha\gamma - \beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma - \beta^2}\left(x + \lambda t + \xi_0\right)\right) - \beta}\right) + \frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{4\alpha\gamma - \beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma - \beta^2}\left(x + \lambda t + \xi_0\right)\right) - \beta\right)^2}\right),$$

$$V_{10}(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)$$

$$+\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{4\alpha\gamma-\beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma-\beta^2}\left(x-\lambda t+\xi_0\right)\right)-\beta}\right)$$
$$+\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma-\beta^2}\left(x-\lambda t+\xi_0\right)\right)-\beta\right)^2}\right), \tag{28}$$

where

$$\lambda=\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}. \tag{29}$$

Case 6. When $\beta^2 > 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{\beta^2-4\alpha\gamma}\tanh((1/2)\sqrt{\beta^2-4\alpha\gamma}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$V_{11}(x,t)=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)$$
$$+\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x+\lambda t+\xi_0\right)\right)-\beta}\right)$$
$$+\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x+\lambda t+\xi_0\right)\right)-\beta\right)^2}\right),$$
$$V_{12}(x,t)=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)$$
$$+\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x-\lambda t+\xi_0\right)\right)-\beta}\right)$$
$$+\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x-\lambda t+\xi_0\right)\right)-\beta\right)^2}\right), \tag{30}$$

where

$$\lambda=\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}. \tag{31}$$

Second Set

$$
\begin{aligned}
a_0 &= \pm\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right),\\
a_1 &= \pm\frac{2\beta\sqrt{30}\sqrt{hc}\gamma}{c},\\
a_2 &= \pm\frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c},\\
a_3 &= 0,\\
a_4 &= 0,\\
\lambda &= \pm\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 360\alpha\gamma\beta^2ch^2 + 45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}.
\end{aligned}
\tag{32}
$$

Case 1. When $\alpha = 0$, $\phi(\xi) = \beta/(-\gamma + \beta e^{-\beta\xi})$, the combined formal single kink solutions of (1) are

$$
\begin{aligned}
V_{13}(x,t) &= \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma}{c\left(-\gamma + \beta e^{-\beta(x+\lambda t+\xi_0)}\right)}\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(-\gamma + \beta e^{-\beta(x+\lambda t+\xi_0)}\right)^2},\\
V_{14}(x,t) &= \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma}{c\left(-\gamma + \beta e^{-\beta(x-\lambda t+\xi_0)}\right)}\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(-\gamma + \beta e^{-\beta(x-\lambda t+\xi_0)}\right)^2},
\end{aligned}
\tag{33}
$$

where

$$
\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{34}
$$

Case 2. When $\beta = 0$, $\alpha > 0$, and $\gamma > 0$, $\phi(\xi) = (\sqrt{\alpha\gamma}/\gamma)\tan(\sqrt{\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$
\begin{aligned}
&V_{15}(x,t)\\
&\quad = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\tan^2\left(\sqrt{\alpha\gamma}\left(x + \lambda t + \xi_0\right)\right),\\
&V_{16}(x,t)\\
&\quad = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\tan^2\left(\sqrt{\alpha\gamma}\left(x - \lambda t + \xi_0\right)\right),
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
\lambda &= \frac{1}{30}\\
&\cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}.
\end{aligned}
\tag{36}
$$

Case 3. When $\beta = 0$, $\alpha < 0$, and $\gamma > 0$, $\phi(\xi) = (\sqrt{-\alpha\gamma}/\gamma)\tanh(-\sqrt{-\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$
\begin{aligned}
&V_{17}(x,t)\\
&\quad = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\left(\sqrt{-\alpha\gamma}\right)^2}{c}\\
&\quad \cdot \tanh^2\left(-\sqrt{-\alpha\gamma}\left(x + \lambda t + \xi_0\right)\right),\\
&V_{18}(x,t)\\
&\quad = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\left(\sqrt{-\alpha\gamma}\right)^2}{c}\\
&\quad \cdot \tanh^2\left(-\sqrt{-\alpha\gamma}\left(x - \lambda t + \xi_0\right)\right),
\end{aligned}
\tag{37}
$$

where

$$\lambda = \frac{1}{30} \cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{38}$$

Case 4. When $\alpha = \beta = 0, \phi(\xi) = -1/\gamma\xi$, the rational solutions of (1) are

$$V_{19}(x,t) = \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x + \lambda t + \xi_0\right)\right)^2},$$

$$V_{20}(x,t) = \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x - \lambda t + \xi_0\right)\right)^2},$$

$$V_{21}(x,t) = \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x + \lambda t + \xi_0\right)\right)^2},$$

$$V_{22}(x,t) = \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x - \lambda t + \xi_0\right)\right)^2},$$

$$V_{23}(x,t) = -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x + \lambda t + \xi_0\right)\right)^2},$$

$$V_{24}(x,t) = -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x - \lambda t + \xi_0\right)\right)^2},$$

$$V_{25}(x,t) = -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x + \lambda t + \xi_0\right)\right)^2},$$

$$V_{26}(x,t) = -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x - \lambda t + \xi_0\right)\right)^2}, \tag{39}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(-30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{40}$$

Case 5. When $\beta^2 \neq 0$ and $\beta^2 = 4\alpha\gamma, \phi(\xi) = -2\alpha(\beta\xi + 2)/\beta^2\xi$, the rational solutions of (1) are

$$\begin{aligned} V_{27}(x,t) &= \frac{1}{30}\frac{15\sqrt{30}\sqrt{hc}\beta^2h + 10bh + \sqrt{30}\sqrt{ch}g}{ch} - \frac{\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x - \lambda t + \xi_0\right) + 2\right)}{c\left(x - \lambda t + \xi_0\right)} + \frac{4\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x - \lambda t + \xi_0\right) + 2\right)^2}{c\left(x - \lambda t + \xi_0\right)^2}, \\ V_{28}(x,t) &= \frac{1}{30}\frac{15\sqrt{30}\sqrt{hc}\beta^2h + 10bh + \sqrt{30}\sqrt{ch}g}{ch} - \frac{\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x + \lambda t + \xi_0\right) + 2\right)}{c\left(x + \lambda t + \xi_0\right)} + \frac{4\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x + \lambda t + \xi_0\right) + 2\right)^2}{c\left(x + \lambda t + \xi_0\right)^2}, \end{aligned} \tag{41}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(-30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{42}$$

Case 6. When $\beta^2 < 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{4\alpha\gamma - \beta^2}\tan((1/2)\sqrt{4\alpha\gamma - \beta^2}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$V_{29}(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{4\alpha\gamma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma - \beta^2}\left(x + \lambda t + \xi_0\right)\right)\right.$$

$$- \beta\Big) + \frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma-\beta^2}\,(x+\lambda t+\xi_0)\right) - \beta\right)^2,$$

$$V_{30}(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma-\beta^2}\,(x-\lambda t+\xi_0)\right) - \beta\right) + \frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma-\beta^2}\,(x-\lambda t+\xi_0)\right) - \beta\right)^2, \quad (43)$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 360\alpha\gamma\beta^2ch^2 + 45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \quad (44)$$

Case 7. When $\beta^2 > 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{\beta^2-4\alpha\gamma}\tanh((1/2)\sqrt{\beta^2-4\alpha\gamma}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$V_{31}(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x+\lambda t+\xi_0)\right) - \beta\right) + \frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x+\lambda t+\xi_0)\right) - \beta\right)^2,$$

$$V_{32}(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x-\lambda t+\xi_0)\right) - \beta\right) + \frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x-\lambda t+\xi_0)\right) - \beta\right)^2, \quad (45)$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 360\alpha\gamma\beta^2ch^2 + 45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \quad (46)$$

4. Summary and Discussion

In conclusion, the improved generalized tanh-coth method is proposed to obtain more general exact solutions of the sixth-order solitary wave equations. It can be observed that the method used is a powerful and more general tool for finding the exact solutions. By using the method, we have obtained new exact solutions in terms of the hyperbolic functions, the trigonometric functions, the exponential functions, and the rational functions. The results reveal that the improved generalized tanh-coth method has significant effects on the wave behavior and can also be applied to other NLEEs in mathematical physics.

The solutions of generalized Riccati equation (7): there are more than seven cases. We can extend the improved generalized tanh-coth method by considering expanding solutions of generalized Riccati equation in this method.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. S. Russell, "Report on Waves," in *Report of the Fourteenth Meeting of the British Association for the Advancement of Science*, John Murray, London, UK, 1844.

[2] Z. Zhang and J. Wu, "Generalized (G'/G)-expansion method and exact traveling wave solutions of the perturbed nonlinear

Schrödinger's equation with Kerr law nonlinearity in optical fiber materials," *Optical and Quantum Electronics*, vol. 49, no. 2, 2017.

[3] A.-M. Wazwaz, "The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 1002–1014, 2007.

[4] O. Guner and A. Bekir, "Traveling wave solutions for time-dependent coefficient nonlinear evolution equations," *Waves in Random and Complex Media. Propagation, Scattering and Imaging*, vol. 25, no. 3, pp. 342–349, 2015.

[5] M. S. Osman, "Nonlinear interaction of solitary waves described by multi-rational wave solutions of the $(2 + 1)$-dimensional Kadomtsev-Petviashvili equation with variable coefficients," *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 87, no. 2, pp. 1209–1216, 2017.

[6] M. S. Islam, K. Khan, and M. A. Akbar, "Application of the improved F-expansion method with Riccati equation to find the exact solution of the nonlinear evolution equations," *Journal of the Egyptian Mathematical Society*, vol. 25, no. 1, pp. 13–18, 2017.

[7] A. Sonmezoglu, M. Ekici, M. Moradi, M. Mirzazadeh, and Q. Zhou, "Exact solitary wave solutions to the new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation," *Optik—International Journal for Light and Electron Optics*, vol. 128, pp. 77–82, 2017.

[8] B. H. Malwe, G. Betchewe, S. Y. Doka, and T. C. Kofane, "Travelling wave solutions and soliton solutions for the nonlinear transmission line using the generalized Riccati equation mapping method," *Nonlinear Dynamics. An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, vol. 84, no. 1, pp. 171–177, 2016.

[9] M. M. Kabir, "Modified Kudryashov method for generalized forms of the nonlinear heat conduction equation," *International Journal of the Physical Sciences*, vol. 6, no. 25, pp. 6061–6064, 2011.

[10] A. Akbulut, M. Kaplan, and F. Tascan, "The investigation of exact solutions of nonlinear partial differential equations by using $\exp(-\phi(\xi))$ method," *Optik—International Journal for Light and Electron Optics*, vol. 132, pp. 382–387, 2017.

[11] S. Sahoo and S. Saha Ray, "Lie symmetry analysis and exact solutions of $(3 + 1)$ dimensional Yu-Toda-Sasa-Fukuyama equation in mathematical physics," *Computers & Mathematics with Applications. An International Journal*, vol. 73, no. 2, pp. 253–260, 2017.

[12] M. Eslami and H. Rezazadeh, "The first integral method for Wu-Zhang system with conformable time-fractional derivative," *Calcolo. A Quarterly on Numerical Analysis and Theory of Computation*, vol. 53, no. 3, pp. 475–485, 2016.

[13] W. Cheng and B. Li, "CRE solvability, exact soliton-cnoidal wave interaction solutions, and nonlocal symmetry for the modified Boussinesq equation," *Advances in Mathematical Physics*, vol. 2016, Article ID 4874392, 7 pages, 2016.

[14] W. Malfliet, "Solitary wave solutions of nonlinear wave equations," *American Journal of Physics*, vol. 60, no. 7, pp. 650–654, 1992.

[15] E. Fan and Y. C. Hona, "Generalized tanh method extended to special types of nonlinear equations," *Zeitschrift für Naturforschung A*, vol. 57, no. 8, 2002.

[16] C. A. Gómez and A. H. Salas, "The Cole-Hopf transformation and improved tanh-coth method applied to new integrable system (KdV6)," *Applied Mathematics and Computation*, vol. 204, no. 2, pp. 957–962, 2008.

[17] A. H. Salas, C. A. Gómez, and J. Castillo, "New solutions for the modified generalized Degasperis-Procesi equation," *Applied Mathematics and Computation*, vol. 215, no. 7, pp. 2608–2615, 2009.

[18] C. A. Gómez, "Closed form solutions for a generalized Benjamin-BONa-Mahony-BURgers equation with higher-order nonlinearity," *Applied Mathematics and Computation*, vol. 234, pp. 618–622, 2014.

[19] G. Hernán Garzón and A. G. S. Cesar, "The generalized tanh-coth method applied to biological model referent to nano-solitons of ionic wave," *Applied Mathematical Sciences*, vol. 9, no. 137–140, pp. 6877–6882, 2015.

[20] M. Torvattanabun and S. Koonprasert, "Variational iteration method combined with the improved generalized tanh-coth method for the generalized $(1 + 1)$-dimensional and $(2 + 1)$-dimensional ito equations," *Far East Journal of Mathematical Sciences*, vol. 100, no. 8, pp. 1339–1355, 2016.

[21] M. A. Christou, "Sixth order solitary wave equations," *Wave Motion. An International Journal Reporting Research on Wave Phenomena*, vol. 71, pp. 18–24, 2017.

Hindawi
Journal of Mathematics
Volume 2017, Article ID 5328246, 9 pages
https://doi.org/10.1155/2017/5328246

Research Article

Interaction of Traveling Curved Fronts in Bistable Reaction-Diffusion Equations in $\mathbb{R}^2$

Nai-Wei Liu[1,2]

[1]*School of Mathematics, Shandong University, Jinan, Shandong 250100, China*
[2]*School of Mathematics and Information Science, Yantai University, Yantai, Shandong 264005, China*

Correspondence should be addressed to Nai-Wei Liu; liunaiwei@aliyun.com

Received 10 February 2017; Accepted 4 April 2017; Published 7 May 2017

Academic Editor: Emir Köksal

We consider the interaction of traveling curved fronts in bistable reaction-diffusion equations in two-dimensional spaces. We first characterize the growth of the traveling curved fronts at infinity; then by constructing appropriate subsolutions and supersolutions, we prove that the solution of the Cauchy problem converges to a pair of diverging traveling curved fronts in $\mathbb{R}^2$ under appropriate initial conditions.

1. Introduction

In the current paper, we consider the following Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u), \quad (x,y) \in \mathbb{R}^2,\ t > 0, \\ u(x,y,0) &= u_0(x,y), \quad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{1}$$

where $u_0 \in C^1(\mathbb{R}^2)$ is a bounded initial function and the function f is of bistable type. Concretely, we assume that f satisfies the following:

(F1) $f \in C^1(\mathbb{R})$ and $f(0) = f(1) = 0$, $f'(0) < 0$ and $f'(1) < 0$,

(F2) $f(s) < 0$ and $f'(s) < 0$ for $s > 1$; $f(s) > 0$ and $f'(s) < 0$ for $s < 0$,

(F3) $\int_0^1 f(s)ds > 0$.

Such an example is

$$f(u) = u(1-u)(u-a), \quad \text{for } 0 < a < \frac{1}{2}. \tag{2}$$

Under assumptions (F1) and (F2), it is clear that there exists a positive constant δ $(0 < \delta < 1/2)$ such that

$$f'(u) \le -\gamma_1, \quad u \in (-\infty, 2\delta) \cup (1-2\delta, \infty), \tag{3}$$

where $\gamma_1 := (1/2)\min\{-f'(0), -f'(1)\} > 0$.

We remark here that the steady states 0 and 1 of (1) are asymptotically stable if (F1)–(F3) hold. In $\mathbb{R}^1$, by letting $u(x-\bar{c}t) = \phi(\xi)$ and $\xi = x - \bar{c}t$ $(\bar{c} > 0)$, one has

$$\begin{gathered} -\phi''(\xi) - \bar{c}\phi'(\xi) - f(\phi) = 0, \quad \phi'(\xi) < 0,\ (\xi \in \mathbb{R}), \\ \lim_{\xi\to-\infty} \phi(\xi) = 1, \\ \lim_{\xi\to\infty} \phi(\xi) = 0. \end{gathered} \tag{4}$$

It is well known that (4) has a planar traveling wave front $\phi(\xi)$ which is unique up to phase shift under assumptions (F1)–(F3), with the unique positive traveling wave speed $\bar{c}$. The traveling wave fronts have been widely studied in many fields, such as biology, chemistry, epidemiology, and physics. One can refer to [1–8] for details. Traveling wave fronts are special solutions of (1), which can be used to characterize the invariant set with respect to transition in spaces.

Without loss of generality, assume that the solution of (1) travels towards x-direction and let

$$u(x, y, t) = V(z, y, t), \quad z = x - ct; \tag{5}$$

then, we can rewrite (1) into

$$V_t - V_{zz} - V_{yy} - cV_z - f(V) = 0, \quad (z, y) \in \mathbb{R}^2,\ t > 0, \tag{6}$$
$$V|_{t=0} = u_0, \quad (z, y) \in \mathbb{R}^2.$$

For convenience, we denote the solution of Cauchy problem (6) with initial function $V(z, y, 0; u_0) = u_0(z, y)$ by $V(z, y, t; u_0)$.

Considering the traveling wave fronts $u(x, y, t) = \Phi(z, y)$ $(z = x - ct)$ of (1) with traveling wave speed c in x-direction, then

$$-\Phi_{zz} - \Phi_{yy} - c\Phi_z - f(\Phi) = 0, \quad \text{in } \mathbb{R}^2. \tag{7}$$

It is obvious that the solution of (7) is a stationary solution of (6). Let $c \geq \bar{c}$; then $\phi((\bar{c}/c)(z \pm m_* y))$ is a solution of (7), where $\phi(\cdot)$ is a solution of (4) and

$$m_* := \frac{\sqrt{c^2 - \bar{c}^2}}{\bar{c}} > 0. \tag{8}$$

We call the solution $\Phi(z, y)$ of (7) *traveling curved front*, since it is nonplanar. By using comparison principle, one has the function

$$\begin{aligned} \varphi(z, y) &:= \max\left\{\phi\left(\frac{\bar{c}}{c}(z - m_* y)\right), \phi\left(\frac{\bar{c}}{c}(z + m_* y)\right)\right\} \\ &= \phi\left(\frac{\bar{c}}{c}(z - m_* |y|)\right) \end{aligned} \tag{9}$$

which is a subsolution of (7) with $\varphi_z(z, y) < 0$ on $\mathbb{R}^2$. By using sub- and supersolutions method, Ninomiya and Taniguchi [9, 10] proved the existence and global stability of traveling curved fronts for (1).

Theorem 1 (see [9, 10]). *Assume that (F1)–(F3) hold. For any $c > \bar{c}$, there exists a traveling curved front $u(x, y, t) = \Phi(z, y)$ $(z = x - ct)$ of (1) such that*

$$\lim_{\mathscr{R} \to +\infty} \sup_{z^2 + y^2 > \mathscr{R}^2} |\Phi(z, y) - \varphi(z, y)| = 0, \tag{10}$$

$$\begin{aligned} &\Phi(z, y) > \varphi(z, y), \\ &0 < \Phi(z, y) < 1, \\ &\Phi_z(z, y) < 0, \\ &\text{for } (z, y) \in \mathbb{R}^2. \end{aligned} \tag{11}$$

Furthermore, if $0 < \delta_1 < 1/2$, there is a constant $\gamma(\delta_1) > 0$ such that

$$\Phi_z(z, y) \leq -\gamma(\delta_1), \quad \text{for } \delta_1 \leq \Phi \leq 1 - \delta_1. \tag{12}$$

If $u_0(z, y)$ satisfies

$$\lim_{\mathscr{R} \to +\infty} \sup_{z^2 + y^2 > \mathscr{R}^2} |u_0(z, y) - \Phi(z, y)| = 0, \tag{13}$$

then

$$\lim_{t \to +\infty} \sup_{(x, y) \in \mathbb{R}^2} |u(x, y, t; u_0) - \Phi(x - ct, y)| = 0, \tag{14}$$

where $u(x, y, t; u_0)$ is the solution of the Cauchy problem (1).

It follows from Theorem 1 that (1) has a unique traveling curved front $\Phi(x - ct, y)$ for each $c > \bar{c}$, which is globally stable in the sense of (14). In fact, there are many mathematical models arising in biology, population dynamics, flame propagation, and disease spread which can be described by traveling curved front. For example, Sheng et al. [11] considered the stability of traveling curved fronts (V-shaped) for Allen-Cahn equations, and they in [11] also proved that the traveling curved fronts (V-shaped) are not asymptotically stable under some perturbations. In another paper, by using comparison principle, Sheng [12] studied the existence and stability of time-periodic traveling curved fronts about bistable reaction-diffusion equations in $\mathbb{R}^3$. In [13], Wang and Bu considered traveling curved fronts (nonplanar) for combustion and degenerate Fisher-KPP type reaction-diffusion equations. Ninomiya and Taniguchi [9, 10] and Taniguchi [14, 15] showed the existence and the stability of traveling curved fronts for Allen-Cahn equations. Furthermore, by constructing some appropriate subsolutions and supersolutions, Hamel et al. [16] considered the existence and the global stability of traveling curved fronts for a model about conical flames. They in [17] established the existence of traveling curved fronts for bistable model by introducing the conical-limiting conditions at infinity. For more interesting results about the existence and stability of traveling curved fronts, one can refer to [18–28].

In addition to the stability results about traveling fronts mentioned above, the interaction between traveling fronts is also an important topic for reaction-diffusion equations. Here, the interaction of traveling fronts means that the solutions of the Cauchy problem converge to a pair of diverging traveling fronts. Recently, there are many results about this problem. Particularly, Fife and McLeod [29, 30] studied the interaction of traveling fronts in one-dimensional space when $t \to +\infty$. Indeed, they in [29, 30] proved that the solutions of the Cauchy problem converge to a single traveling front, a pair of diverging traveling fronts, and a stacked combination of traveling fronts in $\mathbb{R}^1$, respectively. Based on comparison principle, Chen [3] developed the squeeze technique to study the interaction and the exponential stability of traveling wave solution for bistable reaction-diffusion equations. Furthermore, Roquejoffre [31] expanded the results in [29] to infinite cylinders. In another paper, Bebernes et al. [32] proved that the solution converges to a pair of diverging traveling fronts in cylindrical domains. We also remark here that there is another form of interaction between traveling fronts, which can be described by the so-called *entire solutions*. Entire solutions can be used to imply the dynamics of two traveling fronts as $t \to -\infty$; one can refer to [33–37] for related works.

However, the interaction of traveling curved fronts of reaction-diffusion equations in whole spaces $\mathbb{R}^2$ is still open. Since two traveling curved fronts traveling towards opposite directions always interact with each other, a natural issue is that whether we can expect that the solution of (1) converges to a pair of diverging traveling curved fronts in $\mathbb{R}^2$ under some appropriate initial conditions, which behaves as the interaction of traveling curved fronts. The current paper is devoted to resolving this problem for bistable reaction-diffusion equations in $\mathbb{R}^2$.

In this paper, based on comparison principle, we first construct appropriate sub- and supersolutions and then show that the solution of (1) converges to a pair of diverging traveling curved fronts, which will be done in Section 3. Before doing those, by using the asymptotic decay of planar traveling wave fronts, we give some asymptotic estimates for traveling curved fronts at infinity and list the main result in Section 2.

2. Preliminaries and Main Result

In this section, we first study the asymptotic behavior of traveling curved front $\Phi(z, y)$ of (1) as $z \to -\infty$ by using the result of the exponential convergence of one-dimension traveling wave solution $\phi(\xi)$ of (4) at infinity. In fact, it follows from [38] that there exist positive constants $\mathscr{A}$ and $\mathscr{B}$ such that

$$\begin{aligned} \phi(\xi) &= \mathscr{A}e^{-\tilde{\mu}\xi} + o\left(e^{-\tilde{\mu}\xi}\right), \quad \text{as } \xi \longrightarrow +\infty, \\ 1-\phi(\xi) &= \mathscr{B}e^{\tilde{\lambda}\xi} + o\left(e^{\tilde{\lambda}\xi}\right), \quad \text{as } \xi \longrightarrow -\infty, \end{aligned} \tag{15}$$

where $\tilde{\mu} = (\sqrt{\bar{c}^2 - 4f'(0)} + \bar{c})/2$ and $\tilde{\lambda} = (\sqrt{\bar{c}^2 - 4f'(1)} - \bar{c})/2$. From [34], we see that the planar traveling wave front ϕ of (4) satisfies

$$1-\phi(\xi) \le ke^{\tilde{\lambda}\xi}, \quad \xi \le 0 \tag{16}$$

for some $k > 0$ and $\tilde{\lambda}$ defined above.

Under conditions (F1)–(F3) and (3), there exists a constant γ_2 with $\gamma_2 > \gamma_1 > 0$, such that

$$f(u) \le \gamma_2(1-u) \tag{17}$$

for $0 < 1-u \le \delta$ with δ as in (3). Furthermore, by virtue of (12), we have

$$\Phi_z(z, y) \le -\gamma(\delta) := -\gamma_3 < 0 \tag{18}$$

for $\delta \le \Phi \le 1-\delta$. Since the traveling wave front $\phi(\xi)$ of (4) possesses invariance up to translation, we assume that traveling wave front $\phi(\xi)$ satisfies

$$\phi(0) = \theta, \quad \theta \ge 1 - \frac{\delta}{2}, \tag{19}$$

and the constant k in (16) satisfies

$$k \le \min\left\{\frac{\gamma_1\delta}{4(\gamma_2-\gamma_1)}, \frac{\delta}{4}\right\}. \tag{20}$$

We take three positive constants q_0, μ, and M satisfying

$$\begin{aligned} &\frac{\delta}{4} \le q_0 \le \frac{\delta}{2}, \\ &0 < \mu \le \min\left\{\frac{\gamma_1}{2}, \lambda c\right\}, \\ &M \ge \max\{M_1, M_2\}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} M_1 &:= \frac{k(L_2+\gamma_2) + (L_2+\mu)q_0}{\gamma_3\mu}, \\ M_2 &:= \frac{(L_3+\mu)q_0}{\gamma_3\mu}, \\ L_2 &:= \max_{-\delta\le u\le 1-\delta}\left|f'(u)\right|, \\ L_3 &:= \max_{-\delta\le u\le 1+\delta}\left|f'(u)\right|. \end{aligned} \tag{22}$$

By a translation in the ξ-direction, we next take

$$\tilde{\phi}(\xi) = \phi(\xi - M), \tag{23}$$

where M is defined in (21). Then, we have

$$1 - \tilde{\phi}(\xi + M) = 1 - \phi(\xi) \le ke^{\tilde{\lambda}\xi}, \quad \xi \le 0, \tag{24}$$

$$\tilde{\phi}(M) = \phi(0) = \theta, \quad \theta \ge 1 - \frac{\delta}{2}, \tag{25}$$

by view of (16) and (19).

In the following, we consider planar traveling wave front $\tilde{\phi}(\xi)$ satisfying (24) and (25) instead of the solution $\phi(\xi)$ of (4) and assume that Theorem 1 holds with $\tilde{\phi}(\xi)$ instead of $\phi(\xi)$ in the definition of $\varphi(z, y)$. For convenience, in the rest of the paper we drop the tilde of $\tilde{\phi}$ and denote $\tilde{\phi}(\xi)$ also by $\phi(\xi)$.

By using the asymptotic behavior of planar traveling wave fronts of (4), we immediately obtain the following lemma.

Lemma 2. *Assume that f satisfies (F1)–(F3). Then there exist some positive constants λ, k, and C, such that the traveling curved front $\Phi(z, y)$ defined in Theorem 1 satisfies*

$$\begin{aligned} 0 \le 1 - \Phi(z+M, y) \le ke^{\lambda(z - m_*|y|)}, \\ z \le 0, \ y \in \mathbb{R}, \end{aligned} \tag{26}$$

$$\left|\Phi_z(z, y)\right| \le C, \quad (z, y) \in \mathbb{R}^2, \tag{27}$$

where M is defined in (21) and

$$\lambda := \frac{\bar{c}}{c}\tilde{\lambda} = \frac{\bar{c}\sqrt{\bar{c}^2 - 4f'(1)} - \bar{c}^2}{2c} > 0. \tag{28}$$

Furthermore, there is

$$\lim_{\mathscr{R}\to+\infty}\ \sup_{|z-m_*|y||>\mathscr{R}} \Phi_z(z, y) = 0. \tag{29}$$

Proof. It follows from (9), (11), and (23) that

$$\begin{aligned} 0 &\leq 1-\Phi(z+M,y) \leq 1-\varphi(z+M,y) \\ &= 1-\phi\left(\frac{\overline{c}}{c}(z+M-m_*|y|)\right) \\ &\leq 1-\phi\left(\frac{\overline{c}}{c}(z-m_*|y|)+M\right). \end{aligned} \tag{30}$$

Thus (26) holds for $z \leq 0$ by (24).

Inequality (27) follows from the standard elliptic estimates. Next, we prove that (29) holds. In fact, if (29) is not true, there exist $\varepsilon_1 > 0$ and $\{(z_n, y_n)\}_{n=1}^{\infty}$ satisfying

$$\begin{gathered} \lim_{n\to\infty} |z_n + m_* |y_n|| = \infty, \\ \Phi_z(z_n, y_n) \geq \varepsilon_1. \end{gathered} \tag{31}$$

Define

$$\Phi_n(z,y) = \Phi(z+z_n, y+y_n), \quad \text{in } B_0, \tag{32}$$

where

$$B_0 := \left\{(z,y) \in \mathbb{R}^2 \mid z^2 + y^2 < C\right\}, \tag{33}$$

with $C > 0$ a given constant. By extracting subsequence of Φ_n and denoting the subsequence also by Φ_n, we have

$$\Phi_n(z,y) \longrightarrow \Phi^*(z,y), \quad \text{in } C^2_{\mathrm{loc}}\left(\mathbb{R}^2\right), \tag{34}$$

where $\Phi^*(z,y)$ is a solution of (7). On the other hand, by view of (10) and (11), we have

$$\begin{gathered} \lim_{n\to\infty} \sup_{\overline{B_0}} |\Phi_n(z,y) - 1| = 0, \\ \lim_{n\to\infty} \sup_{\overline{B_0}} |f(\Phi_n(z,y))| = 0. \end{gathered} \tag{35}$$

Thus the strong maximum principle implies

$$\frac{\partial}{\partial z}\Phi^*(z,y) = 0, \quad \text{in } B_0. \tag{36}$$

Then,

$$\frac{\partial}{\partial z}\Phi_n(z,y) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{37}$$

which contradicts the assumption $\Phi_z(z_n, y_n) \geq \varepsilon_1 > 0$. Thus, we complete the proof. □

Our main result is the following.

Theorem 3. *For every $c > \overline{c}$, let $\Phi(\cdot, y)$ be the traveling curved front of (1) defined in Theorem 1 with speed c. Assume that (F1)–(F3) hold. Then if $u_0(x,y) \in (0,1)$ satisfies*

$$\begin{gathered} u_0(x,y) \geq \Phi(|x|, y), \\ \lim_{\mathscr{R}\to+\infty} \sup_{x^2+y^2>\mathscr{R}^2} |u_0(x,y) - \Phi(|x|, y)| = 0, \end{gathered} \tag{38}$$

there exist positive constants $q, \mu > 0$ and ζ, such that, for all $(x,y,t) \in \mathbb{R}^2\times[0,+\infty)$, the solution $u(x,y,t;u_0)$ of (1) satisfies

$$\begin{aligned} &\Phi(x-ct+\zeta,y) + \Phi(-x-ct+\zeta,y) - 1 - qe^{-\mu t} \\ &\quad \leq u(x,y,t;u_0) \\ &\quad \leq \Phi(x-ct-\zeta,y) + \Phi(-x-ct-\zeta,y) - 1 \\ &\qquad + qe^{-\mu t}. \end{aligned} \tag{39}$$

Furthermore, one has

$$\lim_{t\to+\infty} |u(x,y,t;u_0) - 1| = 0 \tag{40}$$

locally uniformly with respect to $(x,y) \in \mathbb{R}^2$.

Remark 4. Inequality (39) implies that the x-profile of $u(x,y,t;u_0)$ approaches that of the traveling curved fronts. In particular, it shows that the domain in which u is close to 1 is expanding at the speed of c. The phase shift ζ is a positive constant which will be defined in the proof of Theorem 3. The similar stability about traveling curved front in cylinder domain is treated in [32].

In the last of this section, we give the definitions of subsolution and supersolutions for (1) in $\mathbb{R}^2 \times (0,+\infty)$.

Definition 5. If a function $\underline{u}(x,y,t) \in C^{2,1}(\mathbb{R}^2 \times (0,+\infty), \mathbb{R})$ and satisfies

$$\begin{aligned} \frac{\partial \underline{u}(x,y,t)}{\partial t} &\leq \Delta \underline{u}(x,y,t) + f(\underline{u}(x,y,t)), \\ &\qquad (x,y) \in \mathbb{R}^2,\ t > 0, \end{aligned} \tag{41}$$

then $\underline{u}(x,y,t)$ is called a subsolution for (1) in $\mathbb{R}^2 \times (0,+\infty)$. Similarly, by reversing the inequality in (41), we can define a supersolution $\overline{u}(x,y,t)$ for (1).

3. Proof of Theorem 3

In this section, we prove the main result by constructing appropriate sub- and supersolutions. In the following lemma, we construct a subsolution for (1).

Lemma 6. *Assume that (F1)–(F3) hold. Let $c > \overline{c}$. Then the function*

$$\begin{aligned} \underline{\phi}(x,y,t) := {} & \Phi\left(x-ct+M\left(1-e^{-\mu t}\right), y\right) \\ & + \Phi\left(-x-ct+M\left(1-e^{-\mu t}\right), y\right) - 1 \\ & - q_0 e^{-\mu t} \end{aligned} \tag{42}$$

is a subsolution of (1) on $t \in (0,\infty)$, where $\Phi(\cdot, y)$ is traveling curved front of (1) defined in Theorem 1 and $q_0, \mu > 0$ are constants defined in (21).

Proof. Define

$$\mathcal{F}\left(\underline{\phi}(x,y,t)\right) := \frac{\underline{\phi}(x,y,t)}{\partial t} - \Delta\underline{\phi}(x,y,t) - f\left(\underline{\phi}(x,y,t)\right). \tag{43}$$

By using the above prepared results, direct calculations give

$$\begin{aligned}
\mathcal{F}\left(\underline{\phi}\right) &= \left(M\mu e^{-\mu t} - c\right)\Phi_z\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad + \left(M\mu e^{-\mu t} - c\right)\Phi_z\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad + q_0\mu e^{-\mu t} - \Phi_{zz}\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad - \Phi_{zz}\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) - \Phi_{yy}\left(x - ct\right. \\
&\quad \left. + M\left(1 - e^{-\mu t}\right), y\right) - \Phi_{yy}\left(-x - ct\right. \\
&\quad \left. + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad - f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right. \\
&\quad \left. + \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) - 1 - q_0 e^{-\mu t}\right) \\
&= M\mu e^{-\mu t}\Phi_z\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad + M\mu e^{-\mu t}\Phi_z\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad + q_0\mu e^{-\mu t} + f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\quad + f\left(\Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\quad - f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right. \\
&\quad \left. + \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) - 1 - q_0 e^{-\mu t}\right).
\end{aligned} \tag{44}$$

If $x \geq 0$, we consider two cases $\Phi(x-ct+M(1-e^{-\mu t}), y) \in [0,\delta] \cup [1-\delta, 1]$ and $\Phi(x - ct + M(1 - e^{-\mu t}), y) \in [\delta, 1-\delta]$, respectively.

Case A ($\Phi(x - ct + M(1 - e^{-\mu t}), y) \in [1-\delta, 1]$). By virtue of (3), (21), and (25), we have

$$\begin{aligned}
& f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\quad - f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right. \\
&\quad \left. + \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) - 1 - q_0 e^{-\mu t}\right) \\
&\quad \leq -\gamma_1\left(1 - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right. \\
&\quad \left. + q_0 e^{-\mu t}\right),
\end{aligned} \tag{45}$$

since $0 < 1 - \Phi(-x - ct + M(1 - e^{-\mu t}), y) + q_0 e^{-\mu t} \leq 1 - \phi(M) + q_0 \leq \delta$. By using (17) and (25) and the fact that $0 < 1 - \Phi(-x - ct + M(1 - e^{-\mu t}), y) \leq 1 - \phi(M) \leq \delta/2$ for $x \geq 0$, we have

$$\begin{aligned}
& f\left(\Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\quad \leq \gamma_2\left(1 - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right).
\end{aligned} \tag{46}$$

Thus, we have

$$\begin{aligned}
& \mathcal{F}\left(\underline{\phi}\right) \\
&\quad \leq -\gamma_1\left(1 - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) + q_0 e^{-\mu t}\right) \\
&\qquad + \gamma_2\left(1 - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\qquad + q_0\mu e^{-\mu t} \\
&\quad \leq \left(\gamma_2 - \gamma_1\right)\left(1 - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\qquad - \left(\gamma_1 - \mu\right) q_0 e^{-\mu t} \\
&\quad \leq k\left(\gamma_2 - \gamma_1\right) e^{\lambda(-x-ct-m_*|y|)} - \left(\gamma_1 - \mu\right) q_0 e^{-\mu t} \\
&\quad \leq k\left(\gamma_2 - \gamma_1\right) e^{-\lambda ct} - \left(\gamma_1 - \mu\right) q_0 e^{-\mu t} \leq 0.
\end{aligned} \tag{47}$$

In the last inequality, we have used the facts (20), (21), and (26).

By a similar argument, we have $F(\underline{\phi}) \leq 0$ for $\Phi(x - ct + M(1 - e^{-\mu t}), y) \in [0, \delta]$ with $x \geq 0$.

Case B ($\Phi(x - ct + M(1 - e^{-\mu t}), y) \in [\delta, 1-\delta]$). In a similar way as above, for $x \geq 0$, we have

$$\begin{aligned}
& f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) \\
&\quad - f\left(\Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right. \\
&\quad \left. + \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) - 1 - q_0 e^{-\mu t}\right) \\
&\quad \leq L_2\left(1 - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right. \\
&\quad \left. + q_0 e^{-\mu t}\right),
\end{aligned} \tag{48}$$

where $L_2 := \max_{-\delta \leq u \leq 1-\delta}|f'(u)|$. Particularly, we have (46) in this case. Lastly, due to (18), we have

$$\begin{aligned}
& \Phi_z'\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\
&\quad + \Phi_z'\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) \leq -\gamma_3.
\end{aligned} \tag{49}$$

Combining (46), (48), and (49), we have

$$\begin{aligned}
\mathcal{F}\left(\underline{\phi}\right) &\leq -\gamma_3 M\mu e^{-\mu t} + L_2\left(1\right. \\
&\quad \left. - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) + q_0 e^{-\mu t}\right) + \gamma_2\left(1\right. \\
&\quad \left. - \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right)\right) + q_0\mu e^{-\mu t}
\end{aligned}$$

$$\leq -\gamma_3 M\mu e^{-\mu t} + (L_2+\gamma_2)\Big(1 - \Phi\left(-x-ct+M\left(1-e^{-\mu t}\right), y\right)\Big) + (L_2+\mu) \cdot q_0 e^{-\mu t} \leq -\gamma_3 M\mu e^{-\mu t} + k(L_2+\gamma_2)e^{-\lambda ct} + (L_2 + \mu) q_0 e^{-\mu t}. \tag{50}$$

Consequently, (21) implies $\mathscr{F}(\underline{\phi}) \leq 0$ for this case.

Similarly, we can prove $\mathscr{F}(\underline{\phi}) \leq 0$ when $x < 0$. Thus, we have showed that $\underline{\phi}(x,y,t)$ is a subsolution of (1) on $t \in [0,\infty)$. □

In order to construct a supersolution for (1), we introduce the following lemmas.

Lemma 7. *Assume that (F1)–(F3) hold. Let $c > \bar{c}$; then for any given constant $N > 0$, the function*

$$\phi_N^1(x,y,t) := \Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N, y\right) + q_0 e^{-\mu t} \tag{51}$$

is a supersolution of (1) on $t \in (0,+\infty)$, where $\Phi(\cdot,y)$ is traveling curved front of (1) as in Theorem 1 and $q_0, \mu > 0$ are constants defined in (21).

Proof. As the proof of Lemma 6, we need only to prove that the right hand of (43) is nonnegative for the function $\phi_N^1(x,y,t)$ for $(x,y,t) \in \mathbb{R}^2 \times (0,+\infty)$. By a similar argument, direct calculations give

$$\begin{aligned} \mathscr{F}\left(\phi_N^1\right) &= \left(-M\mu e^{-\mu t} - c\right)\Phi_z\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right) \\ &\quad - q_0\mu e^{-\mu t} - \Phi_{zz}\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right) \\ &\quad - \Phi_{yy}\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right) \\ &\quad - f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)+q_0e^{-\mu t}\right) \\ &= -M\mu e^{-\mu t}\Phi_z\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right) \\ &\quad - q_0\mu e^{-\mu t} \\ &\quad + f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)\right) \\ &\quad - f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)+q_0e^{-\mu t}\right). \end{aligned} \tag{52}$$

To complete the proof, we consider two cases $\Phi(x-ct-M(1-e^{-\mu t})-N,y) \in [0,\delta]\cup[1-\delta,1]$ and $\Phi(x-ct-M(1-e^{-\mu t})-N,y) \in [\delta, 1-\delta]$, respectively.

For $\Phi(x-ct-M(1-e^{-\mu t})-N,y) \in [0,\delta]\cup[1-\delta,1]$, we just consider the case $\Phi(x-ct-M(1-e^{-\mu t})-N,y) \in [1-\delta,1]$. Since $\Phi(x-ct-M(1-e^{-\mu t})-N,y)+q_0e^{-\mu t} \in [1-\delta,1+\delta]$, it follows from (3) that

$$\begin{aligned} & f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)+q_0e^{-\mu t}\right) \\ &\quad - f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)\right) \\ &\leq -\gamma_1 q_0 e^{-\mu t}. \end{aligned} \tag{53}$$

By virtue of (11) and (21), we have

$$\begin{aligned} &\mathscr{F}\left(\phi_N^1\right) \\ &\geq -M\mu e^{-\mu t}\Phi_z\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right) \\ &\quad - q_0\mu e^{-\mu t} + \gamma_1 q_0 e^{-\mu t} \geq q_0(\gamma_1-\mu)e^{-\mu t} \geq 0. \end{aligned} \tag{54}$$

For $\Phi(x-ct-M(1-e^{-\mu t})-N,y) \in [\delta,1-\delta]$, we have

$$\begin{aligned} & f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)\right) \\ &\quad - f\left(\Phi\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right)+q_0e^{-\mu t}\right) \\ &\geq -L_3 q_0 e^{-\mu t}, \end{aligned} \tag{55}$$

where $L_3 := \max_{-\delta\leq u\leq 1+\delta}|f'(u)|$. Particularly, (18) implies

$$\Phi_z'\left(x-ct-M\left(1-e^{-\mu t}\right)-N,y\right) \leq -\gamma_3. \tag{56}$$

Therefore, by (21) we get

$$\begin{aligned} \mathscr{F}\left(\phi_N^1\right) &\geq \gamma_3 M\mu e^{-\mu t} - q_0\mu e^{-\mu t} - L_3 q_0 e^{-\mu t} \\ &\geq \gamma_3 M\mu e^{-\mu t} - (L_3+\mu) q_0 e^{-\mu t} \geq 0. \end{aligned} \tag{57}$$

Thus, $\phi_N^1(x,y,t)$ defined by (51) is a supersolution of (1). □

In a similar way, we prove the following lemma.

Lemma 8. *Assume that (F1)–(F3) hold. Let $c > \bar{c}$; then for any given constant $N > 0$, the function*

$$\phi_N^2(x,y,t) := \Phi\left(-x-ct-M\left(1-e^{-\mu t}\right)-N, y\right) + q_0 e^{-\mu t} \tag{58}$$

is a supersolution of (1) on $t \in (0,+\infty)$, where $\Phi(\cdot,y)$ is traveling curved front of (1) as in Theorem 1 and $q_0, \mu > 0$ are constants defined in (21).

Remark 9. Let $c > \bar{c}$ and $N > 0$; it follows from Lemmas 7 and 8 that the function

$$\overline{\phi}_N(x,y,t) := \Phi\left(|x|-ct-M\left(1-e^{-\mu t}\right)-N, y\right) + q_0 e^{-\mu t} \tag{59}$$

is a supersolution of (1) on $t \in (0,+\infty)$.

To complete the proof of Theorem 3, we establish the following comparison result.

Lemma 10. *Let $\underline{\phi}(x, y, t)$ and $\overline{\phi}_N(x, y, t)$ be defined by (42) and (59), respectively, and $u_0(x, y) \in (0, 1)$ satisfies (38). Then, there exists $N > 0$ such that*

$$\underline{\phi}(x, y, t) \leq u(x, y, t; u_0) \leq \overline{\phi}_N(x, y, t) \tag{60}$$

for all $(x, y, t) \in \mathbb{R}^2 \times [0, +\infty)$.

Proof. By (38) and the definition of (42), when $t = 0$, direct calculations give

$$\begin{aligned}\underline{\phi}(x, y, 0) &= \Phi(x, y) + \Phi(-x, y) - 1 - q_0 \leq \Phi(x, y) \\ &\leq u_0(x, y)\end{aligned} \tag{61}$$

for $x \geq 0$. Similarly, we have $\underline{\phi}(x, y, 0) \leq u_0(x, y)$ for $x < 0$. Therefore, the maximum principle for parabolic equations shows $\underline{\phi}(x, y, t) \leq u(x, y, t; u_0)$ for $(x, y, t) \in \mathbb{R}^2 \times [0, +\infty)$.

Similarly, by the definition of (59), when $t = 0$

$$\overline{\phi}_N(x, y, 0) = \Phi(|x| - N, y) + q_0. \tag{62}$$

By (38), we have that, for $\varepsilon = q_0$ and $N_1 \geq 0$, there exists $\Lambda > 0$ such that

$$\begin{aligned}u_0(x, y) &\leq \Phi(|x|, y) + q_0 \leq \Phi(|x| - N_1, y) + q_0 \\ &\leq \overline{\phi}_{N_1}(x, y, 0)\end{aligned} \tag{63}$$

in $\{(x, y) \mid x^2 + y^2 > \Lambda^2\}$, since $\Phi_z(z, y) < 0$.

In the range of $\{(x, y) \mid x^2 + y^2 \leq \Lambda^2\}$, we have $|x| \leq \Lambda$. Thus, by choosing $N_2 \geq \Lambda - M$ and using (20), (21), and (26), we have

$$\begin{aligned}\overline{\phi}_{N_2}(x, y, 0) &\geq 1 - ke^{\lambda(|x| - M - N_2 - m_*|y|)} + q_0 \\ &\geq 1 - ke^{\lambda(\Lambda - M - (\Lambda - M) - m_*|y|)} + q_0 \\ &\geq 1 - k + q_0 \geq 1\end{aligned} \tag{64}$$

for $(x, y) \in \{(x, y) \mid x^2 + y^2 \leq \Lambda^2\}$. Consequently, we have $\overline{\phi}_{N_2}(x, y, 0) \geq u_0(x, y) \in (0, 1)$ in this range.

Combining (63) and (64) and taking $N \geq \max\{N_1, N_2\}$, we conclude that $u_0(x, y) \leq \overline{\phi}_N(x, y, 0)$ for $(x, y) \in \mathbb{R}^2$. Then the maximum principle for parabolic equations derives that (60) holds. □

Proof of Theorem 3. It follows from Lemma 10 that (60) holds with $N \geq \max\{N_1, N_2\}$. Thus, by (11), we obtain

$$\begin{aligned}\underline{\phi}(x, y, t) &= \Phi\left(x - ct + M\left(1 - e^{-\mu t}\right), y\right) \\ &\quad + \Phi\left(-x - ct + M\left(1 - e^{-\mu t}\right), y\right) - 1 \\ &\quad - q_0 e^{-\mu t} \\ &\geq \Phi(x - ct + \zeta_1, y) + \Phi(-x - ct + \zeta_1, y) \\ &\quad - 1 - q_0 e^{-\mu t}\end{aligned} \tag{65}$$

for all $(x, y, t) \in \mathbb{R}^2 \times [0, +\infty)$, if $\zeta_1 \geq M$. On the other hand, for $x \geq 0$, we have

$$\begin{aligned}\overline{\phi}_N(x, y, t) &= \Phi\left(x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad + q_0 e^{-\mu t} \\ &\leq \Phi\left(x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad + \Phi\left(-x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad - 1 + q_0 e^{-\mu t} + 1 \\ &\quad - \Phi\left(-x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\leq \Phi\left(x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad + \Phi\left(-x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad - 1 + q_0 e^{-\mu t} + ke^{\lambda(-x - ct - M - N - m_*|y|)} \\ &\leq \Phi\left(x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad + \Phi\left(-x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad - 1 + q_0 e^{-\mu t} + ke^{-\lambda ct} \\ &\leq \Phi\left(x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad + \Phi\left(-x - ct - M\left(1 - e^{-\mu t}\right) - N, y\right) \\ &\quad - 1 + (q_0 + k) e^{-\mu t} \\ &\leq \Phi(x - ct - \zeta_2, y) + \Phi(-x - ct - \zeta_2, y) \\ &\quad - 1 + qe^{-\mu t}\end{aligned} \tag{66}$$

if $\zeta_2 \geq M + N$ and $q \geq (q_0 + k)$. By a similar argument, for $x < 0$, (66) also holds.

Thus, if $u_0(x, y) \in (0, 1)$ satisfies (38), by taking $\zeta = \max\{\zeta_1, \zeta_1\}$ and $q \geq (q_0 + k)$, we obtain that (39) holds, where q_0 and $\mu > 0$ are defined in (21).

The asymptotic behavior (40) immediately follows from (39). This completes the proof of Theorem 3. □

4. Discussion

In the current paper, we have proved that the solutions of the bistable reaction-diffusion equations converge to a pair of diverging traveling curved fronts in $\mathbb{R}^2$. It means that the solution $u(x, y, t, u_0)$ of (1) with initial function $u_0(x, y)$ satisfied (38) behaving as two traveling curved fronts traveling towards opposite directions and approaching each other. Our result is different from the stability results in [9–11, 22, 26, 27]. Indeed, the interaction between traveling wave fronts plays an important role in the study of reaction-diffusion equations in $\mathbb{R}^2$, which is crucially related to the pattern formation problem, and there are important applications in chemical, physical, biological systems; see, for example, [39–41].

At last, we note here that the global exponential stability of traveling curved fronts in the sense of Theorem 3 is a difficult problem, since the level set of the traveling curved fronts $\Phi(z, y)$ of (1) have two asymptotic directions as $|z| \to +\infty$, and both directions make an angle with the negative y-axis, which is different from the case of planar traveling fronts (see [20]). We will leave it for a further study. Moreover, how the solution of (1) approaches a “stacked” combination of traveling curved fronts just as the study in Fife and McLeod [29] is also an interesting problem but remains open.

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

The author is very grateful to Dr. Wei-Jie Sheng for helpful discussions. The author’s work was partially supported by NSF of China (11371179 and 11401513) and by China Postdoctoral Science Foundation Funded Project (2014M560546).

References

[1] D. G. Aronson and H. F. Weinberger, “Multidimensional nonlinear diffusion arising in population genetics,” *Advances in Mathematics*, vol. 30, no. 1, pp. 33–76, 1978.

[2] N. F. Britton, *Reaction-Diffusion Equations and Their Applications to Biology*, Academic Press, San Diego, SC, USA, 1986.

[3] X. Chen, “Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations,” *Advances in Differential Equations*, vol. 2, no. 1, pp. 125–160, 1997.

[4] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, vol. 28 of *Lecture Notes in Biomathematics*, Springer, Berlin, Germany, 1979.

[5] J. D. Murray, *Mathematical Biology*, Springer, Berlin, Germany, 1989.

[6] P. Pelcé, *Dynamics of Curved Fronts*, Academic Press Inc., San Diego, SC, USA, 1988.

[7] A. I. Volpert, V. A. Volpert, and V. A. Volpert, *Traveling Wave Solutions of Parabolic Systems*, vol. 140 of *Translations of Mathematical Monographs*, American Mathematical Society, Rhode Island, RI, USA, 1994.

[8] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Springer, New York, NY, USA, 1996.

[9] H. Ninomiya and M. Taniguchi, “Existence and global stability of traveling curved fronts in the Allen-Cahn equations,” *Journal of Differential Equations*, vol. 213, no. 1, pp. 204–233, 2005.

[10] H. Ninomiya and M. Taniguchi, “Global stability of traveling curved fronts in the Allen-Cahn equations,” *Discrete and Continuous Dynamical Systems. Series A*, vol. 15, no. 3, pp. 819–832, 2006.

[11] W. Sheng, W. Li, and Z. Wang, “Multidimensional stability of V-shaped traveling fronts in the Allen-Cahn equation,” *Science China. Mathematics*, vol. 56, no. 10, pp. 1969–1982, 2013.

[12] W.-J. Sheng, “Time periodic traveling curved fronts of bistable reaction–diffusion equations in $\mathbb{R}^3$,” *Annali di Matematica Pura ed Applicata. Series IV*, vol. 196, no. 2, pp. 617–639, 2017.

[13] Z.-C. Wang and Z.-H. Bu, “Nonplanar traveling fronts in reaction-diffusion equations with combustion and degenerate Fisher-KPP nonlinearities,” *Journal of Differential Equations*, vol. 260, no. 7, pp. 6405–6450, 2016.

[14] M. Taniguchi, “Traveling fronts of pyramidal shapes in the Allen-Cahn equations,” *SIAM Journal on Mathematical Analysis*, vol. 39, no. 1, pp. 319–344, 2007.

[15] M. Taniguchi, “The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen-Cahn equations,” *Journal of Differential Equations*, vol. 246, no. 5, pp. 2103–2130, 2009.

[16] F. Hamel, R. Monneau, and J.-M. Roquejoffre, “Stability of travelling waves in a model for conical flames in two space dimensions,” *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série*, vol. 37, no. 3, pp. 469–506, 2004.

[17] F. Hamel, R. Monneau, and J.-M. Roquejoffre, “Existence and qualitative properties of multidimensional conical bistable fronts,” *Discrete and Continuous Dynamical Systems. Series A*, vol. 13, no. 4, pp. 1069–1096, 2005.

[18] X. Chen, J.-S. Guo, F. Hamel, H. Ninomiya, and J.-M. Roquejoffre, “Traveling waves with paraboloid like interfaces for balanced bistable dynamics,” *Annales de l’Institut Henri Poincaré. Analyse Non Linéaire*, vol. 24, no. 3, pp. 369–393, 2007.

[19] C. Gui, “Symmetry of traveling wave solutions to the Allen-Cahn equation in $\mathbb{R}^2$,” *Archive for Rational Mechanics and Analysis*, vol. 203, no. 3, pp. 1037–1065, 2012.

[20] F. Hamel, R. Monneau, and J.-M. Roquejoffre, “Asymptotic properties and classification of bistable fronts with Lipschitz level sets,” *Discrete and Continuous Dynamical Systems*, vol. 14, no. 1, pp. 75–92, 2006.

[21] Y. Kurokawa and M. Taniguchi, “Multi-dimensional pyramidal travelling fronts in the Allen-Cahn equations,” *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, vol. 141, no. 5, pp. 1031–1054, 2011.

[22] W.-J. Sheng, W.-T. Li, and Z.-C. Wang, “Periodic pyramidal traveling fronts of bistable reaction-diffusion equations with time-periodic nonlinearity,” *Journal of Differential Equations*, vol. 252, no. 3, pp. 2388–2424, 2012.

[23] W.-J. Sheng, “Time periodic traveling curved fronts of bistable reaction-diffusion equations in $\mathbb{R}^N$,” *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 54, pp. 22–30, 2016.

[24] M. Taniguchi, “Multi-dimensional traveling fronts in bistable reaction-diffusion equations,” *Discrete and Continuous Dynamical Systems. Series A*, vol. 32, no. 3, pp. 1011–1046, 2012.

[25] M. Taniguchi, “An $(N-1)$-dimensional convex compact set gives an N-dimensional traveling front in the Allen-Cahn equation,” *SIAM Journal on Mathematical Analysis*, vol. 47, no. 1, pp. 455–476, 2015.

[26] Z.-C. Wang and J. Wu, “Periodic traveling curved fronts in reaction-diffusion equation with bistable time-periodic nonlinearity,” *Journal of Differential Equations*, vol. 250, no. 7, pp. 3196–3229, 2011.

[27] Z.-C. Wang, “Traveling curved fronts in monotone bistable systems,” *Discrete and Continuous Dynamical Systems. Series A*, vol. 32, no. 6, pp. 2339–2374, 2012.

[28] Z.-C. Wang, “Cylindrically symmetric travelling fronts in a periodic reaction–diffusion equation with bistable nonlinearity,” *Proceedings of the Royal Society of Edinburgh. Section A. Mathematics*, vol. 145, no. 5, pp. 1053–1090, 2015.

[29] P. C. Fife and J. B. McLeod, “The approach of solutions of nonlinear diffusion equations to travelling front solutions,” *Archive for Rational Mechanics and Analysis*, vol. 65, no. 4, pp. 335–361, 1977.

[30] P. C. Fife and J. B. McLeod, "A phase plane discussion of convergence to travelling fronts for nonlinear diffusion," *Archive for Rational Mechanics and Analysis*, vol. 75, no. 4, pp. 281–314, 1981.

[31] J.-M. Roquejoffre, "Eventual monotonicity and convergence to travelling fronts for the solutions of parabolic equations in cylinders," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 14, no. 4, pp. 499–552, 1997.

[32] J. W. Bebernes, C. Li, and Y. Li, "Travelling fronts in cylinders and their stability," *The Rocky Mountain Journal of Mathematics*, vol. 27, no. 1, pp. 123–150, 1997.

[33] X. Chen and J.-S. Guo, "Existence and uniqueness of entire solutions for a reaction-diffusion equation," *Journal of Differential Equations*, vol. 212, no. 1, pp. 62–84, 2005.

[34] J.-S. Guo and Y. Morita, "Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations," *Discrete and Continuous Dynamical Systems*, vol. 12, no. 2, pp. 193–212, 2005.

[35] F. Hamel and N. Nadirashvili, "Entire solutions of the KPP equation," *Communications on Pure and Applied Mathematics*, vol. 52, no. 10, pp. 1255–1276, 1999.

[36] W.-T. Li, N.-W. Liu, and Z.-C. Wang, "Entire solutions in reaction-advection-diffusion equations in cylinders," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 90, no. 5, pp. 492–504, 2008.

[37] N.-W. Liu, W.-T. Li, and Z.-C. Wang, "Entire solutions of reaction-advection-diffusion equations with bistable nonlinearity in cylinders," *Journal of Differential Equations*, vol. 246, no. 11, pp. 4249–4267, 2009.

[38] H. Berestycki and L. Nirenberg, "Travelling fronts in cylinders," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 9, no. 5, pp. 497–572, 1992.

[39] S.-I. Ei, "The motion of weakly interacting pulses in reaction-diffusion systems," *Journal of Dynamics and Differential Equations*, vol. 14, no. 1, pp. 85–137, 2002.

[40] S.-I. Ei, M. Mimura, and M. Nagayama, "Pulse-pulse interaction in reaction-diffusion systems," *Physica D. Nonlinear Phenomena*, vol. 165, no. 3-4, pp. 176–198, 2002.

[41] T. Kawahara and M. Tanaka, "Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation," *Physics Letters. A*, vol. 97, no. 8, pp. 311–314, 1983.

Hindawi
Journal of Mathematics
Volume 2017, Article ID 6901467, 14 pages
https://doi.org/10.1155/2017/6901467

Research Article

A Mixed Discontinuous Galerkin Approximation of Time Dependent Convection Diffusion Optimal Control Problem

Qingjin Xu and Zhaojie Zhou

College of Mathematical Sciences, Shandong Normal University, Jinan, China

Correspondence should be addressed to Zhaojie Zhou; zhouzhaojie@sdnu.edu.cn

Received 4 November 2016; Accepted 9 January 2017; Published 21 February 2017

Academic Editor: Ram Jiwari

In this paper, we investigate a mixed discontinuous Galerkin approximation of time dependent convection diffusion optimal control problem with control constraints based on the combination of a mixed finite element method for the elliptic part and a discontinuous Galerkin method for the hyperbolic part of the state equation. The control variable is approximated by variational discretization approach. A priori error estimates of the state, adjoint state, and control are derived for both semidiscrete scheme and fully discrete scheme. Numerical example is given to show the effectiveness of the numerical scheme.

1. Introduction

The objective of this paper mainly focuses on developing a mixed discontinuous Galerkin scheme for the following control constrained optimal control problem governed by a transient convection diffusion equation:

$$\begin{aligned}
\min_{u\in K} \quad & J(y,u) \\
& := \frac{1}{2}\int_0^T \int_\Omega \left(y(\mathbf{x},t) - y_d(\mathbf{x},t)\right)^2 dx\,dt \\
& \quad + \frac{\gamma}{2}\int_0^T \int_\Omega u^2(\mathbf{x},t)\,dx\,dt \\
\text{subject to} \quad & \frac{\partial y}{\partial t} - \varepsilon\Delta y - \nabla\cdot(\boldsymbol{\beta} y) + \alpha y = f + u, \\
& \qquad\qquad \text{in } \Omega_T := \Omega\times(0,T), \\
& y = 0, \quad \text{on } \Gamma_T := \partial\Omega\times(0,T), \\
& y(\mathbf{x},0) = y_0(\mathbf{x}), \quad \text{in } \Omega.
\end{aligned} \tag{1}$$

More details will be specified later.

This kind of problem plays an important role in many fields, such as air pollution ([1]) and waste water treatment ([2]). In recent years the research of numerical method for this kind of problem forms a hot topic. Lots of literatures are devoted to developing effective numerical methods for this kind of problem. In [3–6] the stabilization method such as local projection stabilization, SUPG, and continuous interior penalty method are investigated. The discontinuous Galerkin approximation including primary discontinuous Galerkin method and local discontinuous Galerkin method is addressed in [7–12]. In [13] the authors discuss the characteristic finite element approximation of transient convection diffusion optimal control problem. For more references, one can refer to [14].

In this paper, we investigate a mixed discontinuous Galerkin approximation of transient convection diffusion optimal control problem with control constraints. This scheme is based on the combination of a discontinuous Galerkin method for the hyperbolic part and a mixed finite element method for the elliptic part of the state equation. Variational discretization approach is utilized to approximate the control variable. The work of this paper is motivated by [8, 15] where a similar scheme was proposed for convection diffusion equations and stationary convection diffusion optimal control problems, respectively. Similar to other discontinuous Galerkin methods, this scheme is also locally conservative, which makes it much suitable for problems where conservation is important, for example, for time dependent convection diffusion problems. Moreover, when the diffusion coefficient tends to zero, this scheme reduces to the classical discontinuous Galerkin method. Thus it inherits the stabilizing features of discontinuous Galerkin methods. We derive a priori error estimates of the state, adjoint state, and control for both semidiscrete scheme and fully discrete scheme. Numerical experiment is carried out to show the performance of our scheme.

The rest of the paper is organized as follows: In Section 2, semidiscrete and fully discrete mixed discontinuous Galerkin scheme are defined for control constrained optimal control problems governed by the time dependent convection diffusion equations. In Section 3, a priori error estimates of the semidiscrete scheme are derived. In Section 4, we derive a priori error estimates of the fully discrete scheme. Finally numerical example is given to illustrate the theoretical findings.

2. Mixed Discontinuous Galerkin Scheme

Consider the following optimal control problem with control constraints:

$$\begin{aligned} &\min_{u\in K} \quad J(y,u) \\ &\text{subject to} \quad \frac{\partial y}{\partial t} - \varepsilon\Delta y - \nabla\cdot(\boldsymbol{\beta} y) + \alpha y = f + u, \\ &\qquad\qquad \text{in } \Omega_T := \Omega\times(0,T), \\ &y = 0, \quad \text{on } \Gamma_T := \partial\Omega\times(0,T), \\ &y(\mathbf{x},0) = y_0(\mathbf{x}), \quad \text{in } \Omega, \end{aligned} \tag{2}$$

where Ω is a convex polygon with piecewise smooth boundary $\partial\Omega$. Here K is the admissible set defined by

$$\begin{aligned} K = \{ &q \in L^2(\Omega_T) : a \le q(x,t) \\ &\le b \text{ a.e. in } \Omega_T \text{ with } a,b\in \mathrm{R},\ a\le b \}. \end{aligned} \tag{3}$$

α and f are given functions, $\varepsilon > 0$ is a constant, and $\boldsymbol{\beta}$ is a given vector valued function, which satisfies $\alpha - (1/2)\nabla\cdot\boldsymbol{\beta} \ge \widetilde{\gamma} > 0$ with a constant $\widetilde{\gamma} > 0$.

To define a mixed discontinuous Galerkin scheme for (2), we introduce a new variable:

$$\mathbf{q} = -\varepsilon^{1/2}\nabla y. \tag{4}$$

Then the optimal control problem (2) can be rewritten as

$$\begin{aligned} &\min_{u\in K} \quad J(y,u) \\ &y_t + \nabla\cdot(\varepsilon^{1/2}\mathbf{q}) - \nabla\cdot(\boldsymbol{\beta} y) + \alpha y = f + u, \\ &\qquad\qquad \text{in } \Omega_T, \\ &\mathbf{q} = -\varepsilon^{1/2}\nabla y, \quad \text{in } \Omega_T, \\ &y = 0, \quad \text{on } \Gamma_T. \end{aligned} \tag{5}$$

Let T^h be a regular triangulation of Ω with element τ and $\overline{\Omega} = \bigcup_{\tau\in T^h}\overline{\tau}$. Let $h = \max_{\tau\in T^h} h_\tau$, where h_τ denotes the diameter of the element τ. We define the following spaces:

$$\begin{aligned} \mathbf{V} &= \{ \mathbf{v} \in (L^2(\Omega))^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega) \}, \\ W &= \{ w \in L^2(\Omega) : w|_\tau \in H^1(\tau),\ \forall\tau\in T^h \}, \\ U &= L^2(\Omega). \end{aligned} \tag{6}$$

For simplicity, we set

$$\begin{aligned} A(\mathbf{q},\mathbf{v}) &= (\mathbf{q},\mathbf{v}), \quad \mathbf{q},\mathbf{v}\in\mathbf{V}, \\ B(\mathbf{q},w) &= (\operatorname{div}(\varepsilon^{1/2}\mathbf{q}), w), \quad \mathbf{q}\in\mathbf{V},\ w\in W, \\ D(y,w) &= \sum_{\tau\in T^h}\left(\int_\tau y\boldsymbol{\beta}\cdot\nabla w - \int_{\partial\tau_-} y_+[w]\,\mathbf{n}\cdot\boldsymbol{\beta}\,ds\right) \\ &\quad + (\alpha y, w), \quad y,w\in W, \end{aligned} \tag{7}$$

where

$$\partial\tau_- = \{ l \in \partial\tau,\ \mathbf{n}\cdot\boldsymbol{\beta}|_l < 0 \}, \tag{8}$$

$\mathbf{n}$ is the outward norm direction on $\partial\tau$,

$$\begin{aligned} [w] &= w_+ - w_-, \\ w_+(x) &= \lim_{t\to 0^+} w(x + t\boldsymbol{\beta}), \\ w_-(x) &= \lim_{t\to 0^-} w(x + t\boldsymbol{\beta}), \end{aligned} \tag{9}$$

and $[w] = w_+$ on $\partial\tau$ when $\partial\tau_- \subset \partial\Omega$. Then the weak formulation for the optimal control problem (5) reads as follows: finding $(\mathbf{q}, y, u) \subset \mathbf{V}\times W\times K$ such that

$$\min_{u\in K} \quad J(y,u) \tag{10}$$

$$\begin{aligned} \text{subject to} \quad &(y_t, w) + B(\mathbf{q},w) + D(y,w) \\ &= (f+u, w), \quad \forall w\in W \\ &A(\mathbf{q},\mathbf{v}) - B(\mathbf{v},y) = 0, \quad \forall\mathbf{v}\in\mathbf{V}, \\ &y(\mathbf{x},0) = y_0(\mathbf{x}), \quad \forall\mathbf{x}\in\Omega. \end{aligned} \tag{11}$$

Standard arguments techniques imply that optimal control problem (10)-(11) admits a unique solution and the following first-order optimality condition holds:

$$\begin{aligned} &(y_t, w) + B(\mathbf{q},w) + D(y,w) = (f+u, w), \\ &\qquad\qquad \forall w\in W, \\ &A(\mathbf{q},\mathbf{v}) - B(\mathbf{v},y) = 0, \quad \forall\mathbf{v}\in\mathbf{V}, \\ &-(z_t,\psi) - B(\mathbf{p},\psi) + D(\psi,z) = (y - y_d, \psi), \\ &\qquad\qquad \forall\psi\in W, \\ &A(\mathbf{p},\boldsymbol{\varphi}) + B(\boldsymbol{\varphi},z) = 0, \quad \forall\boldsymbol{\varphi}\in\mathbf{V}, \\ &\int_0^T (\gamma u + z, \widetilde{v} - u)\,dt \ge 0, \quad \forall\widetilde{v}\in K, \end{aligned}$$

$$
\begin{aligned}
&y(\mathbf{x},0) = y_0(\mathbf{x}),\\
&z(\mathbf{x},T) = 0,\\
&\qquad\qquad \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{12}
$$

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the Raviart-Thomas element space of the lowest order associated with a triangular or rectangular mesh T^h of Ω (see [16] for details). Then the approximation scheme of (10)-(11) is as follows: finding $(\mathbf{q}_h, y_h, u_h) \in \mathbf{V}_h \times W_h \times K$ such that

$$
\begin{aligned}
&\min_{u_h \in K} \quad J(y,u)\\
&\text{subject to} \quad (y_{h,t}, w_h) + B(\mathbf{q}_h, w_h) + D(y_h, w_h)\\
&\qquad = (f + u_h, w_h), \quad \forall w_h \in W_h,\\
&\qquad A(\mathbf{q}_h, \mathbf{v}_h) - B(\mathbf{v}_h, y_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\\
&\qquad y_h(\mathbf{x},0) = \tilde{y}_0, \quad \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{13}
$$

Here $\tilde{y}_0$ is an approximation of the initial value $y_0(\mathbf{x})$. Variational discretization approach is used for the control u.

Similar to continuous case we derive the following semidiscrete optimality conditions:

$$
\begin{aligned}
&(y_{h,t}, w_h) + B(\mathbf{q}_h, w_h) + D(y_h, w_h) = (f + u_h, w_h),\\
&\qquad \forall w_h \in W_h,\\
&A(\mathbf{q}_h, \mathbf{v}_h) - B(\mathbf{v}_h, y_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\\
&-(z_{h,t}, \psi_h) - B(\mathbf{p}_h, \psi_h) + D(\psi_h, z_h) = (y_h - y_d, \psi_h),\\
&\qquad \forall \psi_h \in W_h,\\
&A(\mathbf{p}_h, \boldsymbol{\varphi}_h) + B(\boldsymbol{\varphi}_h, z_h) = 0,\\
&\qquad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h,\\
&\int_0^T (\gamma u_h + z_h, \tilde{v}_h - u_h)\, dt \ge 0, \quad \forall \tilde{v}_h \in K,\\
&y_h(\mathbf{x},0) = \tilde{y}_0,\\
&z_h(\mathbf{x},T) = 0,\\
&\qquad \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{14}
$$

To define a fully discrete scheme we introduce a time partition. Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a time grid with $k = t_n - t_{n-1}$, $n = 1, 2, \ldots, N$. Let $I_n = (t_{n-1}, t_n]$ be a half-open interval. We write $\phi_n := \phi(t_n)$ for a smooth ϕ.

We use backward Euler scheme for time discretization. Then a fully discrete scheme of (10)-(11) is characterized as follows: finding $(\mathbf{q}_h^n, y_h^n, u_h^n) \in \mathbf{V}_h \times W_h \times K$ such that

$$
\begin{aligned}
&\min_{u_h^n \in K} \quad J_{h,k}(y_h^n, u_h^n)\\
&\qquad := \frac{1}{2}\sum_{n=1}^{N} k\left(\|y_h^n - y_d^n\|_{0,\Omega}^2 + \gamma \|u_h^n\|_{0,\Omega}^2\right)\\
&\text{subject to} \quad \left(\frac{y_h^n - y_h^{n-1}}{k}, w_h\right) + B(\mathbf{q}_h^n, w_h)\\
&\qquad + D(y_h^n, w_h) = (f^n + u_h^n, w_h),\\
&\qquad\qquad \forall w_h \in W_h,\\
&\qquad A(\mathbf{q}_h^n, \mathbf{v}_h) - B(\mathbf{v}_h, y_h^n) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\\
&\qquad y_h^0 = \tilde{y}_0, \quad \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{15}
$$

To obtain the fully discrete first-order optimality condition we define a Lagrange functional as follows:

$$
\begin{aligned}
&L(\mathscr{Y}_h, \mathcal{Q}_h, \mathscr{Z}_h, \mathscr{P}_h, \mathscr{U}_h)\\
&\quad := k\sum_{n=1}^{N}\Bigg[\left(f^n + u_h^n - \frac{y_h^n - y_h^{n-1}}{k}, z_h^{n-1}\right)\\
&\quad - B\left(\mathbf{q}_h^n, z_h^{n-1}\right) - D\left(y_h^n, z_h^{n-1}\right) + B\left(\mathbf{p}_h^{n-1}, y_h^n\right)\\
&\quad - A\left(\mathbf{q}_h^n, \mathbf{p}_h^{n-1}\right)\Bigg] + J_{h,k}(y_h^n, u_h^n).
\end{aligned} \tag{16}
$$

Here $\mathscr{Y}_h = (y_h^1, y_h^2, \ldots, y_h^N)^T$, $\mathcal{Q}_h = (\mathbf{q}_h^1, \mathbf{q}_h^2, \ldots, \mathbf{q}_h^N)^T$, $\mathscr{Z}_h = (z_h^0, z_h^1, \ldots, z_h^{N-1})^T$, $\mathscr{P}_h = (\mathbf{p}_h^0, \mathbf{p}_h^1, \ldots, \mathbf{p}_h^{N-1})^T$, and $\mathscr{U}_h = (u_h^1, u_h^2, \ldots, u_h^N)^T$. Then we derive the discrete first-order optimality condition:

$$
\begin{aligned}
&\left(\frac{y_h^n - y_h^{n-1}}{k}, w_h\right) + B(\mathbf{q}_h^n, w_h) + D(y_h^n, w_h)\\
&\quad = (f^n + u_h^n, w_h), \quad \forall w_h \in W_h,\\
&A(\mathbf{q}_h^n, \mathbf{v}_h) - B(\mathbf{v}_h, y_h^n) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\\
&\left(\frac{z_h^{n-1} - z_h^n}{k}, \psi_h\right) - B\left(\mathbf{p}_h^{n-1}, \psi_h\right) + D\left(\psi_h, z_h^{n-1}\right)\\
&\quad = (y_h^n - y_d^n, \psi_h), \quad \forall \psi_h \in W_h,\\
&A\left(\mathbf{p}_h^{n-1}, \boldsymbol{\varphi}_h\right) + B\left(\boldsymbol{\varphi}_h, z_h^{n-1}\right) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h,\\
&\left(\gamma u_h^n + z_h^{n-1}, \tilde{v} - u_h^n\right) \ge 0, \quad \forall \tilde{v} \in K,\\
&y_h^0 = \tilde{y}_0,\\
&z_h^N = 0,\\
&\qquad \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{17}
$$

3. Semidiscrete Error Estimate

The goal of this section is to prove the semidiscrete error estimates for the state, the adjoint state, and the control. We firstly decompose $y - y_h$ and $z - z_h$ as

$$
\begin{aligned}
y - y_h &= y(u) - y_h(u) + y_h(u) - y_h,\\
z - z_h &= z(u) - z_h(u) + z_h(u) - z_h,
\end{aligned} \tag{18}
$$

where $y_h(u)$ and $z_h(u)$ satisfy the following auxiliary problems:

$$\begin{aligned}
&(y_{h,t}(u), w_h) + B(\mathbf{q}_h(u), w_h) + D(y_h(u), w_h) \\
&\quad = (f + u, w_h), \\
&A(\mathbf{q}_h(u), \mathbf{v}_h) - B(\mathbf{v}_h, y_h(u)) = 0, \\
&-(z_{h,t}(u), \psi_h) - B(\mathbf{p}_h(u), \psi_h) + D(\psi_h, z_h(u)) \\
&\quad = (y_h(u) - y_d, \psi_h), \\
&A(\mathbf{p}_h(u), \boldsymbol{\varphi}_h) + B(\boldsymbol{\varphi}_h, z_h(u)) = 0, \\
&y_h(u)(\mathbf{x}, 0) = \tilde{y}_0, \\
&z_h(u)(\mathbf{x}, T) = 0.
\end{aligned} \tag{19}$$

For the following analysis we introduce a new norm:

$$\begin{aligned}
\|y\|_*^2 &= \tilde{\gamma}\|y\|_{0,\Omega}^2 + \frac{1}{2}\sum_{\tau \in T^h}\int_{\partial\tau_-}[y]^2|\mathbf{n}\cdot\boldsymbol{\beta}|\,ds \\
&\quad + \frac{1}{2}\int_{\partial\Omega\setminus(\bigcup\partial\tau_-)} y_-^2|\mathbf{n}\cdot\boldsymbol{\beta}|\,ds.
\end{aligned} \tag{20}$$

Following [8] we have $D(y, y) \geq \|y\|_*^2$.

Let $(\tilde{y}, \tilde{\mathbf{q}}) \in W_h \times \mathbf{V}_h$ and $(\tilde{z}, \tilde{\mathbf{p}}) \in W_h \times \mathbf{V}_h$ be the projections defined by

$$\begin{aligned}
&B(\mathbf{q} - \tilde{\mathbf{q}}, w_h) + D(y - \tilde{y}, w_h) = 0, \quad \forall w_h \in W_h, \\
&A(\mathbf{q} - \tilde{\mathbf{q}}, \mathbf{v}_h) - B(\mathbf{v}_h, y - \tilde{y}) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&D(\psi_h, z - \tilde{z}) - B(\mathbf{p} - \tilde{\mathbf{p}}, \psi_h) = 0, \quad \forall \psi_h \in W_h, \\
&A(\mathbf{p} - \tilde{\mathbf{p}}, \boldsymbol{\varphi}_h) + B(\boldsymbol{\varphi}_h, z - \tilde{z}) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h,
\end{aligned} \tag{22}$$

According to [8] we have the following.

Lemma 1. *If $(\tilde{y}, \tilde{\mathbf{q}})$ and $(\tilde{z}, \tilde{\mathbf{p}})$ satisfy (21)-(22), respectively, then we have*

$$\begin{aligned}
&\|\mathbf{q} - \tilde{\mathbf{q}}\|_{0,\Omega}^2 + \|y - \tilde{y}\|_*^2 \leq Ch\left(\|\mathbf{q}\|_{1/2,\Omega}^2 + \|y\|_{1,\Omega}^2\right), \\
&\|\mathbf{p} - \tilde{\mathbf{p}}\|_{0,\Omega}^2 + \|z - \tilde{z}\|_*^2 \leq Ch\left(\|\mathbf{p}\|_{1/2,\Omega}^2 + \|z\|_{1,\Omega}^2\right).
\end{aligned} \tag{23}$$

Lemma 2. *If $(\tilde{y}, \tilde{\mathbf{q}})$ and $(\tilde{z}, \tilde{\mathbf{p}})$ satisfy (21)-(22), respectively, then the following estimates hold:*

$$\begin{aligned}
&\|\mathbf{q}_t - \tilde{\mathbf{q}}_t\|_{0,\Omega}^2 + \|y_t - \tilde{y}_t\|_*^2 \leq Ch\left(\|\mathbf{q}_t\|_{1/2,\Omega}^2 + \|y_t\|_{1,\Omega}^2\right), \\
&\|\mathbf{p}_t - \tilde{\mathbf{p}}_t\|_{0,\Omega}^2 + \|z_t - \tilde{z}_t\|_*^2 \leq Ch\left(\|\mathbf{p}\|_{1/2,\Omega}^2 + \|z\|_{1,\Omega}^2\right).
\end{aligned} \tag{24}$$

Proof. Differentiating (21) about t, we have

$$\begin{aligned}
&B(\mathbf{q}_t - \tilde{\mathbf{q}}_t, w_h) + D(y_t - \tilde{y}_t, w_h) = 0, \quad \forall w_h \in W_h, \\
&A(\mathbf{q}_t - \tilde{\mathbf{q}}_t, \mathbf{v}_h) - B(\mathbf{v}_h, y_t - \tilde{y}_t) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{25}$$

Set $P_h : W \to W_h$ and $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ to be the interpolation operators for the standard RT element space (see, e.g., [17]) such that

$$\begin{aligned}
&(\psi - P_h\psi, w_h) = 0, \quad \forall w_h \in W_h, \\
&(\operatorname{div}(\boldsymbol{\phi} - \Pi_h\boldsymbol{\phi}), w_h) = 0, \quad \forall w_h \in W_h.
\end{aligned} \tag{26}$$

Then we have the following approximation properties:

$$\begin{aligned}
&\|\psi - P_h\psi\|_{s,\Omega} \leq Ch^{1-s}\|\psi\|_{1,\Omega}, \\
&\qquad\qquad s = 0, 1, \ \forall \psi \in H_0^1(\Omega), \\
&\|\boldsymbol{\phi} - \Pi_h\boldsymbol{\phi}\|_{0,\Omega} \leq Ch^{\gamma}|\boldsymbol{\phi}|_{\gamma,\Omega}, \\
&\qquad\qquad \frac{1}{2} \leq \gamma < 1, \ \forall \boldsymbol{\phi} \in \left(H_0^{\gamma}(\Omega)\right)^2.
\end{aligned} \tag{27}$$

Let

$$\begin{aligned}
&\mathbf{q} - \tilde{\mathbf{q}} = \mathbf{q} - \Pi_h\mathbf{q} + \Pi_h\mathbf{q} - \tilde{\mathbf{q}} = \eta_{\mathbf{q}} + \xi_{\mathbf{q}}, \\
&y - \tilde{y} = y - P_hy + P_hy - \tilde{y} = \eta_y + \xi_y.
\end{aligned} \tag{28}$$

Then we derive

$$\begin{aligned}
&B(\eta_{\mathbf{q},t}, w_h) + B(\xi_{\mathbf{q},t}, w_h) + D(\eta_{y,t}, w_h) \\
&\quad + D(\xi_{y,t}, w_h) = 0, \quad \forall w_h \in W_h, \\
&A(\eta_{\mathbf{q},t}, \mathbf{v}_h) + A(\xi_{\mathbf{q},t}, \mathbf{v}_h) - B(\mathbf{v}_h, \eta_{y,t}) \\
&\quad - B(\mathbf{v}_h, \xi_{y,t}) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{29}$$

By the definition of interpolation operators P_h and Π_h we obtain

$$\begin{aligned}
&B(\eta_{\mathbf{q},t}, w_h) = 0, \quad \forall w_h \in W_h, \\
&B(\mathbf{v}_h, \eta_{y,t}) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{30}$$

Setting $w_h = \xi_{y,t}$ and $\mathbf{v}_h = \xi_{\mathbf{q},t}$ leads to

$$\begin{aligned}
&A(\xi_{\mathbf{q},t}, \xi_{\mathbf{q},t}) + D(\xi_{y,t}, \xi_{y,t}) \\
&\quad = A(\eta_{\mathbf{q},t}, -\xi_{\mathbf{q},t}) + D(\eta_{y,t}, -\xi_{y,t}),
\end{aligned} \tag{31}$$

namely,

$$\|\xi_{\mathbf{q},t}\|_{0,\Omega}^2 + \|\xi_{y,t}\|_*^2 \leq A(\eta_{\mathbf{q},t}, -\xi_{\mathbf{q},t}) + D(\eta_{y,t}, -\xi_{y,t}). \tag{32}$$

Since the function in W_h is piecewise constant, using the definition of $D(\cdot,\cdot)$, we have for all $w \in W$ and $\theta \in W_h$

$$\begin{aligned}
D(w, \theta) &= \sum_{\tau \in T^h}\left(\int_{\tau} w\beta\cdot\nabla\theta - \int_{\partial}\tau_-w_+[\theta]\,\mathbf{n}\cdot\boldsymbol{\beta}\,ds\right) \\
&\quad + (\alpha w, \theta) \\
&= 0 - \sum_{\tau \in T^h}\int_{\partial\tau_-} w_+[\theta]\,\mathbf{n}\cdot\boldsymbol{\beta}\,ds + (\alpha w, \theta).
\end{aligned} \tag{33}$$

Therefore, we get the estimate as follows:

$$\begin{aligned}\left\|\xi_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\xi_{y,t}\right\|_{*}^{2} &\leq C(\delta)\left(\left\|\eta_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\eta_{y,t}\right\|_{0,\Omega}^{2}\right)\\ &+C(\delta)\sum_{\tau}\int_{\partial\tau_{-}}\left(\eta_{y,t}\right)_{+}^{2}|\mathbf{n}\cdot\boldsymbol{\beta}|\,ds+C\delta\left(\left\|\xi_{\mathbf{q},t}\right\|_{0,\Omega}^{2}\right.\\ &\left.+\left\|\xi_{y,t}\right\|_{0,\Omega}^{2}+\sum_{\tau}\int_{\partial\tau_{-}}\left[\xi_{y,t}\right]^{2}|\mathbf{n}\cdot\boldsymbol{\beta}|\,ds\right).\end{aligned} \tag{34}$$

Note that

$$\begin{aligned}\left\|\mathbf{q}-\Pi_{h}\mathbf{q}\right\|_{0,\Omega} &\leq Ch^{1/2}\|\mathbf{q}\|_{1/2,\Omega},\\ \left\|y-P_{h}y\right\|_{0,\Omega} &\leq Ch\|y\|_{1,\Omega},\end{aligned} \tag{35}$$

and according to [18] we have

$$\begin{aligned}\sum_{\tau\in T^{h}}\int_{\partial\tau_{-}}\left(\eta_{y,t}\right)_{+}^{2}|\mathbf{n}\cdot\boldsymbol{\beta}|\,ds &\leq C\sum_{\tau\in T^{h}}\left\|\eta_{y,t}\right\|_{0,\partial\tau}^{2}\\ &\leq C\sum_{\tau\in T^{h}}\left\|\eta_{y,t}\right\|_{0,\tau}\left\|\eta_{y,t}\right\|_{1,\tau}\leq Ch\left\|y_{t}\right\|_{1,\Omega}^{2}.\end{aligned} \tag{36}$$

Then we derive

$$\begin{aligned}\left\|\xi_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\xi_{y,t}\right\|_{*}^{2} &\leq C(\delta)h\left(\left\|\mathbf{q}_{t}\right\|_{1/2,\Omega}^{2}+\left\|y_{t}\right\|_{1,\Omega}^{2}\right)\\ &+C\delta\left(\left\|\xi_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\xi_{y,t}\right\|_{*}^{2}\right).\end{aligned} \tag{37}$$

By setting δ small enough we obtain

$$\left(\left\|\xi_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\xi_{y,t}\right\|_{*}^{2}\right)^{1/2}\leq Ch^{1/2}\left(\left\|\mathbf{q}_{t}\right\|_{1/2,\Omega}+\left\|y_{t}\right\|_{1,\Omega}\right). \tag{38}$$

Using triangle inequality we have

$$\begin{aligned}&\left\|\mathbf{q}_{t}-\widetilde{\mathbf{q}}_{t}\right\|_{0,\Omega}^{2}+\left\|y_{t}-\widetilde{y}_{t}\right\|_{*}^{2}\\ &\quad\leq\left\|\eta_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\eta_{y,t}\right\|_{*}^{2}+\left\|\xi_{\mathbf{q},t}\right\|_{0,\Omega}^{2}+\left\|\xi_{y,t}\right\|_{*}^{2},\end{aligned} \tag{39}$$

which implies the first theorem result. Similar to the above proof we have

$$\left\|\mathbf{p}_{t}-\widetilde{\mathbf{p}}_{t}\right\|_{0,\Omega}^{2}+\left\|z_{t}-\widetilde{z}_{t}\right\|_{*}^{2}\leq Ch\left(\left\|\mathbf{p}_{t}\right\|_{1/2,\Omega}^{2}+\left\|z_{t}\right\|_{1,\Omega}^{2}\right). \tag{40}$$

Using the above estimates we can derive the following. □

Theorem 3. *Let $(y,\mathbf{q},z,\mathbf{p})$ and $(y_h(u),\mathbf{q}_h(u),z_h(u),\mathbf{p}_h(u))$ be the solutions of (12) and (19), respectively. Then we have*

$$\left|\!\left|\!\left|\left(y-y_{h}(u),\mathbf{q}-\mathbf{q}_{h}(u)\right)\right|\!\right|\!\right|_{+}\leq Ch^{1/2}. \tag{41}$$

Here

$$\left|\!\left|\!\left|\left(y_{h},\mathbf{q}_{h}\right)\right|\!\right|\!\right|_{+}^{2}=\left\|y_{h}\right\|_{0,\Omega}^{2}+\int_{0}^{T}\left\|y_{h}\right\|_{*}^{2}dt+\int_{0}^{T}\left\|\mathbf{q}_{h}\right\|_{0,\Omega}^{2}dt \tag{42}$$

and the constant C depends on the norms of y and $\mathbf{q}$.

Proof. Using (12) and (19) we derive the following error equations:

$$\begin{aligned}&\left(y_{t}-y_{h,t}(u),w_{h}\right)+B\left(\mathbf{q}-\mathbf{q}_{h}(u),w_{h}\right)\\ &\quad+D\left(y-y_{h}(u),w_{h}\right)=0,\quad\forall w_{h}\in W_{h},\\ &A\left(\mathbf{q}-\mathbf{q}_{h}(u),\mathbf{v}_{h}\right)-B\left(\mathbf{v}_{h},y-y_{h}(u)\right)=0,\\ &\qquad\forall\mathbf{v}_{h}\in\mathbf{V}_{h}.\end{aligned} \tag{43}$$

We split the errors $y-y_h(u)$ and $\mathbf{q}-\mathbf{q}_h(u)$ into

$$\begin{aligned}y-y_{h}(u)&=y-\widetilde{y}+\widetilde{y}-y_{h}(u),\\ \mathbf{q}-\mathbf{q}_{h}(u)&=\mathbf{q}-\widetilde{\mathbf{q}}+\widetilde{\mathbf{q}}-\mathbf{q}_{h}(u),\end{aligned} \tag{44}$$

where $(\widetilde{y},\widetilde{\mathbf{q}})$ is defined in (21). Then we have

$$\begin{aligned}&\left(y_{t}-\widetilde{y}_{t},w_{h}\right)+\left(\widetilde{y}_{t}-y_{h,t}(u),w_{h}\right)\\ &\quad+B\left(\widetilde{\mathbf{q}}-\mathbf{q}_{h}(u),w_{h}\right)+D\left(\widetilde{y}-y_{h}(u),w_{h}\right)\\ &\quad+A\left(\widetilde{\mathbf{q}}-\mathbf{q}_{h}(u),\mathbf{v}_{h}\right)-B\left(\mathbf{v}_{h},\widetilde{y}-y_{h}(u)\right)=0.\end{aligned} \tag{45}$$

Setting $\chi=\widetilde{y}-y_h(u)$ and $\boldsymbol{\sigma}=\widetilde{\mathbf{q}}-\mathbf{q}_h(u)$ gives

$$\begin{aligned}&\left(\chi_{t},w_{h}\right)+B\left(\boldsymbol{\sigma},w_{h}\right)+D\left(\chi,w_{h}\right)+A\left(\boldsymbol{\sigma},\mathbf{v}_{h}\right)\\ &\quad-B\left(\mathbf{v}_{h},\chi\right)=-\left(y_{t}-\widetilde{y}_{t},w_{h}\right).\end{aligned} \tag{46}$$

Choosing $w_h=\chi$ and $\mathbf{v}_h=\boldsymbol{\sigma}$ in the above equation leads to

$$\left(\chi_{t},\chi\right)+D\left(\chi,\chi\right)+A\left(\boldsymbol{\sigma},\boldsymbol{\sigma}\right)=-\left(y_{t}-\widetilde{y}_{t},\chi\right). \tag{47}$$

Since $D(\chi,\chi)\geq\|\chi\|_*^2$, then we deduce by Young inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\chi\|_{0,\Omega}^{2}+\|\chi\|_{*}^{2}+\|\boldsymbol{\sigma}\|_{0,\Omega}^{2}\\ &\quad\leq\delta\|\chi\|_{0,\Omega}^{2}+C(\delta)\left\|y_{t}-\widetilde{y}_{t}\right\|_{0,\Omega}^{2}.\end{aligned} \tag{48}$$

By setting δ small enough we derive

$$\frac{1}{2}\frac{d}{dt}\|\chi\|_{0,\Omega}^{2}+\frac{1}{2}\|\chi\|_{*}^{2}+\|\boldsymbol{\sigma}\|_{0,\Omega}^{2}\leq C\left\|y_{t}-\widetilde{y}_{t}\right\|_{0,\Omega}^{2}. \tag{49}$$

Note that $\chi(0)=0$. Integrating (49) from $0\to t$ leads to

$$\|\chi(t)\|_{0,\Omega}^{2}+\int_{0}^{t}\|\chi\|_{*}^{2}+\int_{0}^{t}\|\sigma\|_{0,\Omega}^{2}\leq C\int_{0}^{t}\left\|y_{t}-\widetilde{y}_{t}\right\|_{0,\Omega}^{2}. \tag{50}$$

Similarly, integrating (49) from $0\to T$ yields

$$\begin{aligned}&\|\chi(T)\|_{0,\Omega}^{2}+\int_{0}^{T}\|\chi\|_{*}^{2}+\int_{0}^{T}\|\boldsymbol{\sigma}\|_{0,\Omega}^{2}\\ &\quad\leq C\int_{0}^{T}\left\|y_{t}-\widetilde{y}_{t}\right\|_{0,\Omega}^{2},\end{aligned} \tag{51}$$

which implies

$$\int_{0}^{T}\|\chi\|_{*}^{2}+\int_{0}^{T}\|\boldsymbol{\sigma}\|_{0,\Omega}^{2}\leq C\int_{0}^{T}\left\|y_{t}-\widetilde{y}_{t}\right\|_{0,\Omega}^{2}. \tag{52}$$

Combining (50) and (52) we deduce that

$$\|\chi(t)\|_{0,\Omega}^2 + \int_0^T \|\chi\|_*^2 + \int_0^T \|\boldsymbol{\sigma}\|_{0,\Omega}^2 \le C\int_0^T \|y_t - \tilde{y}_t\|_{0,\Omega}^2. \tag{53}$$

By Lemma 2 we further derive

$$\|\chi(t)\|_{0,\Omega}^2 + \int_0^T \|\chi\|_*^2 + \int_0^T \|\boldsymbol{\sigma}\|_{0,\Omega}^2 \le Ch\left(\int_0^T \|\mathbf{q}_t\|_{1/2,\Omega}^2 + \int_0^T \|y_t\|_{1,\Omega}^2\right). \tag{54}$$

Then by triangle inequality and Lemma 1 we obtain

$$\begin{aligned} |\!|\!|(y - y_h(u), \mathbf{q} - \mathbf{q}_h(u))|\!|\!|_+^2 \le Ch\Bigg(&\|\mathbf{q}\|_{1/2,\Omega}^2 + \|y\|_{1,\Omega}^2 \\ &+ \int_0^T \|\mathbf{q}\|_{1/2,\Omega}^2 + \int_0^T \|y\|_{1,\Omega}^2 + \int_0^T \|\mathbf{q}_t\|_{1/2,\Omega}^2 \\ &+ \int_0^T \|y_t\|_{1,\Omega}^2\Bigg). \end{aligned} \tag{55}$$

This implies the theorem result. □

Theorem 4. *Let $(y, \mathbf{q}, z, \mathbf{p})$ and $(y_h(u), \mathbf{q}_h(u), z_h(u), \mathbf{p}_h(u))$ be the solutions of (12) and (19), respectively. Then we have*

$$|\!|\!|(z - z_h(u), \mathbf{p} - \mathbf{p}_h(u))|\!|\!|_+ \le Ch^{1/2}. \tag{56}$$

Here the constant C depends on the norms of y, z and $\mathbf{q}, \mathbf{p}$.

Proof. Using (12) and (19) we derive the following error equations:

$$\begin{aligned} &-(z_t - z_{h,t}(u), \psi_h) - B(\mathbf{p} - \mathbf{p}_h(u), \psi_h) \\ &\quad + D(\psi_h, z - z_h(u)) = (y - y_h(u), \psi_h), \quad \forall \psi_h \in W_h \\ &A(\mathbf{p} - \mathbf{p}_h(u), \boldsymbol{\varphi}_h) + B(\boldsymbol{\varphi}_h, z - z_h(u)) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h. \end{aligned} \tag{57}$$

We rewrite the errors $z - z_h(u)$ and $\mathbf{p} - \mathbf{p}_h(u)$ as

$$\begin{aligned} z - z_h(u) &= z - \tilde{z} + \tilde{z} - z_h(u), \\ \mathbf{p} - \mathbf{p}_h(u) &= \mathbf{p} - \tilde{\mathbf{p}} + \tilde{\mathbf{p}} - \mathbf{p}_h(u), \end{aligned} \tag{58}$$

where $(\tilde{z}, \tilde{\mathbf{p}})$ is defined in (22). Then we have

$$\begin{aligned} &-(z_t - \tilde{z}_t, \psi_h) - (\tilde{z}_t - z_{h,t}(u), \psi_h) \\ &\quad - B(\tilde{\mathbf{p}} - \mathbf{p}_h(u), \psi_h) + D(\psi_h, \tilde{z} - z_h(u)) \\ &\quad + A(\tilde{\mathbf{p}} - \mathbf{p}_h(u), \boldsymbol{\varphi}_h) + B(\boldsymbol{\varphi}_h, \tilde{z} - z_h(u)) \\ &= (y - y_h(u), \psi_h). \end{aligned} \tag{59}$$

Setting $\xi = \tilde{z} - z_h(u)$ and $\boldsymbol{\rho} = \tilde{\mathbf{p}} - \mathbf{p}_h(u)$ we have

$$\begin{aligned} &-(\xi_t, \psi_h) - B(\boldsymbol{\rho}, \psi_h) + D(\psi_h, \xi) + A(\boldsymbol{\rho}, \boldsymbol{\varphi}_h) \\ &\quad + B(\boldsymbol{\varphi}_h, \xi) = (z_t - \tilde{z}_t, \psi_h) + (y - y_h(u), \psi_h). \end{aligned} \tag{60}$$

Taking $\psi_h = \xi$ and $\boldsymbol{\varphi}_h = \boldsymbol{\rho}$ yields

$$\begin{aligned} &-(\xi_t, \xi) + D(\xi, \xi) + A(\boldsymbol{\rho}, \boldsymbol{\rho}) \\ &\quad = (z_t - \tilde{z}_t, \xi) + (y - y_h(u), \xi). \end{aligned} \tag{61}$$

Since $D(\xi, \xi) \ge \|\xi\|_*^2$, then

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_{0,\Omega}^2 + \|\xi\|_*^2 + \|\boldsymbol{\rho}\|_{0,\Omega}^2 \\ &\quad \le |(z_t - \tilde{z}_t, \xi)| + |(y - y_h(u), \xi)|. \end{aligned} \tag{62}$$

Using *Hölder* inequality and Young inequality we derive

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_{0,\Omega}^2 + \|\xi\|_*^2 + \|\boldsymbol{\rho}\|_{0,\Omega}^2 \\ &\quad \le \delta\|\xi\|_{0,\Omega}^2 + C(\delta)\|z_t - \tilde{z}_t\|_{0,\Omega}^2 \\ &\quad\quad + C(\delta)\|y - y_h(u)\|_{0,\Omega}^2. \end{aligned} \tag{63}$$

Choosing δ small enough we obtain

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\|\xi(t)\|_{0,\Omega}^2 + \frac{1}{2}\|\xi\|_*^2 + \|\boldsymbol{\rho}\|_{0,\Omega}^2 \\ &\quad \le C(\delta)\|z_t - \tilde{z}_t\|_{0,\Omega}^2 + C(\delta)\|y - y_h(u)\|_{0,\Omega}^2. \end{aligned} \tag{64}$$

Integrating (64) with respect to t from $t \to T$ yields

$$\begin{aligned} &\|\xi(t)\|_{0,\Omega}^2 + \int_t^T \|\xi\|_*^2 + \int_t^T \|\boldsymbol{\rho}\|_{0,\Omega}^2 \\ &\quad \le C\int_0^T \|z_t - \tilde{z}_t\|_{0,\Omega}^2 + C\int_0^T \|y - y_h(u)\|_{0,\Omega}^2, \end{aligned} \tag{65}$$

where $\xi(T) = 0$ was used. In an analogue way, integrating (64) from $0 \to T$ results in

$$\begin{aligned} &\frac{1}{2}\|\xi(0)\|_{0,\Omega}^2 + \int_0^T \|\xi\|_*^2 + \int_0^T \|\boldsymbol{\rho}\|_{0,\Omega}^2 \\ &\quad \le C\int_0^T \|z_t - \tilde{z}_t\|_{0,\Omega}^2 + C\int_0^T \|y - y_h(u)\|_{0,\Omega}^2. \end{aligned} \tag{66}$$

We further have

$$\begin{aligned} \int_0^T \|\xi\|_*^2 + \int_0^T \|\boldsymbol{\rho}\|_{0,\Omega}^2 &\le C\int_0^T \|z_t - \tilde{z}_t\|_{0,\Omega}^2 \\ &\quad + C\int_0^T \|y - y_h(u)\|_{0,\Omega}^2. \end{aligned} \tag{67}$$

Combining (65) and (67) we deduce that

$$\begin{aligned} &\|\xi(t)\|_{0,\Omega}^2 + \int_0^T \|\xi\|_*^2 + \int_0^T \|\boldsymbol{\rho}\|_{0,\Omega}^2 \\ &\quad \le C\int_0^T \|z_t - \tilde{z}_t\|_{0,\Omega}^2 + C\int_0^T \|y - y_h(u)\|_{0,\Omega}^2. \end{aligned} \tag{68}$$

Then we can derive the theorem result by using Lemmas 1 and 2 and Theorem 8 and triangle inequality. □

Lemma 5. *Let $(y_h, \mathbf{q}_h, z_h, \mathbf{p}_h)$ and $(y_h(u), \mathbf{q}_h(u), z_h(u), \mathbf{p}_h(u))$ be the solutions of (14) and (19), respectively. Then we have*

$$\begin{aligned} |||(y_h - y_h(u), \mathbf{q}_h - \mathbf{q}_h(u))|||_+^2 &\le C\int_0^T \|u - u_h\|_{0,\Omega}^2, \\ |||(z_h - z_h(u), \mathbf{p}_h - \mathbf{p}_h(u))|||_+^2 &\le C\int_0^T \|u - u_h\|_{0,\Omega}^2. \end{aligned} \tag{69}$$

Proof. Using (19) along with (14) leads to

$$\begin{aligned} &(y_{h,t} - y_{h,t}(u), w_h) + B(\mathbf{q}_h - \mathbf{q}_h(u), w_h) \\ &\quad + D(y_h - y_h(u), w_h) = (u_h - u, w_h), \\ &\qquad\qquad \forall w_h \in W_h, \end{aligned} \tag{70}$$

$$A(\mathbf{q}_h - \mathbf{q}_h(u), \mathbf{v}_h) - B(\mathbf{v}_h, y_h - y_h(u)) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Setting $Y_h = y_h - y_h(u)$ and $\mathbf{Q}_h = \mathbf{q}_h - \mathbf{q}_h(u)$ and choosing $w_h = Y_h$ and $\mathbf{v}_h = \mathbf{Q}_h$ lead to

$$(Y_{h,t}, Y_h) + A(\mathbf{Q}_h, \mathbf{Q}_h) + D(Y_h, Y_h) = (u_h - u, Y_h). \tag{71}$$

By Young inequality we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|Y_h\|_{0,\Omega}^2 + \|\mathbf{Q}_h\|_{0,\Omega}^2 + \|Y_h\|_*^2 \\ &\quad \le C(\delta)\|u - u_h\|_{0,\Omega}^2 + C\delta\|Y_h\|_{0,\Omega}^2. \end{aligned} \tag{72}$$

Similar to the proof of Lemmas 1 and 2 we have

$$\begin{aligned} &\|Y_h\|_{0,\Omega}^2 + \int_0^T \|\mathbf{Q}_h\|_{0,\Omega}^2 + \int_0^T \|Y_h\|_*^2 \\ &\quad \le C\int_0^T \|u - u_h\|_{0,\Omega}^2. \end{aligned} \tag{73}$$

By (17) and (19) we derive

$$\begin{aligned} &-(z_{h,t} - z_{h,t}(u), \psi_h) - B(\mathbf{p}_h - \mathbf{p}_h(u), \psi_h) \\ &\quad + D(\psi_h, z_h - z_h(u)) = (Y_h, \psi_h), \quad \forall \psi_h \in W_h, \\ &A(\mathbf{p}_h - \mathbf{p}_h(u), \boldsymbol{\varphi}_h) - B(\boldsymbol{\varphi}_h, z_h - z_h(u)) = 0, \\ &\qquad\qquad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h. \end{aligned} \tag{74}$$

Setting $Z_h = z_h - z_h(u)$ and $\mathbf{P}_h = \mathbf{p}_h - \mathbf{p}_h(u)$ and choosing $\psi_h = Z_h$ and $\boldsymbol{\varphi}_h = \mathbf{P}_h$ yield

$$-(Z_{h,t}, Z_h) + A(\mathbf{P}_h, \mathbf{P}_h) + D(Z_h, Z_h) = (Y_h, Z_h). \tag{75}$$

Furthermore, we derive by Young inequality

$$\begin{aligned} &-\frac{1}{2}\|Z_h\|_{0,\Omega}^2 + \|\mathbf{P}_h\|_{0,\Omega}^2 + \|Z_h\|_*^2 \\ &\quad \le C(\delta)\|Y_h\|_{0,\Omega}^2 + C\delta\|Z_h\|_{0,\Omega}^2. \end{aligned} \tag{76}$$

Similar to the proof of Lemmas 1 and 2 we have

$$\begin{aligned} \|Z_h\|_{0,\Omega}^2 + \int_0^T \|\mathbf{P}_h\|_{0,\Omega}^2 + \int_0^T \|Z_h\|_*^2 &\le C\int_0^T \|Y_h\|_{0,\Omega}^2 \\ &\le C\int_0^T \|u - u_h\|_{0,\Omega}^2. \end{aligned} \tag{77}$$

Then the theorem result follows from (73) and (77). □

Lemma 6. *Assume that $(y, \mathbf{q}, z, \mathbf{p}, u)$ and $(y_h, \mathbf{q}_h, z_h, \mathbf{p}_h, u_h)$ be the solutions of (11) and (14), respectively. Let*

$$J_h'(u)(v - u) = \int_0^T (\gamma u + z_h(u), v - u), \tag{78}$$

where $z_h(u)$ is the solution of (19). Then the following estimate holds:

$$J_h'(v)(v - u) - J_h'(u)(v - u) \ge \gamma\|v - u\|_{L^2(0,T;L^2(\Omega))}^2. \tag{79}$$

Proof. By the definitions of $(y_h(u), \mathbf{q}_h(u))$, $(z_h(u), \mathbf{p}_h(u))$ we have

$$\begin{aligned} &(y_{h,t}(v) - y_{h,t}(u), w_h) + B(\mathbf{q}_h(v) - \mathbf{q}_h(u), w_h) \\ &\quad + D(y_h(v) - y_h(u), w_h) = (v - u, w_h), \\ &\qquad\qquad \forall w_h \in W_h, \\ &A(\mathbf{q}_h(v) - \mathbf{q}_h(u), \mathbf{v}_h) - B(\mathbf{v}_h, y_h(v) - y_h(u)) = 0, \\ &\qquad\qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ &(z_{h,t}(u) - z_{h,t}(v), \psi_h) - B(\mathbf{p}_h(v) - \mathbf{p}_h(u), \psi_h) \\ &\quad + D(\psi_h, z_h(v) - z_h(u)) = (y_h(v) - y_h(u), \psi_h), \\ &A(\mathbf{p}_h(v) - \mathbf{p}_h(u), \boldsymbol{\varphi}_h) + B(\boldsymbol{\varphi}_h, z_h(v) - z_h(u)) = 0. \end{aligned} \tag{80}$$

Note that

$$\begin{aligned} &J_h'(v)(v - u) - J_h'(u)(v - u) \\ &\quad = \int_0^T (\gamma v + z_h(v), v - u) - \int_0^T (\gamma v + z_h(u), v - u) \\ &\quad = \int_0^T (\gamma(v - u), v - u) \\ &\qquad + \int_0^T (z_h(v) - z_h(u), v - u) \\ &\quad = \gamma\|v - u\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\qquad + \int_0^T (z_h(v) - z_h(u), v - u). \end{aligned} \tag{81}$$

By setting $w_h = z_h(v) - z_h(u)$, $\mathbf{v}_h = \mathbf{p}_h(v) - \mathbf{p}_h(u)$ and $\psi_h = y_h(v) - y_h(u)$, $\boldsymbol{\varphi}_h = \mathbf{q}_h(v) - \mathbf{q}_h(u)$ in the above equations, respectively, we can prove

$$\begin{aligned}\int_0^T (z_h(v) - z_h(u), v - u) &= \int_0^T (v - u, z_h(v) - z_h(u)) \\ &= \int_0^T [D(y_h(v) - y_h(u), z_h(v) - z_h(u)) \\ &\quad + B(\mathbf{q}_h(v) - \mathbf{q}_h(u), z_h(v) - z_h(u)) \\ &\quad + (y_h, t(v) - y_h, t(u), z_h(v) - z_h(u)) \\ &\quad + A(\mathbf{q}_h(v) - \mathbf{q}_h(u), \mathbf{p}_h(v) - \mathbf{p}_h(u)) \\ &\quad - B(\mathbf{p}_h(v) - \mathbf{p}_h(u), y_h(v) - y_h(u))].\end{aligned} \tag{82}$$

Since $(y_h(v) - y_h(u))(0) = 0$, $(z_h(v) - z_h(u))(T) = 0$, and

$$\begin{aligned}\int_0^T (y_{h,t}(v) - y_{h,t}(u), z_h(v) - z_h(u)) &= -\int_0^T (y_h(v) - y_h(u), z_{h,t}(v) - z_{h,t}(u)) \\ &\quad + (y_h(v) - y_h(u), z_h(v) - z_h(u))\Big|_0^T = 0\end{aligned} \tag{83}$$

we can prove that

$$\begin{aligned}\int_0^T (z_h(v) - z_h(u), v - u) &= \int_0^T (y_h(v) - y_h(u), y_h(v) - y_h(u)) \ge 0.\end{aligned} \tag{84}$$

Therefore, we derive

$$J_h'(v)(v - u) - J_h'(u)(v - u) \ge \gamma \|v - u\|^2_{L^2(0,T;L^2(\Omega))}. \tag{85}$$

□

Lemma 7. *Let $(y, \mathbf{q}, z, \mathbf{p}, u)$ and $(y_h, \mathbf{q}_h, z_h, \mathbf{p}_h, u_h)$ be the solutions of (11) and (14), respectively. Then we have*

$$\|u - u_h\|_{L^2(0,T,L^2(\Omega))} \le Ch^{1/2}. \tag{86}$$

Proof. By Lemma 6 we have

$$\begin{aligned}\gamma \|u - u_h\|^2_{L^2(0,T,L^2(\Omega))} &\le (J_h'(u) - J_h'(u_h), u - u_h) \\ &= \int_0^T (\alpha u + z_h(u), u - u_h)\, dt \\ &\quad - \int_0^T (\alpha u_h + z_h, u - u_h)\, dt \\ &= \int_0^T (\alpha u + z, u - u_h)\, dt \\ &\quad + \int_0^T (z_h(u) - z, u - u_h)\, dt \\ &\quad + \int_0^T (\alpha u_h + z_h, u_h - u)\, dt \\ &\le 0 + \int_0^T (z_h(u) - z, u - u_h)\, dt + 0 \\ &\le \|z_h(u) - z\|_{L^2(0,T,L^2(\Omega))} \|u - u_h\|_{L^2(0,T,L^2(\Omega))}.\end{aligned} \tag{87}$$

Using the result of Theorem 4 yields

$$\begin{aligned}\|u - u_h\|_{L^2(0,T,L^2(\Omega))} &\le C \|z_h(u) - z\|_{L^2(0,T,L^2(\Omega))} \\ &\le Ch^{1/2}.\end{aligned} \tag{88}$$

□

Theorem 8. *Let $(y, \mathbf{q}, z, \mathbf{p})$ and $(y_h(u), \mathbf{q}_h(u), z_h(u), \mathbf{p}_h(u))$ be the solutions of (12) and (14), respectively. Then we have*

$$\begin{aligned}&\|u - u_h\|_{L^2(0,T,L^2(\Omega))} + |||(y - y_h, \mathbf{q} - \mathbf{q}_h)|||_+ \\ &\quad + |||(z - z_h, \mathbf{p} - \mathbf{p}_h)|||_+ \le Ch^{1/2}.\end{aligned} \tag{89}$$

Proof. By Lemma 2 and (88) we derive

$$\begin{aligned}|||(y - y_h, \mathbf{q} - \mathbf{q}_h)|||_+^2 &\le |||(y - y_h(u), \mathbf{q} - \mathbf{q}_h(u))|||_+^2 \\ &\quad + |||(y_h(u) - y_h, \mathbf{q}_h(u) - \mathbf{q}_h)|||_+^2 \le Ch\left(\|\mathbf{q}\|^2_{1/2,\Omega}\right. \\ &\quad \left.+ \|y\|^2_{1,\Omega}\right) + C\int_0^T \|u - u_h\|^2_{0,\Omega} + Ch\left(\int_0^T \|\mathbf{q}\|^2_{1/2,\Omega}\right. \\ &\quad \left.+ \int_0^T \|y\|^2_{1,\Omega} + \int_0^T \left\|\frac{\partial \mathbf{q}}{\partial t}\right\|^2_{1/2,\Omega} + \int_0^T \left\|\frac{\partial y}{\partial t}\right\|^2_{1,\Omega}\right).\end{aligned} \tag{90}$$

Combining Lemma 5 and (90) yields

$$\begin{aligned}|||(z - z_h, \mathbf{p} - \mathbf{p}_h)|||_+^2 &\le |||(z - z_h(u), \mathbf{p} - \mathbf{p}_h(u))|||_+^2 \\ &\quad + |||(z_h(u) - z_h, \mathbf{p}_h(u) - \mathbf{p}_h)|||_0^2 \le Ch\left(\|\mathbf{p}\|^2_{1/2,\Omega}\right. \\ &\quad \left.+ \|z\|^2_{1,\Omega} + \int_0^T \|\mathbf{p}\|^2_{1/2,\Omega} + \int_0^T \|z\|^2_{1,\Omega}\right) \\ &\quad + Ch\left(\int_0^T \left\|\frac{\partial \mathbf{p}}{\partial t}\right\|^2_{1/2,\Omega} + \int_0^T \left\|\frac{\partial z}{\partial t}\right\|^2_{1,\Omega} + \int_0^T \|\mathbf{q}\|^2_{1/2,\Omega}\right. \\ &\quad \left.\int_0^T \|y\|^2_{1,\Omega}\right) + C\int_0^T \|u - u_h\|^2_{0,\Omega}.\end{aligned} \tag{91}$$

Combining (88)–(91) we can derive the theorem result. □

4. Fully Discrete Error Estimate

In this section we will prove the error estimates for the fully discrete scheme. For this purpose we firstly introduce the following auxiliary problems:

$$\begin{aligned}
&\left(\frac{y_h^n(u)-y_h^{n-1}(u)}{k}, w_h\right)+B\left(\mathbf{q}_h^n(u), w_h\right) \\
&\quad +D\left(y_h^n(u), w_h\right)=\left(f^n+u^n, w_h\right), \quad \forall w_h \in W_h, \\
&A\left(\mathbf{q}_h^n(u), \mathbf{v}_h\right)-B\left(\mathbf{v}_h, y_h^n(u)\right)=0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
&\left(\frac{z_h^{n-1}(u)-z_h^n(u)}{k}, \psi_h\right)-B\left(\mathbf{p}_h^{n-1}(u), \psi_h\right) \\
&\quad +D\left(\psi_h, z_h^{n-1}(u)\right)=\left(y_h^n(u)-y_d^n, \psi_h\right), \\
&\qquad \forall \psi_h \in W_h, \\
&A\left(\mathbf{p}_h^{n-1}(u), \boldsymbol{\varphi}_h\right)+B\left(\boldsymbol{\varphi}_h, z_h^{n-1}(u)\right)=0, \quad \forall \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
&y_h^0(u)=\tilde{y}_0, \\
&z_h^N(u)=0, \\
&\qquad \forall \mathbf{x} \in \Omega.
\end{aligned} \tag{92}$$

Lemma 9. *Let $(y, \mathbf{q}, z, \mathbf{p})$ and $(y_h^n(u), \mathbf{q}_h^n(u), z_h^{n-1}(u), \mathbf{p}_h^{n-1}(u))$ be the solutions of (12) and (92), respectively. Then we have*

$$|||\left(y-y_h(u), \mathbf{q}-\mathbf{q}_h(u)\right)|||_{\diamond} \leq C\left(h^{1/2}+k\right), \tag{93}$$

where

$$|||(y, \mathbf{q})|||_{\diamond}^2=\left\|y^n\right\|_{0,\Omega}^2+k \sum_{n=1}^N\left\|y^n\right\|_*^2+k \sum_{n=1}^N\left\|\mathbf{q}^n\right\|_{0,\Omega}^2. \tag{94}$$

Proof. Using (12) and (92) we derive the following error equations:

$$\begin{aligned}
&\left(\frac{\partial y^n}{\partial t}-\frac{y_h^n(u)-y_h^{n-1}(u)}{k}, w_h\right)+B\left(\mathbf{q}^n-\mathbf{q}_h^n(u), w_h\right) \\
&\quad +D\left(y^n-y_h^n(u), w_h\right)=0, \quad \forall w_h \in W_h, \\
&A\left(\mathbf{q}^n-\mathbf{q}_h^n(u), \mathbf{v}_h\right)-B\left(\mathbf{v}_h, y^n-y_h^n(u)\right)=0, \\
&\qquad \forall \mathbf{q} \in \mathbf{V_h}.
\end{aligned} \tag{95}$$

Setting $\chi=\tilde{y}-y_h(u)$, $\rho=y-\tilde{y}$, and $\boldsymbol{\sigma}=\tilde{\mathbf{q}}-\mathbf{q}_h(u)$ and combining the definition of $\tilde{y}$ and $\tilde{\mathbf{q}}$ we have

$$\begin{aligned}
&\left(\frac{\chi^n-\chi^{n-1}}{k}, w_h\right)+B\left(\boldsymbol{\sigma}^n, w_h\right)+D\left(\chi^n, w_h\right) \\
&\quad=-\left(\frac{\partial y^n}{\partial t}-\frac{y^n-y^{n-1}}{k}, w_h\right)-\left(\frac{\rho^n-\rho^{n-1}}{k}, w_h\right), \\
&A\left(\boldsymbol{\sigma}^n, \mathbf{v}_h\right)-B\left(\mathbf{v}_h, \chi^n\right)=0.
\end{aligned} \tag{96}$$

Testing (96) with $w_h=\chi^n$ and $\mathbf{v}_h=\boldsymbol{\sigma}^n$ yields

$$\begin{aligned}
&\left(\frac{\chi^n-\chi^{n-1}}{k}, \chi^n\right)+A\left(\boldsymbol{\sigma}^n, \boldsymbol{\sigma}^n\right)+D\left(\chi^n, \chi^n\right) \\
&\quad=-\left(\frac{\partial y^n}{\partial t}-\frac{y^n-y^{n-1}}{k}, \chi^n\right)-\left(\frac{\rho^n-\rho^{n-1}}{k}, \chi^n\right) \\
&\quad=D_1+D_2.
\end{aligned} \tag{97}$$

By Taylor expansion with integral reminder we deduce that

$$\begin{aligned}
\frac{\partial y^n}{\partial t}-\frac{y^n-y^{n-1}}{k}&=\frac{1}{k} \int_{t_{n-1}}^{t_n}\left(t-t_{n-1}\right) \frac{\partial^2 y}{\partial t^2} d t, \\
\rho^n-\rho^{n-1}&=\int_{t_{n-1}}^{t_n} \frac{\partial \rho}{\partial t} d t.
\end{aligned} \tag{98}$$

By *Hölder* inequality we obtain

$$\begin{aligned}
\left\|\frac{\partial y^n}{\partial t}-\frac{y^n-y^{n-1}}{k}\right\|_{0,\Omega}^2 &\leq k \int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{0,\Omega}^2 d t, \\
\left\|\rho^n-\rho^{n-1}\right\|_{0,\Omega}^2 &\leq k \int_{t_{n-1}}^{t_n}\left\|\frac{\partial \rho}{\partial t}\right\|_{0,\Omega}^2 d t.
\end{aligned} \tag{99}$$

Then by Young inequality we derive

$$\begin{aligned}
D_1 &\leq C(\delta) k \int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{0,\Omega}^2 d t+\delta\left\|\chi^n\right\|_{0,\Omega}^2, \\
D_2 &\leq C(\delta) \frac{1}{k} \int_{t_{n-1}}^{t_n}\left\|\frac{\partial \rho}{\partial t}\right\|_{0,\Omega}^2 d t+\delta\left\|\chi^n\right\|_{0,\Omega}^2.
\end{aligned} \tag{100}$$

Choosing δ to be small enough leads to

$$\begin{aligned}
&\frac{1}{2k}\left(\left\|\chi^n\right\|_{0,\Omega}^2-\left\|\chi^{n-1}\right\|_{0,\Omega}^2\right)+\left\|\boldsymbol{\sigma}^n\right\|_{0,\Omega}^2+\frac{1}{2}\left\|\chi^n\right\|_*^2 \\
&\quad \leq C k \int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{0,\Omega}^2 d t+C \frac{1}{k} \int_{t_{n-1}}^{t_n}\left\|\frac{\partial \rho}{\partial t}\right\|_{0,\Omega}^2 d t.
\end{aligned} \tag{101}$$

Multiplying (114) by $2k$ and summing up with respect to n from 1 to m yield

$$\begin{aligned}
&\left\|\chi^m\right\|_{0,\Omega}^2+\tau \sum_{n=1}^m\left\|\boldsymbol{\sigma}^n\right\|_{0,\Omega}^2+\tau \sum_{n=1}^m\left\|\chi^n\right\|_*^2 \\
&\quad \leq C k^2 \int_0^{t_m}\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{0,\Omega}^2 d t+C \int_0^{t_m}\left\|\frac{\partial \rho}{\partial t}\right\|_{0,\Omega}^2 d t.
\end{aligned} \tag{102}$$

Multiplying (101) by $2k$ and summing up with respect to n from 1 to N lead to

$$\begin{aligned}
&\left\|\chi^N\right\|_{0,\Omega}^2+k \sum_{n=1}^N\left\|\boldsymbol{\sigma}^n\right\|_{0,\Omega}^2+k \sum_{n=1}^N\left\|\chi^n\right\|_*^2 \\
&\quad \leq C k^2 \int_0^T\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{0,\Omega}^2 d t+C \int_0^T\left\|\frac{\partial \rho}{\partial t}\right\|_{0,\Omega}^2 d t.
\end{aligned} \tag{103}$$

Collecting above estimates we have

$$\|\chi^m\|_{0,\Omega}^2 + k\sum_{n=1}^{N}\|\boldsymbol{\sigma}^n\|_{0,\Omega}^2 + k\sum_{n=1}^{N}\|\chi^n\|_*^2 \le Ck^2\int_0^T\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{0,\Omega}^2 dt + C\int_0^T\left\|\frac{\partial \rho}{\partial t}\right\|_{0,\Omega}^2 dt. \tag{104}$$

Using above estimates, Lemmas 1 and 2, and triangle inequality we deduce that

$$\begin{aligned}&|||(y - y_h(u), \mathbf{q} - \mathbf{q}_h(u))|||_\diamond \\ &\le Ch^{1/2}\left(\|\mathbf{q}\|_{L^\infty(0,T;H^{1/2}(\Omega))} + \|y\|_{L^\infty(0,T;H^1(\Omega))}\right. \\ &\left.+ \|\mathbf{q}_t\|_{L^2(0,T;H^{1/2}(\Omega))} + \|y_t\|_{L^2(0,T;H^1(\Omega))}\right) \\ &+ Ck\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{105}$$

□

Lemma 10. *Let* $(y, \mathbf{q}, z, \mathbf{p})$ *and* $(y_h^n(u), \mathbf{q}_h^n(u), z_h^{n-1}(u), \mathbf{p}_h^{n-1}(u))$ *be the solutions of (12) and (92), respectively. Then we have*

$$|||(z - z_h(u), \mathbf{p} - \mathbf{p}_h(u))|||_\diamond \le C\left(h^{1/2} + k\right). \tag{106}$$

Proof. Using (12) and (92) we derive the following error equations:

$$\begin{aligned}&\left(-\frac{z_h^{n-1}(u) - z_h^n(u)}{k} - \frac{\partial z^{n-1}}{\partial t}, \psi_h\right) \\ &- B\left(\mathbf{p}^{n-1} - \mathbf{p}_h^{n-1}(u), \psi_h\right) \\ &+ D\left(\psi_h, z^{n-1} - z_h^{n-1}(u)\right) = -\left(y_h^n(u) - y^{n-1}, \psi_h\right) \\ &+ \left(y_d^n - y_d^{n-1}, \psi_h\right), \quad \forall \psi_h \in W_h, \\ &A\left(\mathbf{p}^{n-1} - \mathbf{p}_h^{n-1}(u), \mathbf{v}_h\right) \\ &+ B\left(\mathbf{v}_h, z^{n-1} - z_h^{n-1}(u)\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.\end{aligned} \tag{107}$$

Setting $\xi = \tilde{z} - z_h(u)$, $\eta = z - \tilde{z}$, and $\boldsymbol{\sigma} = \tilde{\mathbf{p}} - \mathbf{p}_h(u)$ and using the definitions of $\tilde{z}$ and $\tilde{\mathbf{p}}$ give

$$\begin{aligned}&\left(\frac{\xi^{n-1} - \xi^n}{k}, \psi_h\right) + B\left(\boldsymbol{\sigma}^{n-1}, \psi_h\right) + D\left(\psi_h, \xi^{n-1}\right) \\ &= -\left(y_h^n(u) - y^{n-1}, \psi_h\right) + \left(y_d^n - y_d^{n-1}, \psi_h\right) \\ &+ \left(\frac{\partial z^{n-1}}{\partial t} + \frac{z^{n-1} - z^n}{k}, \xi^{n-1}\right) \\ &- \left(\frac{\eta^{n-1} - \eta^n}{k}, \xi^{n-1}\right), \\ &A\left(\boldsymbol{\sigma}^{n-1}, \mathbf{v}_h\right) - B\left(\mathbf{v}_h, \xi^{n-1}\right) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.\end{aligned} \tag{108}$$

Choosing $w_h = \xi^{n-1}$ and $\mathbf{v}_h = \boldsymbol{\sigma}^n$ in (108) yields

$$\begin{aligned}&\left(\frac{\xi^{n-1} - \xi^n}{k}, \xi^{n-1}\right) + A\left(\boldsymbol{\sigma}^{n-1}, \boldsymbol{\sigma}^{n-1}\right) + D\left(\xi^{n-1}, \xi^{n-1}\right) \\ &= -\left(y_h^n(u) - y^{n-1}, \xi^{n-1}\right) + \left(y_d^n - y_d^{n-1}, \xi^{n-1}\right) \\ &+ \left(\frac{\partial z^{n-1}}{\partial t} + \frac{z^{n-1} - z^n}{k}, \xi^{n-1}\right) \\ &- \left(\frac{\eta^{n-1} - \eta^n}{k}, \xi^{n-1}\right) = \sum_{i=1}^{4} T_i.\end{aligned} \tag{109}$$

Using *Hölder* inequality and Young inequality we obtain

$$\begin{aligned}T_1 &\le C(\delta)\left(\|y_h^n(u) - y^n\|_{0,\Omega}^2 + k\int_{t_{n-1}}^{t_n}\left\|\frac{\partial y}{\partial t}\right\|_{0,\Omega}^2 dt\right) \\ &+ \delta\left\|\xi^{n-1}\right\|_{0,\Omega}^2, \\ T_2 &\le C(\delta)\,k\int_{t_{n-1}}^{t_n}\left\|\frac{\partial y_d}{\partial t}\right\|_{0,\Omega}^2 dt + \delta\left\|\xi^{n-1}\right\|_{0,\Omega}^2, \\ T_3 &\le C(\delta)\,k\int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2 z}{\partial t^2}\right\|_{0,\Omega}^2 dt + \delta\left\|\xi^{n-1}\right\|_{0,\Omega}^2, \\ T_4 &\le C(\delta)\,\frac{1}{k}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial \eta}{\partial t}\right\|_{0,\Omega}^2 dt + \delta\left\|\xi^{n-1}\right\|_{0,\Omega}^2.\end{aligned} \tag{110}$$

Then by setting δ small enough we have

$$\begin{aligned}&\frac{1}{2k}\left(\left\|\xi^{n-1}\right\|_{0,\Omega}^2 - \|\xi^n\|_{0,\Omega}^2\right) + \left\|\boldsymbol{\sigma}^{n-1}\right\|_{0,\Omega}^2 + \left\|\xi^{n-1}\right\|_*^2 \\ &\le C\|y_h^n(u) - y^n\|_{0,\Omega}^2 + Ck\int_{t_{n-1}}^{t_n}\left\|\frac{\partial y}{\partial t}\right\|_{0,\Omega}^2 dt \\ &+ Ck\int_{t_{n-1}}^{t_n}\left\|\frac{\partial y_d}{\partial t}\right\|_{0,\Omega}^2 dt + Ck\int_{t_{n-1}}^{t_n}\left\|\frac{\partial^2 z}{\partial t^2}\right\|_{0,\Omega}^2 dt \\ &+ C\frac{1}{k}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial \eta}{\partial t}\right\|_{0,\Omega}^2 dt.\end{aligned} \tag{111}$$

Multiplying (111) by $2k$ and summing up with respect to n from m to N yield

$$\begin{aligned}&\left\|\xi^{m-1}\right\|_{0,\Omega}^2 + k\sum_{n=m}^{N}\left\|\boldsymbol{\sigma}^{n-1}\right\|_{0,\Omega}^2 + k\sum_{n=m}^{N}\left\|\xi^{n-1}\right\|_*^2 \\ &\le Ck\sum_{n=m}^{N}\|y_h^n(u) - y^n\|_{0,\Omega}^2 + Ck^2\int_{t_m}^{T}\left\|\frac{\partial y}{\partial t}\right\|_{0,\Omega}^2 dt \\ &+ Ck^2\int_{t_m}^{T}\left\|\frac{\partial y_d}{\partial t}\right\|_{0,\Omega}^2 dt + Ck^2\int_{t_m}^{T}\left\|\frac{\partial^2 z}{\partial t^2}\right\|_{0,\Omega}^2 dt \\ &+ C\int_{t_m}^{T}\left\|\frac{\partial \eta}{\partial t}\right\|_{0,\Omega}^2 dt.\end{aligned} \tag{112}$$

Multiplying (111) by $2k$ and summing up with respect to n from 1 to N give

$$\begin{aligned}\|\xi^0\|_{0,\Omega}^2 &+ k\sum_{n=1}^{N}\|\boldsymbol{\sigma}^{n-1}\|_{0,\Omega}^2 + k\sum_{n=1}^{N}\|\xi^{n-1}\|_*^2 \\ &\le Ck\sum_{n=1}^{N}\|y_h^n(u)-y^n\|_{0,\Omega}^2 + Ck^2\int_0^T\left\|\frac{\partial y}{\partial t}\right\|_{0,\Omega}^2 dt \\ &+ Ck^2\int_0^T\left\|\frac{\partial y_d}{\partial t}\right\|_{0,\Omega}^2 dt + Ck^2\int_0^T\left\|\frac{\partial^2 z}{\partial t^2}\right\|_{0,\Omega}^2 dt \\ &+ C\int_0^T\left\|\frac{\partial \eta}{\partial t}\right\|_{0,\Omega}^2 dt.\end{aligned} \tag{113}$$

Collecting (112) and (113) we obtain

$$\begin{aligned}\|\xi^{n-1}\|_{0,\Omega}^2 &+ k\sum_{n=1}^{N}\|\boldsymbol{\sigma}^{n-1}\|_{0,\Omega}^2 + k\sum_{n=1}^{N}\|\xi^{n-1}\|_*^2 \\ &\le Ck\sum_{n=1}^{N}\|y_h^n(u)-y^n\|_{0,\Omega}^2 + Ck^2\int_0^T\left\|\frac{\partial y}{\partial t}\right\|_{0,\Omega}^2 dt \\ &+ Ck^2\int_0^T\left\|\frac{\partial y_d}{\partial t}\right\|_{0,\Omega}^2 dt + Ck^2\int_0^T\left\|\frac{\partial^2 z}{\partial t^2}\right\|_{0,\Omega}^2 dt \\ &+ C\int_0^T\left\|\frac{\partial \eta}{\partial t}\right\|_{0,\Omega}^2 dt.\end{aligned} \tag{114}$$

By Lemmas 1 and 2 and triangle inequality we derive

$$\begin{aligned}&|||(z-z_h(u),\mathbf{p}-\mathbf{p}_h(u))|||_\diamond \\ &\le Ch^{1/2}\left(\|\mathbf{q}\|_{L^\infty(0,T;H^{1/2}(\Omega))} + \|y\|_{L^\infty(0,T;H^1(\Omega))}\right. \\ &\left.+ \|\mathbf{q}_t\|_{L^2(0,T;H^{1/2}(\Omega))} + \|y_t\|_{L^2(0,T;H^1(\Omega))}\right) \\ &+ Ch^{1/2}\left(\|\mathbf{p}\|_{L^\infty(0,T;H^{1/2}(\Omega))} + \|z\|_{L^\infty(0,T;H^1(\Omega))}\right. \\ &\left.+ \|\mathbf{p}_t\|_{L^2(0,T;H^{1/2}(\Omega))} + \|z_t\|_{L^2(0,T;H^1(\Omega))}\right) \\ &+ Ck\left(\left\|\frac{\partial y}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial y_d}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}\right. \\ &\left.+ \left\|\frac{\partial^2 z}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}\right).\end{aligned} \tag{115}$$

□

Lemma 11. *Let $(y_h^n, \mathbf{q}_h^n, z_h^{n-1}, \mathbf{p}_h^{n-1})$ and $(y_h^n(u), \mathbf{q}_h^n(u), z_h^{n-1}(u), \mathbf{p}_h^{n-1}(u))$ be the solutions of (12) and (92), respectively. Then we have*

$$\begin{aligned}&|||(y_h-y_h(u),\mathbf{q}_h-\mathbf{q}_h(u))|||_\diamond^2 \\ &\le Ck\sum_{n=1}^{N}\|u_h^n-u^n\|_{0,\Omega}^2, \\ &|||(z_h-z_h(u),\mathbf{p}_h-\mathbf{p}_h(u))|||_\diamond^2 \le Ck\sum_{n=1}^{N}\|u_h^n-u^n\|_{0,\Omega}^2.\end{aligned} \tag{116}$$

Proof. Using (92) along with (17) leads to

$$\begin{aligned}&\left(\frac{y_h^n-y_h^n(u)-\left(y_h^{n-1}-y_h^{n-1}(u)\right)}{k}, w_h\right) \\ &+ B\left(\mathbf{q}_h^n-\mathbf{q}_h^n(u), w_h\right) \\ &+ D\left(y_h^n-y_h^n(u), w_h\right) = \left(u_h^n-u^n, w_h\right), \\ &A\left(\mathbf{q}_h^n-\mathbf{q}_h^n(u), \mathbf{v}_h\right) - B\left(\mathbf{v}_h, y_h^n-y_h^n(u)\right) = 0.\end{aligned} \tag{117}$$

Choosing $w_h = y_h^n - y_h^n(u)$ and $\mathbf{v}_h = \mathbf{q}_h^n - \mathbf{q}_h^n(u)$ in the above equations and adding the resulting equations together yield

$$\begin{aligned}&\left(\frac{y_h^n-y_h^n(u)-\left(y_h^{n-1}-y_h^{n-1}(u)\right)}{k}, y_h^n-y_h^n(u)\right) \\ &+ B\left(\mathbf{q}_h^n-\mathbf{q}_h^n(u), y_h^n-y_h^n(u)\right) \\ &+ D\left(y_h^n-y_h^n(u), y_h^n-y_h^n(u)\right) \\ &+ A\left(\mathbf{q}_h^n-\mathbf{q}_h^n(u), \mathbf{q}_h^n-\mathbf{q}_h^n(u)\right) \\ &- B\left(\mathbf{q}_h^n-\mathbf{q}_h^n(u), y_h^n-y_h^n(u)\right) \\ &= \left(u_h^n-u^n, y_h^n-y_h^n(u)\right).\end{aligned} \tag{118}$$

Setting $\mathscr{Y}_h = y_h - y_h(u)$ and $\mathscr{Q}_h = \mathbf{q}_h - \mathbf{q}_h(u)$ in the above equation leads to

$$\begin{aligned}&\left(\frac{\mathscr{Y}_h^n-\mathscr{Y}_h^{n-1}}{k}, \mathscr{Y}_h^n\right) + D\left(\mathscr{Y}_h^n, \mathscr{Y}_h^n\right) + A\left(\mathscr{Q}_h^n, \mathscr{Q}_h^n\right) \\ &= \left(u_h^n-u^n, \mathscr{Y}_h^n\right),\end{aligned} \tag{119}$$

and then

$$\begin{aligned}&\left(\frac{\mathscr{Y}_h^n-\mathscr{Y}_h^{n-1}}{k}, \mathscr{Y}_h^n\right) + \|\mathscr{Y}_h^n\|_*^2 + \|\mathscr{Q}_h^n\|_{0,\Omega}^2 \\ &\le \left(u_h^n-u^n, \mathscr{Y}_h^n\right).\end{aligned} \tag{120}$$

Furthermore, we have

$$\begin{aligned}&\frac{1}{2k}\left(\|\mathscr{Y}_h^n\|_{0,\Omega}^2 - \|\mathscr{Y}_h^{n-1}\|_{0,\Omega}^2\right) + \|\mathscr{Y}_h^n\|_*^2 + \|\mathscr{Q}_h^n\|_{0,\Omega}^2 \\ &\le C(\delta)\|u_h^n-u^n\|_{0,\Omega}^2 + \delta\|\mathscr{Y}_h^n\|_{0,\Omega}^2.\end{aligned} \tag{121}$$

Similar to the proof of Lemmas 9 and 10 we have

$$\begin{aligned}&\|\mathscr{Y}_h^n\|_{0,\Omega}^2 + k\sum_{n=1}^{N}\|\mathscr{Y}_h^n\|_*^2 + k\sum_{n=1}^{N}\|\mathscr{Q}_h^n\|_{0,\Omega}^2 \\ &\le Ck\sum_{n=1}^{N}\|u_h^n-u^n\|_{0,\Omega}^2,\end{aligned} \tag{122}$$

Similarly, by (17) and (92) we derive

$$\begin{aligned}
&\left(\frac{z_h^{n-1}-z_h^{n-1}(u)-(z_h^n-z_h^n(u))}{k},\psi_h\right)\\
&\quad -B\left(\mathbf{p}_h^{n-1}-\mathbf{p}_h^{n-1}(u),\psi_h\right)\\
&\quad +D\left(\psi_h,z_h^{n-1}-z_h^{n-1}(u)\right)\\
&\quad =\left(y_h^{n-1}-y_h^{n-1}(u),\psi_h\right),\\
&A\left(\mathbf{p}_h^{n-1}-\mathbf{p}_h^{n-1}(u),\varphi_h\right)+B\left(\varphi_h,z_h^{n-1}-z_h^{n-1}(u)\right)\\
&\quad =0.
\end{aligned}\tag{123}$$

Taking $\psi_h=z_h^{n-1}-z_h^{n-1}(u)$ and $\varphi_h=\mathbf{p}_h^{n-1}-\mathbf{p}_h^{n-1}(u)$ in the above equation, setting $\mathscr{Z}_h^{n-1}=z_h^{n-1}-z_h^{n-1}(u)$ and $\mathscr{P}_h^{n-1}=p_h^{n-1}-p_h^{n-1}(u)$, and using similar argument to (124) give

$$\begin{aligned}
&\left\|\mathscr{Z}_h^{n-1}\right\|_{0,\Omega}^2+k\sum_{n=1}^{N}\left\|\mathscr{Z}_h^{n-1}\right\|_*^2+k\sum_{n=1}^{N}\left\|\mathscr{P}_h^{n-1}\right\|_{0,\Omega}^2\\
&\quad\le Ck\sum_{n=1}^{N}\left\|y_h^n-y_h^n(u)\right\|_{0,\Omega}^2\le Ck\sum_{n=1}^{N}\left\|u_h^n-u^n\right\|_{0,\Omega}^2.
\end{aligned}\tag{124}$$

Then the theorem result follows from (122) and (124). □

Lemma 12. *Let $(y,\mathbf{q},z,\mathbf{p},u)$ and $(y_h^n,\mathbf{q}_h^n,z_h^{n-1},\mathbf{p}_h^{n-1},u_h^n)$ be the solutions of (12) and (92), respectively. Let*

$$\widehat{J}_h'(u)(v-u)=k\sum_{n=1}^{N}\left(\gamma u^n+z_h^{n-1}(u),v^n-u^n\right),\tag{125}$$

where $z_h^{n-1}(u)$ is the solution of (92). Then the following estimate holds:

$$\left(\widehat{J}_h'(v)-\widehat{J}_h'(u),v-u\right)\ge\gamma\,|||v-u|||_{L^2(0,T,L^2(\Omega))}^2.\tag{126}$$

Here $|||v-u|||_{L^2(0,T;L^2(\Omega))}^2:=k\sum_{n=1}^{N}\|v^n-u^n\|^2$.

Proof. Note that

$$\begin{aligned}
&\left(\widehat{J}_h'(v)-\widehat{J}_h'(u),v-u\right)\\
&\quad=k\sum_{n=1}^{N}\left(\gamma v^n+z_h^{n-1}(v)-\gamma u^n-z_h^{n-1}(u),v^n-u^n\right)\\
&\quad=k\sum_{n=1}^{N}\left(\gamma v^n-\gamma u^n,v^n-u^n\right)\\
&\qquad+k\sum_{n=1}^{N}\left(z_h^{n-1}(v)-z_h^{n-1}(u),v^n-u^n\right)\\
&\quad=\gamma\,|||v-u|||_{L^2(0,T,L^2(\Omega))}^2\\
&\qquad+k\sum_{n=1}^{N}\left(z_h^{n-1}(v)-z_h^{n-1}(u),v^n-u^n\right).
\end{aligned}\tag{127}$$

Setting $Y_h^n=y_h^n(v)-y_h^n(u)$, $\mathbf{Q}_h^n=\mathbf{q}_h^n(v)-\mathbf{q}_h^n(u)$, $Z_h^{n-1}=z_h^{n-1}(v)-z_h^{n-1}(u)$, and $\mathbf{P}_h^{n-1}=\mathbf{p}_h^{n-1}(v)-\mathbf{p}_h^{n-1}(u)$, by the definition of $y_h^n(u)$ and $z_h^{n-1}(u)$ we have

$$\begin{aligned}
&\left(\frac{Y_h^n-Y_h^{n-1}}{\tau},w_h\right)+B\left(\mathbf{Q}_h^n,w_h\right)+D\left(Y_h^n,w_h\right)\\
&\quad=\left(v^n-u^n,w_h\right),\\
&A\left(\mathbf{Q}_h^n,\mathbf{v}_h\right)-B\left(\mathbf{v}_h,Y_h^n\right)=0,\\
&\left(\frac{Z_h^{n-1}-Z_h^n}{\tau},\psi_h\right)-B\left(\mathbf{P}_h^n,\psi_h\right)+D\left(\psi_h,Z_h^{n-1}\right)\\
&\quad=\left(Y_h^n,\psi_h\right),\\
&A\left(\mathbf{P}_h^{n-1},\boldsymbol{\varphi}_h\right)+B\left(\boldsymbol{\varphi}_h,Z_h^{n-1}\right)=0.
\end{aligned}\tag{128}$$

Choosing $w_h=Z_h^{n-1}$, $\mathbf{v}_h=\mathbf{P}_h^{n-1}$ and $\psi_h=Y_h^n$, $\boldsymbol{\varphi}_h=\mathbf{Q}_h^n$ in the above equations, respectively, we can easily prove that

$$k\sum_{n=1}^{N}\left(z_h^{n-1}(v)-z_h^{n-1}(u),v^n-u^n\right)\ge0.\tag{129}$$

Therefore, we derive

$$\left(\widehat{J}_h'(v)-\widehat{J}_h'(u),v-u\right)\ge\gamma\,|||v-u|||_{L^2(0,T,L^2(\Omega))}^2.\tag{130}$$

□

Lemma 13. *Let $(y,\mathbf{q},z,\mathbf{p},u)$ and $(y_h^n,\mathbf{q}_h^n,z_h^{n-1},\mathbf{p}_h^{n-1},u_h^n)$ be the solutions of (12) and (92), respectively. Then we have*

$$|||u-u_h|||_{L^2(0,T,L^2(\Omega))}\le Ch^{1/2}+Ck\left\|\frac{\partial z}{\partial t}\right\|_{L^2(0,T,L^2(\Omega))}.\tag{131}$$

Proof. Following Lemma 12 we derive

$$\begin{aligned}
\gamma\,|||u-u_h|||_{L^2(0,T,L^2(\Omega))}^2&\le\left(\widehat{J}_h'(u)-\widehat{J}_h'(u_h),u-u_h\right)\\
&=k\sum_{n=1}^{N}\left(\gamma u^n+z_h^{n-1}(u),u^n-u_h^n\right)_U\\
&\quad-k\sum_{n=1}^{N}\left(\gamma u^n+z_h^{n-1},u^n-u_h^n\right)\\
&=k\sum_{n=1}^{N}\left(\gamma u^n+z^n,u^n-u_h^n\right)_U\\
&\quad+k\sum_{n=1}^{N}\left(z_h^{n-1}(u)-z^n,u^n-u_h^n\right)\\
&\quad+k\sum_{n=1}^{N}\left(\gamma u_h^n+z_h^{n-1},u_h^n-u^n\right)\\
&\le0+k\sum_{n=1}^{N}\left(z_h^{n-1}(u^n)-z^n,u^n-u_h^n\right)+0
\end{aligned}$$

$$= k\sum_{n=1}^{N}\left(z_h^{n-1}\left(u^n\right) - z^{n-1}, u^n - u_h^n\right)$$
$$+ k\sum_{n=1}^{N}\left(z^{n-1} - z^n, u^n - u_h^n\right). \tag{132}$$

Using Young inequality we obtain

$$\left|\left\|u - u_h\right\|\right|_{L^2(0,T,L^2(\Omega))} \leq Ch^{1/2} + Ck\left\|\frac{\partial z}{\partial t}\right\|_{L^2(0,T,L^2(\Omega))}. \tag{133}$$

□

Theorem 14. *Let $(y, \mathbf{q}, z, \mathbf{p}, u)$ and $(y_h^n, \mathbf{q}_h^n, z_h^{n-1}, \mathbf{p}_h^{n-1}, u_h^n)$ be the solutions of (11) and (17), respectively. Then we have*

$$\left|\left\|u - u_h\right\|\right|_{L^2(0,T,L^2(\Omega))} + \left|\left\|\left(y - y_h, \mathbf{q} - \mathbf{q}_h\right)\right\|\right|_{\diamond}$$
$$+ \left|\left\|\left(z - z_h, \mathbf{p} - \mathbf{p}_h\right)\right\|\right|_{\diamond} \leq C\left(h^{1/2} + k\right). \tag{134}$$

Proof. By Lemma 9 and using triangle inequality we derive

$$\left|\left\|\left(y - y_h, \mathbf{q} - \mathbf{q}_h\right)\right\|\right|_{\diamond}^2 \leq \left|\left\|\left(y - y_h(u), \mathbf{q} - \mathbf{q}_h(u)\right)\right\|\right|_{\diamond}^2$$
$$+ \left|\left\|\left(y_h(u) - y_h, \mathbf{q}_h(u) - \mathbf{q}_h\right)\right\|\right|_{\diamond}^2$$
$$\leq Ch\left(\|\mathbf{q}\|_{L^\infty(0,T;H^{1/2}(\Omega))}^2 + \|y\|_{L^\infty(0,T;H^1(\Omega))}^2\right.$$
$$\left. + \|\mathbf{q}_t\|_{L^2(0,T;H^{1/2}(\Omega))}^2 + \|y_t\|_{L^2(0,T;H^1(\Omega))}^2\right)$$
$$+ Ck^2\left\|\frac{\partial^2 y}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}^2 + C\left|\left\|u - u_h\right\|\right|_{L^2(0,T;L^2(\Omega))}^2. \tag{135}$$

By Lemma 10 we deduce that

$$\left|\left\|\left(z - z_h, \mathbf{p} - \mathbf{p}_h\right)\right\|\right|_{\diamond}^2 \leq \left|\left\|\left(z - z_h(u), \mathbf{p} - \mathbf{p}_h(u)\right)\right\|\right|_{\diamond}^2$$
$$+ \left|\left\|\left(z_h(u) - z_h, \mathbf{p}_h(u) - \mathbf{p}_h\right)\right\|\right|_{\diamond}^2$$
$$\leq Ch\left(\|\mathbf{p}\|_{L^\infty(0,T;H^{1/2}(\Omega))}^2 + \|z\|_{L^\infty(0,T;H^1(\Omega))}^2\right.$$
$$+ \|\mathbf{p}_t\|_{L^2(0,T;H^{1/2}(\Omega))}^2 + \|z_t\|_{L^2(0,T;H^1(\Omega))}^2$$
$$\left. + \|y\|_{L^\infty(0,T;H^1(\Omega))}^2\right) + C\left|\left\|u - u_h\right\|\right|_{L^2(0,T;L^2(\Omega))}^2 \tag{136}$$
$$+ Ck^2\left(\|y_t\|_{L^2(0,T;L^2(\Omega))}^2 + \left\|\frac{\partial y_d}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}^2\right.$$
$$\left. + \left\|\frac{\partial^2 z}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}^2\right).$$

Combining Lemma 13, (135), and (136) we arrive at

$$\left|\left\|u - u_h\right\|\right|_{L^2(0,T,L^2(\Omega))} + \left|\left\|\left(y - y_h, \mathbf{q} - \mathbf{q}_h\right)\right\|\right|_{\diamond}$$
$$+ \left|\left\|\left(z - z_h, \mathbf{p} - \mathbf{p}_h\right)\right\|\right|_{\diamond} \leq Ch^{1/2}\left(\|\mathbf{q}\|_{L^\infty(0,T;H^{1/2}(\Omega))}\right.$$
$$+ \|y\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{q}_t\|_{L^2(0,T;H^{1/2}(\Omega))}$$
$$\left. + \|y_t\|_{L^2(0,T;H^1(\Omega))}\right) + Ch^{1/2}\left(\|\mathbf{p}\|_{L^\infty(0,T;H^{1/2}(\Omega))}\right.$$
$$+ \|z\|_{L^\infty(0,T;H^1(\Omega))} + \|\mathbf{p}_t\|_{L^2(0,T;H^{1/2}(\Omega))} \tag{137}$$
$$\left. + \|z_t\|_{L^2(0,T;H^1(\Omega))}\right) + Ck\left(\|y_t\|_{L^2(0,T;L^2(\Omega))}\right.$$
$$+ \left\|\frac{\partial y_d}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\frac{\partial^2 y}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}$$
$$\left. + \left\|\frac{\partial^2 z}{\partial t^2}\right\|_{L^2(0,T;L^2(\Omega))}\right),$$

which completes the theorem. □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to acknowledge the support of Natural Science Foundation of Shandong Province (no. ZR2016JL004).

References

[1] J. Zhu and Q. C. Zeng, "A mathematical theoretical frame for control of air pollution," *Science China Earth Sciences D*, vol. 32, pp. 864–870, 2002.

[2] A. Martínez, C. Rodríguez, and M. E. Vazquez-Mendez, "Theoretical and numerical analysis of an optimal control problem related to wastewater treatment," *SIAM Journal on Control and Optimization*, vol. 38, no. 5, pp. 1534–1553, 2000.

[3] R. Becker and B. Vexler, "Optimal control of the convection-diffusion equation using stabilized finite element methods," *Numerische Mathematik*, vol. 106, no. 3, pp. 349–367, 2007.

[4] M. Braack, "Optimal control in fluid mechanics by finite elements with symmetric stabilization," *SIAM Journal on Control and Optimization*, vol. 48, no. 2, pp. 672–687, 2009.

[5] S. S. Collis and M. Heinkenschloss, "Analysis of the streamline upwind/Petrov Galerkin method applied to the solution of optimal control problems," Tech. Rep. TR02-01, Department of Computational and Applied Mathematics, Rice University, Houston, Tex, USA, 2002.

[6] M. Hinze, N. Yan, and Z. Zhou, "Variational discretization for optimal control governed by convection dominated diffusion equations," *Journal of Computational Mathematics*, vol. 27, no. 2-3, pp. 237–253, 2009.

[7] T. Sun, "Discontinuous Galerkin finite element method with interior penalties for convection diffusion optimal control problem," *International Journal of Numerical Analysis and Modeling*, vol. 7, no. 1, pp. 87–107, 2010.

[8] N. Yan and Z. Zhou, "A RT mixed FEM/DG scheme for optimal control governed by convection diffusion equations," *Journal of Scientific Computing*, vol. 41, pp. 273–299, 2009.

[9] H. Yücel, M. Heinkenschloss, and B. Karasözen, "An adaptive discontinuous Galerkin method for convection dominated distributed optimal control problems," Tech. Rep., Department of Computational and Applied Mathematics, Rice University, Houston, Tex, USA, 2012.

[10] H. Yücel, M. Heinkenschloss, and B. Karasözen, "Distributed optimal control of diffusion-convection-reaction equations using discontinuous Galerkin methods," in *Numerical Mathematics and Advanced Applications*, vol. 2011, pp. 389–397, Springer, Heidelberg, Germany, 2013.

[11] Z. Zhou and N. Yan, "The local discontinuous Galerkin method for optimal control problem governed by convection diffusion equations," *International Journal of Numerical Analysis and Modeling*, vol. 7, no. 4, pp. 681–699, 2010.

[12] Z. Zhou, "A posteriori error estimates for discontinuous Galerkin approximation of non-stationary convection-diffusion optimal control problems," *International Journal of Computer Mathematics*, vol. 93, no. 12, pp. 2106–2123, 2016.

[13] H. Fu, "A characteristic finite element method for optimal control problems governed by convection-diffusion equations," *Journal of Computational and Applied Mathematics*, vol. 235, no. 3, pp. 825–836, 2010.

[14] Z. J. Zhou and N. N. Yan, "A survey of numerical methods for convection-diffusion optimal control problems," *Journal of Numerical Mathematics*, vol. 22, no. 1, pp. 61–85, 2014.

[15] J. Wang and N. Yan, "A parallel domain decomposition procedure for convection diffusion problems," in *Domain Decomposition Methods in Sciences and Engineering (Beijing, 1995)*, pp. 333–339, Wiley, Chichester, UK, 1997.

[16] P. A. Raviart and J. M. Thomas, "A mixed finite element method for 2nd order elliptic problems," in *Mathematical Aspects of Finite Element Methods*, vol. 606 of *Lecture Notes in Mathematics*, pp. 292–315, Springer, Berlin, Germany, 1977.

[17] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, vol. 15 of *Springer Series in Computational Mathematics*, Springer-Verlag, Berlin, Germany, 1991.

[18] C. Johnson and J. Pitkränta, "An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation," *Mathematics of Computation*, vol. 46, no. 173, pp. 1–26, 1986.

Hindawi Publishing Corporation
Journal of Mathematics
Volume 2016, Article ID 7308609, 11 pages
http://dx.doi.org/10.1155/2016/7308609

Research Article

Stability Analysis of the Periodic Solutions of Some Kinds of Predator-Prey Dynamical Systems

Neslihan Nesliye Pelen

Department of Mathematics, Ondokuz Mayis University, Samsun, Turkey

Correspondence should be addressed to Neslihan Nesliye Pelen; nesliyeaykir@gmail.com

Received 29 September 2016; Accepted 18 October 2016

Academic Editor: Ram Jiwari

Analysis of predator-prey dynamical systems that have the functional response which generalizes the other types of functional responses in two dimensions is mainly studied in this paper. The main problems for this study are to detect the if and only if conditions for attaining the periodic solution of the considered system and to find the condition for global asymptotic stability of this solution for some different types of predator-prey systems that are obtained from that system. To get the desired results, some aspects of semigroup theory for stability analysis and coincidence degree theory are used.

1. Introduction

Predator-prey dynamic systems mainly investigate the relationships between the species and the relation of the species with the outer environment. Analysis of such kind of mathematical models is really important because, by using these analytical results, one can see the future of the species. The analytical results for these systems change according to two main issues. The first one is the functional response, which shows the effect of predator on prey and the effect of prey on predator. The second one is being in the periodic environment. In this study, by using these two issues, the necessary and sufficient conditions to have globally attractive or globally asymptotically stable solution for some different types of predator-prey systems are found.

First, let us give some information about the meaning of functional response and the types of functional responses. As it is remembered above, functional responses show how and how much predator gets benefit from the prey and how and how much prey is affected by predator. There are many types of functional responses. Some of them are Holling type, Beddington-DeAngelis type, ratio type, monotype, semiratio type, and so on. The following studies are some examples about predator-prey models with Holling type functional response: [1–3]. Studies that are about Beddington-DeAngelis type functional response are [4–9]. Predator-prey systems with other types of functional responses are studied in [10–13]. In this study, the main aim is to formulize the functional responses in a general form. In other words, the essential object for this study is to be able to express different types of functional responses in one functional response. Therefore, the dynamical properties that are valid for this system become also valid and useful for the other types.

The second issue in that study is to be in a periodic environment. In nature, periodicity can be seen in many different circumstances. For example, many animals ovulate periodically or many insects have periodic life cycle. Therefore, the analysis of the predator-prey dynamic system in a periodic environment is very significant. On the other hand, in population growth model, the significantly studied problem is the stability and global existence of a positive periodic solution in that system. In an autonomous model, the globally stable equilibrium point is the same as the notion globally asymptotically stable positive periodic solution in a nonautonomous system. Hence, the main problem in this study is to determine under which conditions globally attractive positive periodic solution is attained for the systems with different functional responses that are obtained from the system with generalized functional response. Additionally the importance of this issue can be seen from previous studies, since this subject is investigated in those ones. Some of the examples can be given as follows: [3, 10–20].

In study [21], some kinds of impulsive predator-prey systems have been investigated on time scales. Nevertheless, the functional response that affects predator is not used in [21]. Only its effect on prey is considered. Hence, the results of that paper only related to generalized type of semiratio dependent functional responses. However, in this study, the main aim is to obtain a model with a generalized functional response both on prey and on predator. Because of the above reason, this study has important contributions.

On the other hand, the systems with Holling type I and II functional responses which are obtained from the system with generalized functional response are also studied. For the system with Holling type I functional responses, in paper [22], an if and only if condition has been found to obtain the permanence of the given system. However, by using the main result of our study, an if and only if condition can be given for the globally asymptotically stable periodic solution of the considered system. Also in the study of [22], in the predator part, to bound the solution of predator from above an extra term has been used, but we can get rid of that extra term in this study. These are some points that show the importance of our study.

As a result, the primary objective of that study is to generalize the functional responses that act both on predator and on prey. Additionally, the second aim is to find the if and only if condition for the globally asymptotically stable periodic solution of the considered systems with some different types of functional responses.

2. Preliminaries

As preliminary, we use Definition 1, Lemma 1, Theorem 1, and necessary information that is needed for Theorem 1 from [4].

3. Main Result

The following is the main equation for this study:

$$\begin{aligned}\tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) \\ &\quad - \Phi_1(t,\tilde{x}(t),\tilde{y}(t))\,\tilde{y}(t), \\ \tilde{y}'(t) &= -d(t)\tilde{y}(t) + \Phi_2(t,\tilde{x}(t),\tilde{y}(t))\,\tilde{x}(t).\end{aligned} \tag{1}$$

In system (1), Φ_1 satisfies the following: $\Phi_1(t,0,y) = 0$, is continuous and w-periodic, and satisfies the following inequality:

$$\begin{aligned}\gamma_0(t)x^{m_1}(t) + \cdots + \gamma_{m_1-1}(t)x(t) &\le \Phi_1(t,x(t),y(t)) \\ &\le \alpha_0(t)x^{m_1}(t) + \cdots + \alpha_{m_1-1}(t)x(t).\end{aligned} \tag{2}$$

Assumptions on Φ_2 are as follows: $\Phi_2(t,x,0) = 0$, is continuous and w-periodic, and satisfies the following inequality:

$$\Phi_2(t,x(t),y(t)) \le \beta_0(t)y^{m_2} + \cdots + \beta_{m_2-1}(t)y(t). \tag{3}$$

For the above inequalities, α_i, β_j, and γ_j are positive, w-periodic coefficient functions for $i = 1,2,\ldots,m_1-1$ and $j = 1,2,\ldots,m_2-1$. Also, $\Phi_1(t,x(t),y(t))$ and $\Phi_2(t,x(t),y(t)) > 0$ if $x(t)$ and $y(t) > 0$. In addition to these conditions, $\Phi_2(t,\exp(x),\exp(y)) - d\Phi_2(t,\exp(x),\exp(y))/dy > 0$ and $a(t)$, $b(t)$, and $d(t)$ are positive and w-periodic coefficient functions.

(1) $a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t)$ is the specific growth rate of the prey in the absence of predator;

(2) $d(t)$ is the death rate of predator;

(3) $-\Phi_1(t,\tilde{x}(t),\tilde{y}(t))\tilde{y}(t)$ is generalized effect of predator on prey;

(4) $\Phi_2(t,\tilde{x}(t),\tilde{y}(t))\tilde{x}(t)$ is the generalized effect of prey on predator.

Assume that $\tilde{y}(t) = \exp(y(t))$ and $\tilde{x}(t) = \exp(x(t))$. Therefore, system (1) becomes equal to the following system:

$$\begin{aligned}x'(t) &= a(t) - b(t)\exp(x(t)) \\ &\quad - \Phi_1(t,\exp(x(t)),\exp(y(t)))\exp(y(t)-x(t)), \\ y'(t) &= -d(t) \\ &\quad + \Phi_2(t,\exp(x(t)),\exp(y(t)))\exp(x(t)-y(t)).\end{aligned} \tag{4}$$

Theorem 1. *In system (4), all of the coefficient functions satisfy the above conditions. System (4) has at least one w-periodic solution if and only if $\int_0^w a(t)dt > 0$ and $\int_0^w d(t)dt > 0$.*

Proof. When there is at least one w-periodic solution for system (4), then it is apparent that $\int_0^w a(t)dt > 0$ and $\int_0^w d(t)dt > 0$. Since

$$\begin{aligned}\int_0^w a(t)\,dt &= \int_0^w b(t)\exp(x(t)) \\ &\quad + \Phi_1(t,\exp(x(t)),\exp(y(t))) \\ &\quad \cdot \exp(y(t)-x(t))\,dt > 0, \\ \int_0^w d(t)\,dt &= \int_0^w \Phi_2(t,\exp(x(t)),\exp(y(t))) \\ &\quad \cdot \exp(x(t)-y(t))\,dt > 0,\end{aligned} \tag{5}$$

now, for the converse part, assume that $\int_0^w a(t)dt, \int_0^w d(t)dt > 0$ and try to find the fact that system (4) has at least one w-periodic solution.

Let X, Y be the normed vector spaces and let the operators L, N be the same as they are defined in the proof of Theorem 2 in [4]. By using the similar proof techniques of Theorem 2 from [4], then for any open bounded set $\Omega \subset X$, we can obtain that N is L-compact on $\overline{\Omega}$.

The below system is investigated by using the application of Theorem 1 from [4]:

$$\begin{aligned} x'(t) &= \lambda\left[a(t) - b(t)\exp(x(t))\right. \\ &\quad \left.- \Phi_1(t, \exp(x(t)), \exp(y(t)))\exp(y(t) - x(t))\right], \\ y'(t) &= \lambda\left[-d(t)\right. \\ &\quad \left.+ \Phi_2(t, \exp(x(t)), \exp(y(t)))\exp(x(t) - y(t))\right]. \end{aligned} \tag{6}$$

Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in X$ solve system (6). If system (6) is integrated from 0 to w, then

$$\begin{aligned} \int_0^w a(t)\,dt &= \int_0^w b(t)\exp(x(t)) \\ &\quad + \Phi_1(t, \exp(x(t)), \exp(y(t))) \\ &\quad \cdot \exp(y(t) - x(t))\,dt, \\ \int_0^w d(t)\,dt &= \int_0^w \Phi_2(t, \exp(x(t)), \exp(y(t))) \\ &\quad \cdot \exp(x(t) - y(t))\,dt. \end{aligned} \tag{7}$$

By using the first equations of (6) and (7), we get

$$\int_0^w \left|x'(t)\right| dt \le \lambda\left[2\int_0^w a(t)\,dt\right] \le S_1, \tag{8}$$

where $S_1 := 2\int_0^w a(t)dt$.

By using the second equations of (6) and (7), we have

$$\int_0^w \left|y'(t)\right| dt \le \lambda\left[2\int_0^w d(t)\,dt\right] \le S_2; \tag{9}$$

here $S_2 := 2\int_0^w d(t)dt$.

On compact sets, continuous functions attain its maximum and minimum. Therefore, there exist ψ_i, θ_i, $i = 1, 2$, for $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in X$ such that

$$\begin{aligned} x(\theta_1) &= \min_{t\in[0,w]} x(t), \\ x(\psi_1) &= \max_{t\in[0,w]} x(t), \end{aligned} \tag{10}$$

$$\begin{aligned} y(\theta_2) &= \min_{t\in[0,w]} y(t), \\ y(\psi_2) &= \max_{t\in[0,w]} y(t). \end{aligned} \tag{11}$$

The following is obtained by using the first equations of (7) and (11):

$$x(\theta_1) < \tilde{l}_1, \quad \text{where } \tilde{l}_1 := \ln\left(\frac{\int_0^w a(t)\,dt}{\int_0^w b(t)\,dt}\right). \tag{12}$$

Using the first inequality in Lemma 1 from [4] and inequality (8) we have

$$\begin{aligned} x(t) &\le x(\theta_1) + \int_0^w \left|x'(t)\right| dt \\ &\le x(\theta_1) + \left(2\int_0^w a(t)\,dt\right) < P_1 := \tilde{l}_1 + S_1. \end{aligned} \tag{13}$$

Using the first equation of (7), the below result is obtained:

$$\begin{aligned} \int_0^w a(t)\,dt &\ge \int_0^w \left[\gamma_0(t)\exp(x(t))^{m_1} + \cdots\right. \\ &\quad \left.+ \gamma_{m_1-1}(t)\exp(x(t))\right]\exp(y(t) - x(t))\,dt \\ &\ge \int_0^w \gamma_{m_1-1}(t)\exp(y(t) - x(t))\,dt \\ &\ge \exp(y(\theta_2))\int_0^w \gamma_{m_1-1}(t)\,dt \ge \exp(y(\theta_2)) \\ &\quad \cdot \int_0^w \gamma_{m_1-1}(t)\,dt. \end{aligned} \tag{14}$$

Therefore,

$$\exp(y(\theta_2)) \le \frac{\int_0^w d(t)\,dt}{\int_0^w \gamma_{m_1-1}(t)\,dt} := \tilde{l}_2. \tag{15}$$

Using the first inequality in Lemma 1 from [4] and inequality (9), we get

$$\begin{aligned} y(t) &\le y(\theta_2) + \int_0^w \left|y'(t)\right| dt \\ &\le y(\theta_2) + \left(2\int_0^w d(t)\,dt\right) < P_2 := \tilde{l}_2 - S_2. \end{aligned} \tag{16}$$

From the second equation of (7), we have

$$\begin{aligned} \int_0^w d(t)\,dt &\le \int_0^w \left[\beta_0(t)\exp(y(t))^{m_2} + \cdots\right. \\ &\quad \left.+ \beta_{m_2-1}\exp(y(t))\right]\exp(x(t) - y(t))\,dt \\ &\le \int_0^w \left[\beta_0(t)\exp(y(t))^{m_2-1} + \cdots + \beta_{m_2-1}\right] \\ &\quad \cdot \exp(x(t))\,dt \le \exp(x(\psi_1)) \\ &\quad \cdot \int_0^w \left[\beta_0(t)\exp(P_2)^{m_2-1} + \cdots + \beta_{m_2-1}\right] dt. \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} x(\psi_1) &\ge \ln\left(\frac{\int_0^w d(t)\,dt}{\int_0^w \left[\beta_0(t)\exp(P_2)^{m_2-1} + \cdots + \beta_{m_2-1}\right] dt}\right) \\ &:= \tilde{l}_3. \end{aligned} \tag{18}$$

Using the second inequality in Lemma 1 from [4] and inequality (8), we have

$$\begin{aligned} x(t) &\geq x(\psi_1) + \int_0^w \left|x'(t)\right| dt \\ &\geq x(\psi_1) + \left(2\int_0^w a(t)\,dt\right) > P_3 := \tilde{l}_3 + S_1. \end{aligned} \tag{19}$$

Again by using the first equations of (7) and (13), we get

$$\begin{aligned} \int_0^w a(t)\,dt &\leq \int_0^w \left[b(t)\exp(x(t)) \right.\\ &\quad \left. + \alpha_0(t)\exp(x(t))^{m_1} + \cdots + \alpha_{m_1-1}(t)\exp(x(t))\right] \\ &\quad \cdot \exp(y(t) - x(t))\,dt \\ &\leq \int_0^w \left[\alpha_0(t)\exp(x(t))^{m_1-1} + \cdots \right.\\ &\quad \left. + \left(\alpha_{m_1-1}(t) + b(t)\right)\right]\exp(y(t))\,dt \\ &\leq \exp(y(\psi_2))\int_0^w \left[\alpha_0(t)\exp(x(t))^{m_1-1} + \cdots \right.\\ &\quad \left. + \left(\alpha_{m_1-1}(t) + b(t)\right)\right]dt \leq \exp(y(\psi_2)) \\ &\quad \cdot \int_0^w \left[\alpha_0(t)\exp(P_1)^{m_1-1} + \cdots \right.\\ &\quad \left. + \left(\alpha_{m_1-1}(t) + b(t)\right)\right]dt. \end{aligned} \tag{20}$$

Therefore the following is obtained:

$$\begin{aligned} &\exp(y(\psi_2)) \\ &\geq \frac{\int_0^w a(t)\,dt}{\int_0^w \left[\alpha_0(t)\exp(P_1)^{m_1-1} + \cdots + \left(\alpha_{m_1-1}(t) + b(t)\right)\right]dt} \\ &:= \tilde{l}_4. \end{aligned} \tag{21}$$

Using the second inequality in Lemma 1 from [4] and (9), we get

$$\begin{aligned} y(t) &\geq y(\psi_2) - \int_0^w \left|y'(t)\right| dt \\ &\geq y(\psi_2) - \left(2\int_0^w d(t)\,dt\right) > P_4 := \tilde{l}_4 - S_2. \end{aligned} \tag{22}$$

By (13) and (19), we have $\sup_{t\in[0,w]}|x(t)| \leq C_1 := \max\{|P_1|, |P_3|\}$ and by (16) and (22), we have $\sup_{t\in[0,w]}|y(t)| \leq C_2 := \max\{|P_2|, |P_4|\}$. Here, C_1 and C_2 are independent from λ. Let $S = C_1 + C_2 + 1$. In addition to these,

$$\max_{t\in[0,w]} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| < S. \tag{23}$$

Suppose that $\Omega = \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in X : \|\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\| < S\}$ verifies the first requirement of Theorem 1 from [4]. When $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{Ker} L \cap \partial\Omega$, then we have $\|\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\| = S$, where S is a constant. Thus,

$$QN\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\left(\begin{bmatrix} \int_0^w a(s) - b(s)\exp(x) - \Phi_1(s, \exp(x), \exp(y))\exp(y - x)\,ds \\ \int_0^w -d(s) + \Phi_2(t, \exp(x), \exp(y))\exp(x - y)\,ds \end{bmatrix}\right). \tag{24}$$

The operator $J : \operatorname{Im} V \rightarrow \operatorname{Ker} L$ is taken as the identity operator and the homotopy is defined as

$$H_\nu = \nu(JQN) + (1 - \nu)G, \tag{25}$$

where

$$\begin{aligned} &G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \begin{bmatrix} \int_0^w a(s) - b(s)\exp(x)\,ds \\ \int_0^w d(s) - \Phi_2(t, \exp(x), \exp(y))\exp(x - y)\,ds \end{bmatrix}. \end{aligned} \tag{26}$$

Let DJ_G be the determinant of the Jacobian of G. We know that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \operatorname{Ker} L$; then the Jacobian of G is

$$\begin{bmatrix} -e^x \int_0^w b(s)\,ds & 0 \\ \int_0^w e^{(x-y)}\left[\Phi_2(t, \exp(x), \exp(y)) + \frac{d\Phi_2(t, \exp(x), \exp(y))}{dx}\right] ds & -\int_0^w e^{(x-y)}\left[\Phi_2(t, \exp(x), \exp(y)) - \frac{d\Phi_2(t, \exp(x), \exp(y))}{dy}\right] ds \end{bmatrix}. \tag{27}$$

Here sign DJ_G is always positive and this is obtained by the assumption of the theorem. Hence

$$\begin{aligned}\deg(JN, \Omega \cap \operatorname{Ker} L, 0) &= \deg(G, \Omega \cap \operatorname{Ker} L, 0)\\ &= \sum_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in G^{-1}\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right)} \operatorname{sign} DJ_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq 0.\end{aligned} \tag{28}$$

Thus, all the conditions of Theorem 1 from [4] are satisfied. Therefore, system (1) has at least one positive w-periodic solution. □

3.1. Predator-Prey Systems with Different Types of Functional Responses

3.1.1. Predator-Prey System with Holling Type Functional Response. One of the applications of the above result for Theorem 1 is the Holling type functional response from [1].

System (3.13) from [1] for continuous case is

$$\begin{aligned}\tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - \frac{c(t)\tilde{x}(t)\tilde{y}(t)}{1+m(t)\tilde{x}(t)},\\ \tilde{y}'(t) &= -d(t)\tilde{y}(t) + \frac{f(t)\tilde{y}(t)\tilde{x}(t)}{1+m(t)\tilde{x}(t)}.\end{aligned} \tag{29}$$

When $\tilde{x}(t) = \exp(x(t))$ and $\tilde{y}(t) = \exp(y(t))$, we obtain the transformed form of this system as

$$\begin{aligned}x'(t) &= a(t) - b(t)\exp(x(t)) - \frac{c(t)\exp(y(t))}{1+m(t)\exp(x(t))},\\ y'(t) &= -d(t) + \frac{f(t)\exp(x(t))}{1+m(t)\exp(x(t))}.\end{aligned} \tag{30}$$

After assuming the system has positive coefficient functions, it is obvious that we can take

$$\begin{aligned}\Phi_1(t, x(t), y(t)) &= \frac{c(t)x(t)}{1+m(t)x(t)},\\ \Phi_2(t, x(t), y(t)) &= \frac{f(t)y(t)}{1+m(t)x(t)}.\end{aligned} \tag{31}$$

It is also apparent that $\Phi_1(t, x(t), y(t)) \leq c(t)x(t)$ and $\Phi_2(t, x(t), y(t)) \leq f(t)y(t)$. Here if prey does not go to extinction, then there exists $k \in \mathbb{N}$ such that for sufficiently large k, $\Phi_1(t, x(t), y(t)) \geq (1/k)c(t)x(t)$, since $c(t)x(t)/(1+m(t)x(t)) > 0$. Therefore, Theorem 1 has equal statement with the following remark for the system with Holling type functional response.

Remark 2. For system (3.13) from [1] for continuous case, there exists at least one w-periodic solution if and only if the prey does not become extinct.

Remark 3. Here by using proof techniques of Lemma 3 from [4], it can be seen that if predator does not become extinct, then prey also does not become extinct. Therefore, Remark 2 becomes as follows: system (3.13) from [1] for continuous case has w-periodic solution if and only if predator does not become extinct.

Lemma 4. *For system (3.13) from [1] for continuous case, predator becomes extinct if and only if*

$$\int_0^w -d(t) + \frac{f(t)x^*(t)}{1+m(t)x^*(t)} \leq 0, \tag{32}$$

where $x^*(t) = (1-\exp(-\int_0^w a(s)ds))/\int_0^w b(t-s)\exp(-\int_0^s a(t-\tau)d\tau)ds$. *Here* $x^*(t)$ *is the globally attractive unique solution of the system*

$$x(t) = a(t)x(t) - b(t)x^2(t). \tag{33}$$

Proof. Assume that predator goes to extinction. The first equation of the predator-prey system with Holling type functional response is

$$\tilde{x}'(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - \frac{c(t)\tilde{x}(t)\tilde{y}(t)}{1+m(t)\tilde{x}(t)}. \tag{34}$$

Since as t tends to infinity, predator or $\tilde{y}(t) = \exp(y(t))$ tends to zero. Hence, equality (34) tends to the following equation as t tends to infinity:

$$\tilde{x}'(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t). \tag{35}$$

This means that as t tends to infinity the solution of $\tilde{x}(t)$ tends to $x^*(t)$.

Since predator goes to extinction, when we take $\exp(y(t)) = \tilde{y}(t)$, then from the second equation of this transformed version of the system with Holling type functional response the following inequality is obtained as

$$y(t) = \int_0^w -d(t) + \frac{f(t)\exp(x(t))}{1+m(t)\exp(x(t))} \leq 0. \tag{36}$$

Since t tends to infinity, the solution of $\tilde{x}(t) = \exp(x(t))$ tends to $x^*(t)$. Thus, equality (36) tends to

$$y(t) = \int_0^w -d(t) + \frac{f(t)x^*(t)}{1+m(t)x^*(t)}. \tag{37}$$

Since $\int_0^w -d(t) + f(t)\exp(x(t))/(1+m(t)\exp(x(t))) \leq 0$, then also $\int_0^w -d(t) + f(t)x^*(t)/(1+m(t)x^*(t)) \leq 0$.

For the converse, let us assume that $\int_0^w -d(t) + f(t)x^*(t)/(1+m(t)x^*(t)) \leq 0$. Consider the first equation of the system with Holling type functional response, then we obtain

$$\begin{aligned}\tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - \frac{c(t)\tilde{x}(t)\tilde{y}(t)}{1+m(t)\tilde{x}(t)}\\ &\leq a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t).\end{aligned} \tag{38}$$

About the solution of prey, the result $\exp(x(t)) = \tilde{x}(t) \leq x^*(t)$ is obtained by the comparison theorem for ODEs.

Since $\int_0^w -d(t)+f(t)x^*(t)/(1+m(t)x^*(t)) \leq 0$, then $y(t) \leq 0$, which means predator goes to extinction. Hence the proof follows. □

Lemma 5. *In system (3.13) from [1] take $\mathbb{T} = \mathbb{R}$; then this system has at least one w-periodic solution if and only if*

$$\int_0^w -d(t) + \frac{f(t)x^*(t)}{1+m(t)x^*(t)} > 0, \tag{39}$$

where $x^*(t) = (1-\exp(-\int_0^w a(s)ds))/\int_0^w b(t-s)\exp(-\int_0^s a(t-\tau)d\tau)ds$.

Proof. By using Lemma 4 and Remarks 2 and 3, one can prove Lemma 5. □

The permanence definition is taken from [19].

Corollary 6. *For system (3.13) from [1] for continuous case, the solution is permanent if and only if (39) is satisfied.*

Proof. Let us take (39) as satisfied. Therefore, from Lemma 4, predator does not become extinct and from Remark 3, prey does not become extinct also which means that both $\tilde{x}(t)$ and $\tilde{y}(t)$ are bounded from below with a positive number. From [22] (Proposition 3.1), the solution of the prey $\tilde{x}(t)$ is bounded from above. Assume that (39) is satisfied and $\tilde{y}(t)$ is not bounded from above which means that as t tends to infinity $\tilde{y}(t)$ also tends to infinity. Consider the first equation of transformed version of the system with Holling type functional response, which is

$$\begin{aligned} x'(t) &= a(t) - b(t)\exp(x(t)) \\ &\quad - \frac{c(t)\exp(y(t))}{1+m(t)\exp(x(t))}. \end{aligned} \tag{40}$$

Because of the assumption, as t tends to infinity $\tilde{y}(t) = \exp(y(t))$ also tends to infinity; then also $y(t)$ tends to infinity. Since we have found above that $\tilde{x}(t) = \exp(x(t))$ is bounded from above and below with positive constants, then $x(t)$ is bounded from above and below with constants from real numbers. Hence, for sufficiently large T, $x'(t)$ tends to $-\infty$ as t tends to infinity. Therefore, $\tilde{x}(t)$ tends to 0 as t tends to infinity. This is a contradiction with the boundedness of $\tilde{x}(t)$ from below with a positive constant. Therefore, $\tilde{y}(t)$ is bounded from above. Hence the solution of the considered system is permanent.

For the converse, let us assume that the system is permanent. Then prey and predator do not go to extinction; then by Remark 2 the considered system has at least one w-periodic solution. Thus, by using Lemma 5 obviously, the system satisfies (39). □

Theorem 7. *Assume that (39) holds true. Then, the w-periodic solution of system (3.13) from [1] for continuous case is globally attractive or globally asymptotically stable.*

Proof. Proof is similar with the proof of Lemma 5 in [4]. □

Corollary 8. *Consider the same conditions for the coefficient functions of system (3.13) from [1]. There exists w-periodic globally attractive solution for this system if and only if (39) is satisfied.*

Proof. The result is obtained from Theorem 7 and Lemma 5. □

3.1.2. Predator-Prey System with Holling Type II Functional Response. As a second case, consider the following predator-prey system with Holling type II functional response which is the same as system (3.14) from [1] for continuous case:

$$\begin{aligned} \tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - \frac{c(t)\tilde{x}^2(t)\tilde{y}(t)}{1+m(t)\tilde{x}^2(t)}, \\ \tilde{y}'(t) &= -d(t)\tilde{y}(t) + \frac{f(t)\tilde{y}(t)\tilde{x}^2(t)}{1+m(t)\tilde{x}^2(t)}. \end{aligned} \tag{41}$$

When $\tilde{y}(t) = \exp(y(t))$ and $\tilde{x}(t) = \exp(x(t))$, the transformed form of this system is gotten as

$$\begin{aligned} x'(t) &= a(t) - b(t)\exp(x(t)) \\ &\quad - \frac{c(t)\exp(y(t)+x(t))}{1+m(t)\exp(2x(t))}, \\ y'(t) &= -d(t) + \frac{f(t)\exp(2x(t))}{1+m(t)\exp(2x(t))}. \end{aligned} \tag{42}$$

Assume that all the coefficient functions are positive for the above systems. Then

$$\begin{aligned} \Phi_1(t, x(t), y(t)) &= \frac{c(t)x^2(t)}{1+m(t)x^2(t)}, \\ \Phi_2(t, x(t), y(t)) &= \frac{f(t)x(t)y(t)}{1+m(t)x^2(t)}, \end{aligned} \tag{43}$$

$\Phi_1(t, x(t), y(t)) \leq c(t)x^2(t)$ and $\Phi_2(t, x(t), y(t)) \leq f(t)M_x y(t)$, where M_x is the maximum of the solution of system (41) for prey which can be found from Proposition 3.1 in [22]. Here if prey does not go to extinction, then there exists $k \in \mathbb{N}$ such that, for sufficiently large k, $\Phi_1(t, x(t), y(t)) \geq (1/k)c(t)x^2(t)$, since $c(t)x^2(t)/(1+m(t)x(t)) > 0$. Therefore, Theorem 1 has equal statement with the following remark for the systems with Holling type II functional response.

Remark 9. For continuous case of system (3.14) from [1] it has at least one w-periodic solution if and only if the prey does not go to extinction.

By using similar techniques of Lemma 3 from [4], it can be found that if predator does not go to extinction, then prey also does not go to extinction. Therefore, the statement of Remark 9 is the same as the following one: system (3.14) from [1] for continuous case has at least one w-periodic solution if and only if predator does not go to extinction.

Lemma 10. *For system (3.14) from [1], take $\mathbb{T} = \mathbb{R}$. Then predator becomes extinct if and only if*

$$\int_0^w -d(t) + \frac{f(t)x^{*2}(t)}{1+m(t)x^{*2}(t)} \leq 0, \tag{44}$$

where $x^*(t) = (1-\exp(-\int_0^w a(s)ds))/\int_0^w b(t-s)\exp(-\int_0^s a(t-\tau)d\tau)ds$. *Here* $x^*(t)$ *is the unique globally attractive solution of the system*

$$x(t) = a(t)x(t) - b(t)x^2(t). \tag{45}$$

Proof. Proof is similar to the proof Lemma 4. □

Lemma 11. *In system (3.14) from [1] when it is taken as $\mathbb{T} = \mathbb{R}$, the system has at least one w-periodic solution if and only if*

$$\int_0^w -d(t) + \frac{f(t)x^{*2}(t)}{1+m(t)x^{*2}(t)} > 0, \tag{46}$$

where $x^*(t) = (1-\exp(-\int_0^w a(s)ds))/\int_0^w b(t-s)\exp(-\int_0^s a(t-\tau)d\tau)ds$.

Proof. This lemma can be proven by Lemma 10 and Remark 9. □

Corollary 12. *In system (3.14) from [1] when it is taken as $\mathbb{T} = \mathbb{R}$, the solution of the considered system is permanent if and only if (46) is satisfied.*

Proof. The proof is similar to the proof of Corollary 6. □

Theorem 13. *Assume that (46) holds true. Hence, the w-periodic solution of system (3.14) from [1] for continuous case is globally attractive or globally asymptotically stable.*

Proof. This has a similar proof with the proof of Lemma 5 in [4]. □

Corollary 14. *Assume that the same conditions for the coefficient functions of system (3.14) from [1] hold. There exists w-periodic globally attractive solution for this system if and only if (46) is satisfied.*

Proof. By Theorem 13 and Lemma 11, one can get the desired result. □

3.1.3. Predator-Prey System with Beddington-DeAngelis Type Functional Response. Consider the following system which is taken from [4]:

$$\begin{aligned} \tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) \\ &\quad - \frac{c(t)\tilde{y}(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}, \\ \tilde{y}'(t) &= -d(t)\tilde{y}(t) + \frac{f(t)\tilde{x}(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}. \end{aligned} \tag{47}$$

If we take $\exp(x(t)) = \tilde{x}(t)$, then we have the following transformed system:

$$\begin{aligned} x'(t) &= a(t) - b(t)\exp(x(t)) \\ &\quad - \frac{c(t)\exp(y(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}, \\ y'(t) &= -d(t) \\ &\quad + \frac{f(t)\exp(x(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}. \end{aligned} \tag{48}$$

For the above systems, all the coefficient functions are positive. It is obvious that when we take

$$\begin{aligned} \Phi_1(t, x(t), y(t)) &= \frac{c(t)x(t)}{\alpha(t)+\beta(t)x(t)+m(t)y(t)}, \\ \Phi_2(t, x(t), y(t)) &= \frac{f(t)y(t)}{\alpha(t)+\beta(t)x(t)+m(t)y(t)}, \end{aligned} \tag{49}$$

$\Phi_1(t, x(t), y(t)) \leq (c(t)/\alpha(t))x(t)$ and $\Phi_2(t, x(t), y(t)) \leq (f(t)/\alpha(t))y(t)$. Here if prey does not go to extinction, then there exists $k \in \mathbb{N}$ such that, for sufficiently large k, $\Phi_1(t, x(t), y(t)) \geq (1/k)(c(t)/\alpha(t))x(t)$, since $c(t)x(t)/(\alpha(t)+\beta(t)x(t)+m(t)y(t)) > 0$. Hence, Theorem 1 is true for system (47). These findings are the same as the ones in [4].

4. Examples

Example 1.

$$\begin{aligned} x'(t) &= 3 - (\cos(t)+2)\exp(x(t)) - \frac{2\exp(y(t))}{\exp(x(t))+1}, \\ y'(t) &= -(0.1\sin(t)+0.5) + \frac{\exp(x(t))}{\exp(x(t))+1}. \end{aligned} \tag{50}$$

By doing some simple calculations, it can be seen that $x^* > 1$; therefore (39) is satisfied for Example 1, since

$$\int_0^{2\pi} -(0.1\sin(t)+0.5) + \frac{x^*}{x^*+1} > 0. \tag{51}$$

Figure 1 supports this result.

Even if we have changed the initial conditions, after a while still we obtain the same solution which shows the global attractivity of the solutions. Figure 2 supports this result.

Example 2.

$$\begin{aligned} x'(t) &= 3 - (\cos(t)+2)\exp(x(t)) - \frac{2\exp(y(t))}{\exp(x(t))+1}, \\ y'(t) &= -(0.1\sin(t)+4) + \frac{\exp(x(t))}{\exp(x(t))+1}. \end{aligned} \tag{52}$$

Example 2 does not satisfy inequality (39). This is easily seen by doing some simple calculations. Hence predator goes to extinction which satisfies Lemma 4. This result is supported by Figure 3.

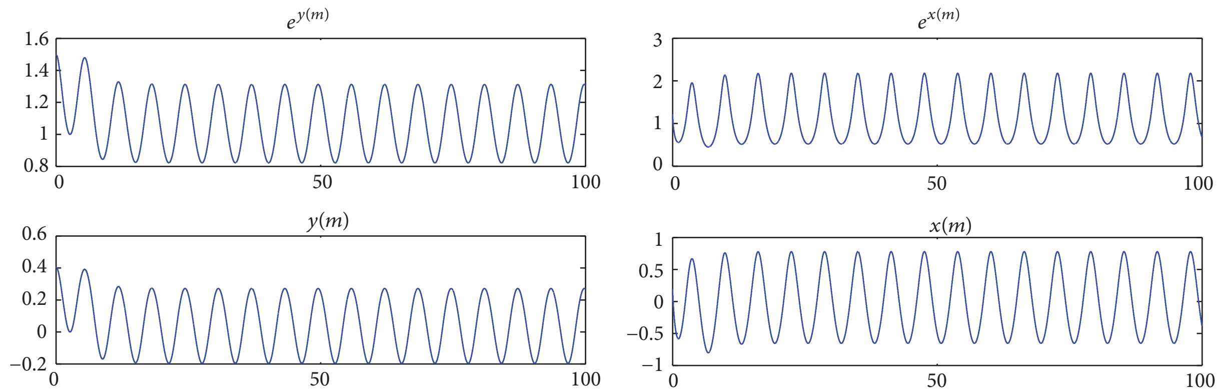


FIGURE 1: $x(0) = 0.2$ and $y(0) = 0.4$.

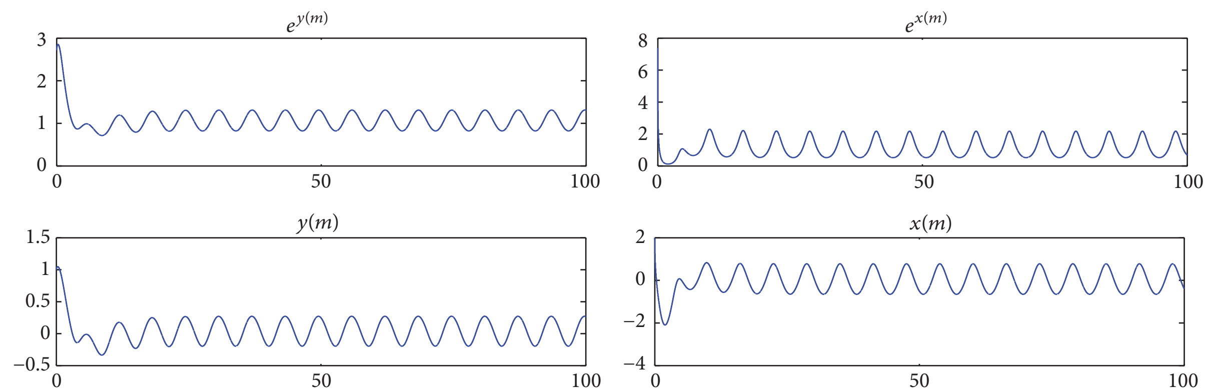


FIGURE 2: $x(0) = 2$ and $y(0) = 1$.

Example 3.

$$x'(t) = 3 - (\cos(t) + 2)\exp(x(t)) - \frac{2\exp(y(t) + x(t))}{\exp(2x(t)) + 1}, \quad y'(t) = -(0.1\sin(t) + 1) + \frac{3\exp(2x(t))}{\exp(2x(t)) + 1}. \tag{53}$$

By doing some simple calculations it can be seen that $x^* > 1$; therefore (46) is satisfied by Example 3, since

$$\int_0^{2\pi} -(0.1\sin(t) + 1) + \frac{3(x^*)^2}{(x^*)^2 + 1} > 0.5. \tag{54}$$

This is supported by Figure 4.

Even if we have changed the initial conditions, after a while, we still obtain the same solution which shows the global attractivity of the solutions. Figure 5 supports this.

Example 4.

$$x'(t) = 3 - (\cos(t) + 2)\exp(x(t)) - \frac{2\exp(y(t) + x(t))}{\exp(2x(t)) + 1}, \quad y'(t) = -(0.1\sin(t) + 6) + \frac{3\exp(2x(t))}{\exp(2x(t)) + 1}. \tag{55}$$

Example 4 does not satisfy inequality (46) and this can be easily seen by doing some simple calculations. Therefore, predator goes to extinction which satisfies Lemma 10 and this is supported by Figure 6.

For the examples of the system with Beddington-DeAngelis functional response, look at the study [4].

5. Discussion

In that study, two important analytical results are found. First, for the periodic solution of the system with generalized functional response, the if and only if condition is able to be found. Therefore, we are able to extend the study in [21] for continuous case. The second one is to be able to find an if and only if

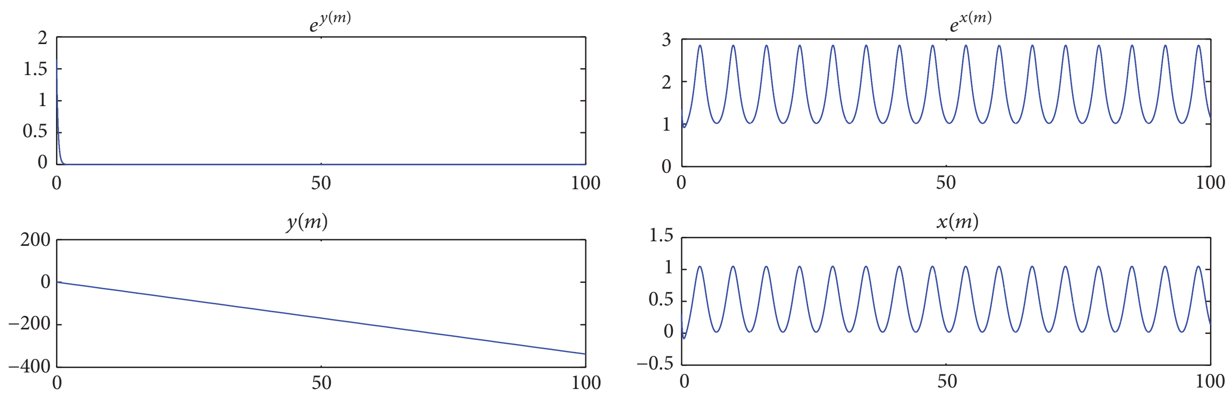


Figure 3: $x(0) = 0.3$ and $y(0) = 0.5$.

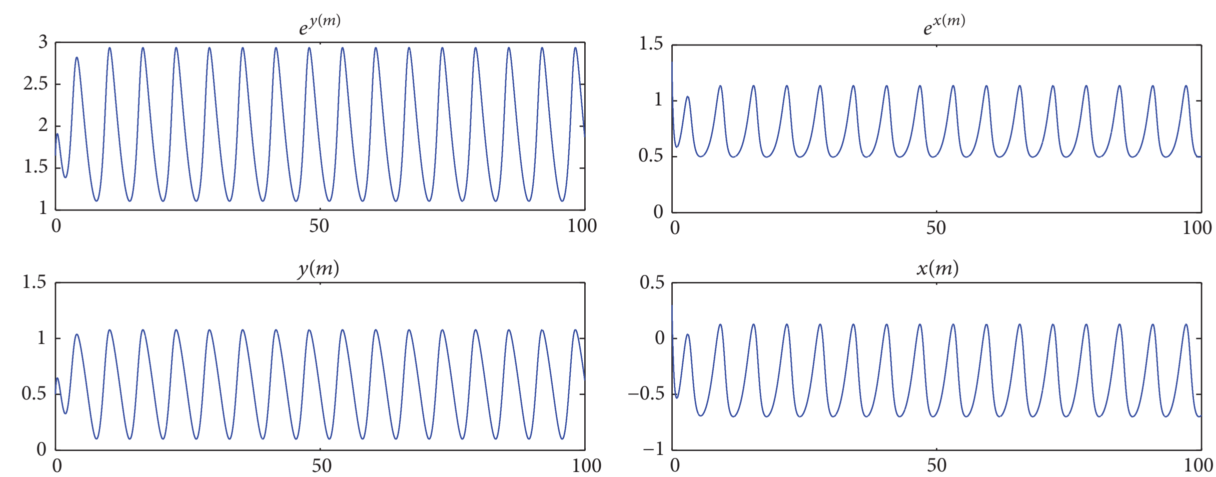


Figure 4: $x(0) = 0.3$ and $y(0) = 0.5$.

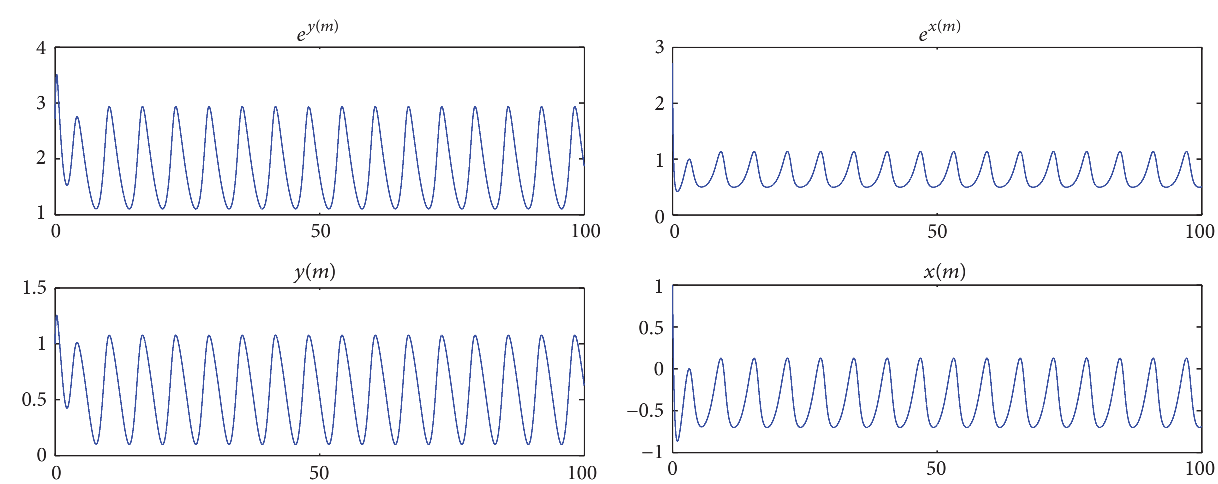


Figure 5: $x(0) = 1$ and $y(0) = 1$.

condition for the globally periodic solution of the predator-prey models with Holling type I and II functional responses. Additionally, the results of that paper is consistent with the findings of the paper [4] when the Beddington-DeAngelis type functional response is considered as an application.

Hence, the allover work in this study is important, since by using the results of that paper, the if and only if condition for at least one periodic solution and the globally attractive periodic solution can be found and these results can be generalized in many different types of functional responses.

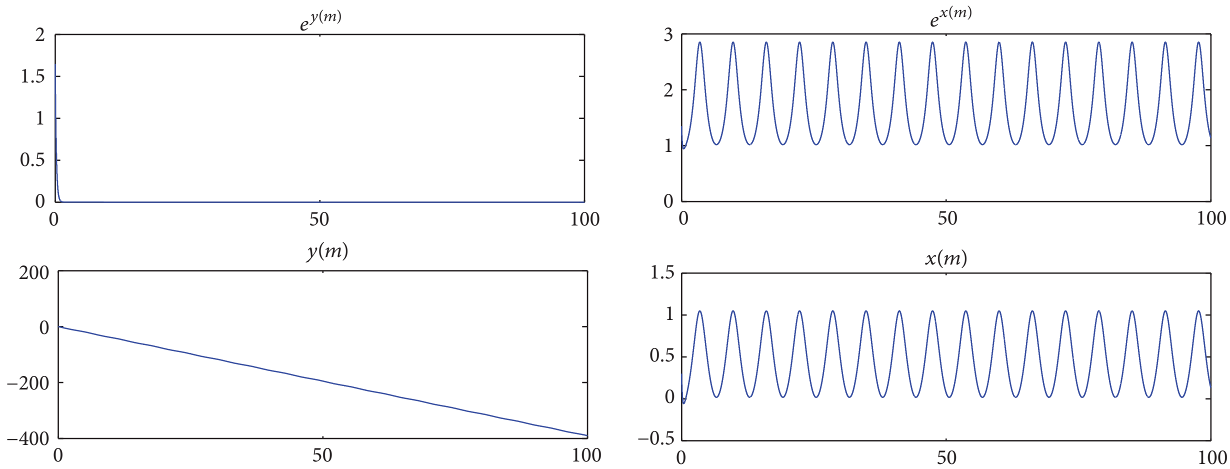


Figure 6: $x(0) = 0.3$ and $y(0) = 0.5$.

The suggested problem for the future works is to find the if and only if condition for the globally attractive periodic solution of the discrete predator-prey dynamic systems. Semigroup theory has been used in the analysis of the system when it has globally attractive periodic solution of the continuous system. For the discrete case, for further studies, the result that is related to the global attractivity of the system is another open problem.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this article.

References

[1] M. Bohner, M. Fan, and J. Zhang, "Existence of periodic solutions in predator-prey and competition dynamic systems," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 7, no. 5, pp. 1193–1204, 2006.

[2] W. Wang, J. Shen, and J. J. Nieto, "Permanence and periodic solution of predator-prey system with holling type functional response and impulses," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 81756, 15 pages, 2007.

[3] R. Xu, M. A. J. Chaplain, and F. A. Davidson, "Periodic solutions for a predator-prey model with Holling-type functional response and time delays," *Applied Mathematics and Computation*, vol. 161, no. 2, pp. 637–654, 2005.

[4] N. N. Pelen, A. F. Güvenilir, and B. Kaymakçalan, "Necessary and sufficient condition for existence of periodic solutions of predator-prey dynamic systems with Beddington-DeAngelis-type functional response," *Advances in Difference Equations*, vol. 2016, no. 1, article 15, 19 pages, 2016.

[5] N. N. Pelen, A. F. Güvenilir, and B. Kaymakçalan, "Behavior of the solutions for predator-prey dynamic systems with Beddington-DeAngelis type functional response on periodic time scales in shifts," *Abstract and Applied Analysis*, vol. 2016, Article ID 1463043, 9 pages, 2016.

[6] A. F. Guvenilir, B. Kaymakalan, and N. Pelen, "Impulsive predator-prey dynamic systems with Beddington-DeAngelis type functional response on the unification of discrete and continuous systems," *Applied Mathematics*, vol. 6, no. 9, pp. 1649–1664, 2015.

[7] C. Xu and M. Liao, "Existence of periodic solutions in a discrete predator-prey system with Beddington-DeAngelis functional responses," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 970763, 18 pages, 2011.

[8] L. Yang, J. Yang, and Q. Zhog, "Periodic solutions for a predator-prey model with Beddington-DeAngelis type functional response on time scales," *General Mathematics Notes*, vol. 3, no. 1, pp. 46–54, 2011.

[9] J. Zhang and J. Wang, "Periodic solutions for discrete predator-prey systems with the Beddington-DeAngelis functional response," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1361–1366, 2006.

[10] M. Fan and K. Wang, "Periodicity in a delayed ratio-dependent predator-prey system," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 1, pp. 179–190, 2001.

[11] M. Fan and Q. Wang, "Periodic solutions of a class of nonautonomous discrete time semi-ratio-dependent predator-prey systems," *Discrete and Continuous Dynamical Systems. Series B*, vol. 4, no. 3, pp. 563–574, 2004.

[12] H.-F. Huo, "Periodic solutions for a semi-ratio-dependent predator-prey system with functional responses," *Applied Mathematics Letters*, vol. 18, no. 3, pp. 313–320, 2005.

[13] Q. Wang, M. Fan, and K. Wang, "Dynamics of a class of nonautonomous semi-ratio-dependent predator-prey systems with functional responses," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 443–471, 2003.

[14] F. Chen, "Permanence and global stability of nonautonomous Lotka-Volterra system with predator-prey and deviating arguments," *Applied Mathematics and Computation*, vol. 173, no. 2, pp. 1082–1100, 2006.

[15] J. Cui and Y. Takeuchi, "Permanence, extinction and periodic solution of predator-prey system with Beddington-DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 317, no. 2, pp. 464–474, 2006.

[16] M. Fan and S. Agarwal, "Periodic solutions for a class of discrete time competition systems," *Nonlinear Studies*, vol. 9, no. 3, pp. 249–261, 2002.

[17] M. Fan and Y. Kuang, "Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 15–39, 2004.

[18] M. Fan and K. Wang, "Global periodic solutions of a generalized n-species Gilpin-Ayala competition model," *Computers & Mathematics with Applications*, vol. 40, no. 10-11, pp. 1141–1151, 2000.

[19] Q. Fang, X. Li, and M. Cao, "Dynamics of a discrete predator-prey system with Beddington-DeAngelis function response," *Applied Mathematics*, vol. 3, no. 4, pp. 389–394, 2012.

[20] H. Li and Y. Takeuchi, "Dynamics of the density dependent predator-prey system with Beddington-DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 374, no. 2, pp. 644–654, 2011.

[21] X. Liu and X. Liu, "Necessary and sufficient conditions for the existence of periodic solutions in a predator-prey model on time scales," *Electronic Journal of Differential Equations*, vol. 2012, no. 199, pp. 1–13, 2012.

[22] J. Cui, "Permanence of predator-prey system with periodic coefficients," *Mathematical and Computer Modelling*, vol. 42, no. 1-2, pp. 87–98, 2005.