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Fractals, Vol. 28, No. 8 (2020) 2040035 (11 pages)
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DOI: 10.1142/S0218348X20400356

ON THE NEW EXPLICIT SOLUTIONS OF THE FRACTIONAL NONLINEAR SPACE-TIME NUCLEAR MODEL

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Received December 29, 2019
Accepted February 17, 2020
Published June 25, 2020

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Abstract

In this research, the analytical and numerical solutions of the fractional nonlinear space-time Phi-four model are investigated by employing two systematic schemes and the B-spline schemes. A new fractional operator definition is applied to this model to convert the model from its fractional formula to an integer-order nonlinear ordinary differential equation. The considered model is of major interest for studying the nuclear interaction, elementary particles in a condensed medium, and propagation of dislocations in crystals. Explicit wave solutions are obtained.

Keywords: The Fractional Nonlinear Space-Time Phi-Four Model; Modified Khater Method; B-Spline Schemes; $\mathcal{ABR}$ Fractional Operator.

1. INTRODUCTION

The fractional nonlinear partial differential equations are classified as a significant branch in mathematical analysis that describes many natural phenomena using differential operators.[1,2] This kind of equation is considered as the natural generalization of the classical integer order partial differential equation.[3–6] It has been used in different fields to formulate these phenomena in mathematical formats. The areas of fluid flow, finance, physics, electromagnetic, acoustics, electrochemistry, fluid mechanics, mathematical biology, bioengineering, and material science, to cite few, have applied fractional equation to study the specific solutions of their models.[7–11] So, many computational and semi-analytical techniques have been derived to find the exact and semi-analytical solutions that help to derive accurate numerical schemes.[12–28]

In this context, several research investigations on the Abel–Volterra integral equations of first and second kinds link to fractional integral equations.[29–33] These fractional equations and other models of the fractional partial differential equations at small scales have gained enough considerable importance and popularity as they present applications in various fields of science. They are considered as a powerful tool to describe many of the natural phenomena.

In this paper, we use recent analytical methods in this field, namely the modified Khater (mK) method,[34–41] and the generalized Kudryashov method. These methods guarantee different forms of closed solutions. Applying these recent methods to the fractional nonlinear space-time Phi-four equation[42–45] leads to computational solutions that are used in the next step to calculate the initial and boundary conditions using the Bspline scheme[46–49] to find the numerical solution. This numerical solution is used to estimate the absolute error value, which shows the accuracy of our obtained solutions. Moreover, the obtained solutions show more novel physical properties for the relativistic electrons, spineless relativistic composite particles, and de Broglie waves.

Phi-four equation has two general forms:

$$(\mathcal{P}^{\pm n})_{t\,t} - a\,(\mathcal{P}^{\pm n})_{x\,x} - \mathcal{P} + \mathcal{P}^{\pm n} = 0, \tag{1}$$

$$(\mathcal{P}^{\pm n})_{t\,t} - a\,(\mathcal{P}^{\pm n})_{x\,x} - \mathcal{P}^{m} + \mathcal{P}^{\pm n} = 0. \tag{2}$$

These two models have kink and anti-kink soliton solutions which are predestined at high velocities of the exact solution,

$$\mathcal{P}(x,t) = \pm\tanh\left(\frac{x - c\,t}{\sqrt{2\,(1-c^2)}}\right). \tag{3}$$

In this work, we explore a special fractional nonlinear space-time case of the general form of Phi-four equation for $n = 1$ given by

$$\mathcal{D}^{2\,\vartheta}_{t\,t}\mathcal{P} - \mathcal{D}^{2\,\vartheta}_{x\,x}\mathcal{P} + \sigma^2\mathcal{P} + \epsilon\,\mathcal{P}^3 = 0. \tag{4}$$

This model can be considered as a special case of the Klein–Gordon equation that governs the dynamics in particle physics. Moreover, the Phi-four model has played an essential role in nuclear and particle physics over the decades where it is used to describe the nuclear element interaction.

In this paper, the Atangana–Baleanu–Riemann ($\mathcal{ABR}$) fractional operator is applied to transform the fractional model to a nonlinear partial differential equation with integer order.[12,13,50]

Definition 1.1. $\mathcal{ABR}$ fractional operator is given by[51–60]

$$^{\mathcal{ABR}}\mathcal{D}^{\alpha}_{a+}\,\mathcal{F}(t) = \frac{\mathcal{B}(\alpha)}{1-\alpha}\,\frac{d}{d\,t}\int_a^t \mathcal{F}(x) \times \mathcal{E}_\alpha\left(\frac{-\alpha\,(t-\alpha)^\alpha}{1-\alpha}\right)dx, \quad (5)$$

where $\mathcal{E}_\alpha$ is the Mittag–Leffler function, and is defined by the following formula:

$$\mathcal{E}_\alpha\left(\frac{-\alpha\,(t-\alpha)^\alpha}{1-\alpha}\right) = \sum_{n=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\right)^n\,(t-x)^{\alpha\,n}}{\Gamma(\alpha\,n+1)}$$

and $\mathcal{B}(\alpha)$ is a normalization function. Thus

$$^{\mathcal{ABR}}\mathcal{D}^{\alpha}_{a+}\,\mathcal{F}(x) = \frac{\mathcal{B}(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n \times \frac{1}{n\,\alpha}\,{}^{RL}\mathcal{I}_a^{\alpha\,n}\,\mathcal{F}(x). \quad (6)$$

The structure of this paper is is a follows: Section 2 employs the mK method and the generalized Kudryashov method to get abundant explicit wave solutions of the fractional nonlinear space-time Phi-four equation; then use we these solutions in the investigation of numerical solutions by using the B-spline schemes. Section 3 investigates the numerical solutions of the model by employing the B-spline schemes based on the obtained analytical solution. Section 4 shows and discusses the obtained results in our research paper. Section 5 concludes our study.

2. ANALYTICAL SOLUTIONS

Here, we apply two computational and three numerical schemes to the fractional nonlinear space-time Telegraph equation as follows: Using the following wave transformation with the definition of $\mathcal{ABR}$ fractional operator

$$\mathcal{T}(x,t) = \mathcal{T}(\Im),\qquad \Im = \frac{(1-\alpha)\left(c\,t^{\alpha(-n)} + \lambda\,x^{\alpha(-n)}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)},$$

in Eq. (4), we get

$$\left(c^2-\lambda^2\right)\mathcal{P}'' - \sigma^2\,\mathcal{P} + \epsilon\,\mathcal{P}^3 = 0. \quad (7)$$

2.1. Explicit Wave Solutions via the mK Method

According to the homogeneous balance rule between the highest derivative and the nonlinear term in Eq. (7), we obtain $n = 1$. Thus, the general solution of Eq. (7) is given by

$$\mathcal{P}(\Im) = \sum_{i=1}^{n} a_i\,\mathcal{K}^{i\mathcal{M}(\Im)} + \sum_{i=1}^{n} b_i\,\mathcal{K}^{-i\mathcal{M}(\Im)} + a_0 = a_1\,\mathcal{K}^{\mathcal{M}(\Im)} + a_0 + b_1\,\mathcal{K}^{-\mathcal{M}(\Im)}, \quad (8)$$

where a_0, a_1 and b_1 are arbitrary constants. Additionally, $\mathcal{M}(\Im)$ is the solution function of

$$\mathcal{M}'(\Im) = \frac{1}{Ln\,(\mathcal{K})}\left[\delta\,\mathcal{K}^{\mathcal{M}(\Im)} + \varrho\,\mathcal{K}^{-\mathcal{M}(\Im)} + \chi\right],$$

where δ, ϱ, and χ are arbitrary constants. Substituting Eq. (8) and its derivatives into Eq. (7) and then collecting all terms of $\mathcal{K}^{i\,\mathcal{M}(\Im)}$, $\{i = -3,-2,-1,0,1,2,3\}$, lead to a system of algebraic equations. Solving this system yields the following:

Family I:

$$\left[a_0 \to \frac{\sigma^{3/2}\chi}{\sqrt{\chi^2\epsilon - 4\delta\epsilon\varrho}}, a_1 \to \frac{2\delta\sigma^{3/2}}{\sqrt{-\epsilon\,(4\delta\varrho-\chi^2)}}, b_1 \to 0, \lambda \to -\frac{\sqrt{4c^2\delta\varrho - c^2\chi^2 - 2\sigma^3}}{\sqrt{4\delta\varrho-\chi^2}}\right],$$

where $[4\delta\varrho - \chi^2 < 0$ and $\sigma \neq 0$, $\delta \neq 0$ and $4c^2\delta\varrho > c^2\chi^2 + 2\sigma^3]$.

Thus, the explicit wave solutions of Eq. (4) are given as follows:

for $[\chi^2 - 4\delta\varrho > 0$ and $\delta \neq 0]$,

$$\mathcal{P}_1(x,t) = -\frac{\sigma^{3/2}\sqrt{\chi^2-4\delta\varrho}}{\sqrt{\epsilon\,(\chi^2-4\delta\varrho)}}\tanh\left(\frac{(1-\alpha)\sqrt{\chi^2-4\delta\varrho}\left(ct^{-\alpha} - \frac{x^{-\alpha}\sqrt{c^2(4\delta\varrho-\chi^2)-2\sigma^3}}{\sqrt{4\delta\varrho-\chi^2}}\right)}{2B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right), \quad (9)$$

$$\mathcal{P}_2(x,t) = -\frac{\sigma^{3/2}\sqrt{\chi^2-4\delta\varrho}}{\sqrt{\epsilon\left(\chi^2-4\delta\varrho\right)}} \coth\left(\frac{(1-\alpha)\sqrt{\chi^2-4\delta\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{c^2(4\delta\varrho-\chi^2)-2\sigma^3}}{\sqrt{4\delta\varrho-\chi^2}}\right)}{2B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right); \tag{10}$$

for $[\delta\varrho < 0,\ \delta \neq 0,\ \varrho \neq 0$ and $\chi = 0]$,

$$\mathcal{P}_3(x,t) = -\frac{\sigma^{3/2}\sqrt{-\delta\varrho}}{\sqrt{-\delta\epsilon\varrho}} \tanh\left(\frac{(1-\alpha)\sqrt{-\delta\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{4c^2\delta\varrho-2\sigma^3}}{2\sqrt{\delta\varrho}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right), \tag{11}$$

$$\mathcal{P}_4(x,t) = -\frac{\sigma^{3/2}\sqrt{-\delta\varrho}}{\sqrt{-\delta\epsilon\varrho}} \coth\left(\frac{(1-\alpha)\sqrt{-\delta\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{4c^2\delta\varrho-2\sigma^3}}{2\sqrt{\delta\varrho}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right); \tag{12}$$

for $[\chi = 0$ and $\varrho = -\delta]$,

$$\mathcal{P}_5(x,t) = -\frac{\sigma^{3/2}\varrho}{\sqrt{\epsilon\varrho^2}} \coth\left(\frac{(1-\alpha)\varrho\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{-4c^2\varrho^2-2\sigma^3}}{2\sqrt{-\varrho^2}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right); \tag{13}$$

for $[\chi = \delta = \kappa$ and $\varrho = 0]$,

$$\mathcal{P}_6(x,t) = -\frac{\kappa\sigma^{3/2}}{\sqrt{\kappa^2\epsilon}} \coth\left(\frac{(1-\alpha)\kappa\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{-c^2\kappa^2-2\sigma^3}}{\sqrt{-\kappa^2}}\right)}{2B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right); \tag{14}$$

for $[\varrho = 0,\ \chi \neq 0,$ and $\delta \neq 0]$,

$$\mathcal{P}_7(x,t) = \frac{\sigma^{3/2}\chi}{\sqrt{\chi^2\epsilon}}\left(-\frac{4}{\delta\exp\left(\frac{(1-\alpha)\chi\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{-c^2\chi^2-2\sigma^3}}{\sqrt{-\chi^2}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)-2}-1\right). \tag{15}$$

Family II:

$$\left[a_0 \to \frac{\sigma^{3/2}\chi}{\sqrt{\chi^2\epsilon-4\delta\epsilon\varrho}}, a_1 \to 0, b_1 \to \frac{2\sigma^{3/2}\varrho}{\sqrt{-\epsilon\left(4\delta\varrho-\chi^2\right)}}, \lambda \to -\frac{\sqrt{4c^2\delta\varrho-c^2\chi^2-2\sigma^3}}{\sqrt{4\delta\varrho-\chi^2}}\right]$$

where $[4\delta\varrho-\chi^2 < 0,\ \sigma \neq 0,\ \varrho \neq 0$ and $4c^2\delta\varrho > c^2\chi^2+2\sigma^3]$.

Thus, the explicit wave solutions of Eq. (4) are given as follows:
for $[\chi^2-4\delta\varrho > 0$ and $\delta \neq 0]$,

$$\mathcal{P}_8(x,t) = \frac{\sigma^{3/2}}{\sqrt{\epsilon(\chi^2-4\delta\varrho)}}\left(\chi-\frac{4\delta\varrho}{\sqrt{\chi^2-4\delta\varrho}\coth\left(\frac{(1-\alpha)\sqrt{\chi^2-4\delta\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{c^2(4\delta\varrho-\chi^2)-2\sigma^3}}{\sqrt{4\delta\varrho-\chi^2}}\right)}{2B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)+\chi}\right)1,$$

$$\mathcal{P}_9(x,t)=\frac{\sigma^{3/2}}{\sqrt{\epsilon(\chi^2-4\delta\varrho)}}\left(\chi-\frac{4\delta\varrho}{\sqrt{\chi^2-4\delta\varrho}\tanh\left(\frac{(1-\alpha)\sqrt{\chi^2-4\delta\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{c^2(4\delta\varrho-\chi^2)-2\sigma^3}}{\sqrt{4\delta\varrho-\chi^2}}\right)}{2B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)+\chi}\right);$$

for $[\delta\varrho<0,\ \delta\neq 0,\ \varrho\neq 0$ and $\chi=0]$,

$$\mathcal{P}_{10}(x,t)=\frac{\sqrt{\delta}\sigma^{3/2}\sqrt{\varrho}}{\sqrt{-\delta\epsilon\varrho}}\cot\left(\frac{(1-\alpha)\sqrt{\delta}\sqrt{\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{4c^2\delta\varrho-2\sigma^3}}{2\sqrt{\delta\varrho}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right), \tag{16}$$

$$\mathcal{T}_P(x,t)=\frac{\delta^{3/2}\sigma^{3/2}\epsilon\varrho^{3/2}}{(-\delta\epsilon\varrho)^{3/2}}\tan\left(\frac{(1-\alpha)\sqrt{\delta}\sqrt{\varrho}\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{4c^2\delta\varrho-2\sigma^3}}{2\sqrt{\delta\varrho}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right); \tag{17}$$

for $[\chi=0$ and $\varrho=-\delta]$,

$$\mathcal{P}_{13}(x,t)=\frac{\sigma^{3/2}\varrho}{\sqrt{\epsilon\varrho^2}}\tanh\left(\frac{(1-\alpha)\varrho\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{-4c^2\varrho^2-2\sigma^3}}{2\sqrt{-\varrho^2}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right); \tag{18}$$

for $[\chi=\frac{\varrho}{2}=\kappa$ and $\delta=0]$,

$$\mathcal{P}_{14}(x,t)=\frac{\sigma^{3/2}}{\sqrt{\kappa^2\epsilon}}\left(\frac{4\kappa}{\exp\left(\frac{(1-\alpha)\kappa\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{-c^2\kappa^2-2\sigma^3}}{\sqrt{-\kappa^2}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)-2}+\kappa\right); \tag{19}$$

for $[\delta=0,\ \chi\neq 0$ and $\varrho\neq 0]$,

$$\mathcal{P}_{15}(x,t)=\frac{\sigma^{3/2}}{\sqrt{\chi^2\epsilon}}\left(\chi-\frac{2\chi\varrho}{\varrho-\chi\exp\left(\frac{(1-\alpha)\chi\left(ct^{-\alpha}-\frac{x^{-\alpha}\sqrt{-c^2\chi^2-2\sigma^3}}{\sqrt{-\chi^2}}\right)}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n\Gamma(1-\alpha n)}\right)}\right). \tag{20}$$

2.2. Explicit Wave Solutions via the Generalized Kudryashov Method

Based on the homogeneous balance value and the general suggested solution by the simplest equation method, we get the general solution of Eq. (7) is given by

$$\mathcal{P}(\Im) = \frac{\sum_{i=0}^{m} a_i\, \mathcal{Q}(\Im)^i}{\sum_{j=0}^{n} b_j\, \mathcal{Q}(\Im)^j} = \frac{a_2\, \mathcal{Q}(\Im)^2 + a_1\, \mathcal{Q}(\Im) + a_0}{b_1\, \mathcal{Q}(\Im) + b_0}, \tag{21}$$

where a_0, a_1, a_2, b_0 and b_1 are arbitrary constants. Additionally, $\mathcal{Q}(\Im)$ is the solution function of $\mathcal{Q}'(\Im) \to \mathcal{Q}(\Im)^2 - \mathcal{Q}(\Im)$. Substituting Eq. (21) and its derivatives into Eq. (7) and then collecting all terms of $\mathcal{Q}^i(\Im)$, $\{i = 0, 1, 2, 3\}$, lead to a system of algebraic equations. Solving this system yields the following:

Family I:

$$\left[\begin{array}{l} a_0 \to \frac{a_1 b_0}{b_1 - 2b_0},\ a_2 \to -\frac{2a_1 b_1}{b_1 - 2b_0}, \epsilon \to \frac{(b_1-2b_0)^2(\lambda^2-c^2)}{2a_1^2}, \\ \quad \sigma \to \frac{\sqrt[3]{\lambda^2-c^2}}{\sqrt[3]{2}}, \\ \quad \text{where } (b_1 - 2b_0 \neq 0,\ a_1 \neq 0,\ b_0 \neq 0, \\ \qquad \text{and } \lambda^2 \neq c^2) \end{array}\right].$$

Thus, the solitary wave solutions of Eq. (4) are given by

$$\mathcal{P}_{16}(x,t) = \frac{a_1}{2b_0 - b_1}\left(\frac{2}{A\exp\left(\frac{(1-\alpha)(ct^{-\alpha}+\lambda x^{-\alpha})}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)}\right)+1} - 1\right). \tag{22}$$

Family II:

$$\left[\begin{array}{l} a_0 \to 0, a_2 \to -2a_1, b_0 \to 0, \epsilon \to \frac{b_1^2(\lambda^2-c^2)}{2a_1^2}, \sigma \to \frac{(-1)^{2/3}\sqrt[3]{\lambda^2-c^2}}{\sqrt[3]{2}} \\ \quad \text{where } (a_1 \neq 0,\ b_1 \neq 0 \text{ and } \lambda^2 \neq c^2) \end{array}\right].$$

Thus, the solitary wave solutions of Eq. (4) are given by

$$\mathcal{P}_{17}(x,t) = \frac{a_1}{b_1}\left(1 - \frac{2}{A\exp\left(\frac{(1-\alpha)(ct^{-\alpha}+\lambda x^{-\alpha})}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)}\right)+1}\right). \tag{23}$$

Family III:

$$\left[\begin{array}{l} a_0 \to 0, a_2 \to -a_1, b_1 \to -2b_0, \epsilon \to \frac{8(b_0^2\lambda^2 - b_0^2 c^2)}{a_1^2}\sigma \to \sqrt[3]{c^2-\lambda^2} \\ \quad \text{where } (a_1 \neq 0,\ b_1 \neq 0 \text{ and } \lambda^2 \neq c^2) \end{array}\right].$$

Thus, the solitary wave solutions of Eq. (4) are given by

$$\mathcal{P}_{18}(x,t) = \frac{a_1 A\exp\left(\frac{(1-\alpha)(ct^{-\alpha}+\lambda x^{-\alpha})}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)}\right)}{b_0\left(A^2\exp\left(\frac{2(1-\alpha)(ct^{-\alpha}+\lambda x^{-\alpha})}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)}\right)-1\right)}. \tag{24}$$

Family IV:

$$\left[\begin{array}{l} a_1 \to 0, a_2 \to -4a_0, b_1 \to 2b_0, \epsilon \to \frac{b_0^2(\lambda^2-c^2)}{2a_0^2}, \sigma \to \frac{\sqrt[3]{\lambda^2-c^2}}{\sqrt[3]{2}} \\ \quad \text{where } (a_1 \neq 0,\ b_1 \neq 0 \text{ and } \lambda^2 \neq c^2) \end{array}\right].$$

Thus, the solitary wave solutions of Eq. (4) are given by

$$\mathcal{P}_{19}(x,t) = \frac{a_0}{b_0}\left(1 - \frac{2}{A\exp\left(\frac{(1-\alpha)(ct^{-\alpha}+\lambda x^{-\alpha})}{B(\alpha)\sum_{n=0}^{\infty}\left(-\frac{\alpha}{1-\alpha}\right)^n \Gamma(1-\alpha n)}\right)+1}\right). \tag{25}$$

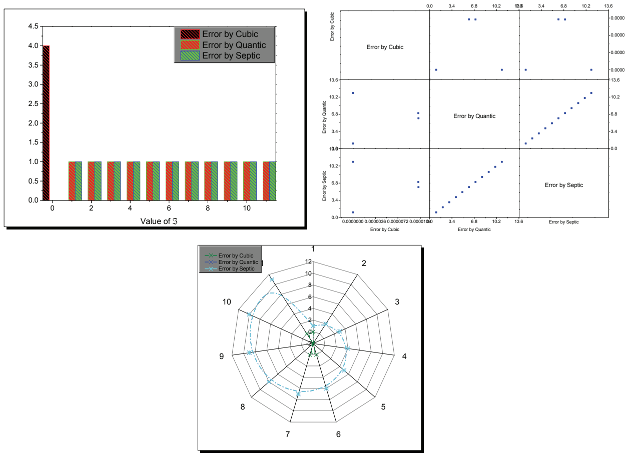

Fig. 1 Relation between absolute errors of the modified Khater method and the B-spline schemes based on Table 1. It shows the superiority of Quintic B-spline scheme over the other used methods.

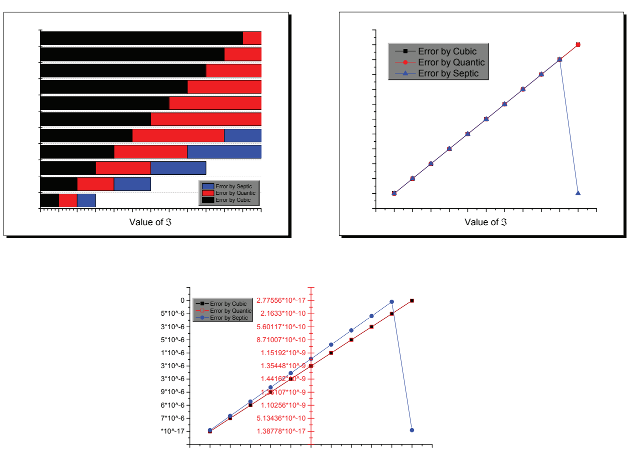

Fig. 2 Relation between absolute errors of the generalized Kudryashov method and the B-spline schemes based on Table 2. It shows the superiority of Quintic B-spline scheme over the other used methods.

3. NUMERICAL SOLUTIONS

Applying the B-spline schemes (cubic, quintic and septic) on Eq. (7) with the following conditions according to (16) and (22), respectively $[\mathcal{P}(\Im) = -\frac{1}{2}\tanh\left(\frac{\Im}{2}\right),\ \delta = 1,\ c = 7,\ \lambda = -\sqrt{51},\ \sigma = 1,\ \chi = 5,\ \epsilon = 4,\ \varrho = 6]$ and $[\mathcal{P}(\Im) = \frac{1}{4}\left(\frac{2}{2e^{\Im}+1} - 1\right),\ \sigma = 2^{2/3},\ \epsilon = 64,\ a_1 = -1,\ A = 2,\ b_0 = -3,\ b_1 = -2,\ c = 1,\ \lambda = 3]$ leads to obtaining the data shown in Tables 1 and 2.

4. RESULTS AND DISCUSSION

This section discusses the obtained solutions of the fractional nonlinear space-time Phi-four equation.

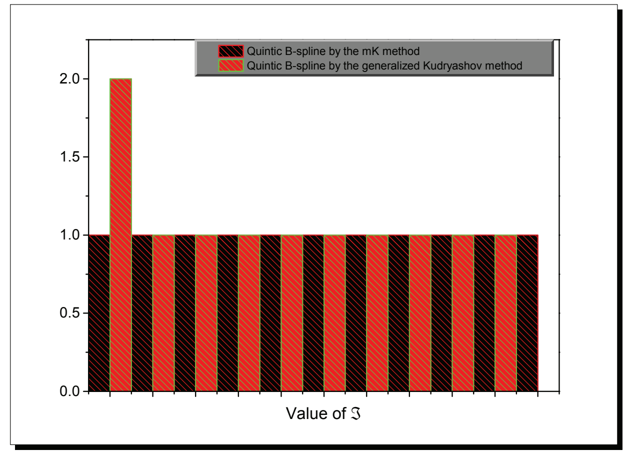

Fig. 3 Absolute errors that are obtained by the septic B-spline scheme in the mK method and generalized Kudryashov method, respectively.

- **Analytical solutions**
  We successfully applied the mK method and the generalized Kudryashov method on the fractional Phi-four equation after transforming it to the integer-order nonlinear ordinary differential equation by using the $\mathcal{ABR}$ fractional operator definition and based on the obtained solutions, we can summarize the outcomes of comparison between both the methods as follows:

  (1) The mK method obtained many forms of the solution, while the generalized Kudryashov method was just able to get only one form of solutions.
  (2) Equation (15) is equal to Eq. (23) when $\left[\varrho = \sigma = 1,\ b_1 = \sqrt{\epsilon}\,a_1,\ A = \frac{\chi\, b_1}{a_1}\right]$. So, the mK method covers the generalized Kudryashov method, which means the superiority of the mK method over the simplest equation method.

- **Numerical solutions**

  We successfully applied B-spline schemes to the fractional nonlinear space-time Phi-four equation and Tables 5 and 3 show the values of absolute errors that are calculated between the exact and approximate solutions that are obtained by the above mentioned numerical schemes. Moreover, the tables show the superior accuracy of the septic B-spline

Table 1 Absolute Errors between Exact Solution that is Obtained by the mK Method (9) and the Numerical Solutions that are Obtained by Using the Cubic, Quintic, and Septic Schemes Respectively.

	B-spline		
	Error by Cubic Scheme	Error by Quintic Scheme	Error by Septic Scheme
0	0	1.73472×10^{-18}	8.67362×10^{-19}
0.1	2.94126×10^{-6}	1.26792×10^{-9}	2.46196×10^{-12}
0.2	5.66251×10^{-6}	3.07264×10^{-9}	5.5194×10^{-12}
0.3	7.95404×10^{-6}	4.39398×10^{-9}	7.22998×10^{-12}
0.4	9.62589×10^{-6}	5.34474×10^{-9}	8.61458×10^{-12}
0.5	1.05157×10^{-5}	5.76834×10^{-9}	8.98245×10^{-12}
0.6	1.04946×10^{-5}	5.6305×10^{-9}	8.63704×10^{-12}
0.7	9.47007×10^{-6}	4.88248×10^{-9}	7.27898×10^{-12}
0.8	7.38715×10^{-6}	3.62414×10^{-9}	5.62606×10^{-12}
0.9	4.22633×10^{-6}	1.57609×10^{-9}	2.54421×10^{-12}
1	0	2.77556×10^{-17}	2.77556×10^{-17}

Table 2 Absolute Errors between Exact Solution that is Obtained by the Generalized Kudryashov Method (22) and the Numerical Solutions that are Obtained by Using the Cubic, Quintic, and Septic Schemes Respectively.

	B-spline		
	Error by Cubic Scheme	Error by Quintic Scheme	Error by Septic Scheme
0	2.77556×10^{-17}	1.38778×10^{-17}	0
0.1	2.16427×10^{-6}	5.13436×10^{-10}	3.11987×10^{-13}
0.2	3.79486×10^{-6}	1.10256×10^{-9}	5.01363×10^{-13}
0.3	4.88869×10^{-6}	1.36107×10^{-9}	3.44974×10^{-13}
0.4	5.45943×10^{-6}	1.44162×10^{-9}	1.48992×10^{-13}
0.5	5.53463×10^{-6}	1.35448×10^{-9}	1.03167×10^{-13}
0.6	5.15241×10^{-6}	1.15192×10^{-9}	3.14415×10^{-13}
0.7	4.35815×10^{-6}	8.71007×10^{-10}	4.23744×10^{-13}
0.8	3.20123×10^{-6}	5.60117×10^{-10}	4.56635×10^{-13}
0.9	1.73215×10^{-6}	2.1633×10^{-10}	2.47274×10^{-13}
1	0	2.77556×10^{-17}	0

scheme among the three used schemes. Additionally, the absolute error by the generalized Kudryashov method is smaller than that obtained by the mK method (Figs. 1–3).

5. CONCLUSION

In this paper, $\mathcal{ABR}$ fractional definition was used to convert the fractional form to integer order and then the mK and generalized Kudryashov methods were successfully implemented on the fractional nonlinear space-time Phi-four model. Moreover, the numerical solutions obtained by using the B-spline schemes were discussed. The performance of the used method shows the power, capacity, and ability for applying to other nonlinear partial differential equations with integer order and fractional order.

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