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Inequalities for the Euclidean Operator Radius of n -Tuple Operators and Operator Matrices in Hilbert C^* -Modules

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Abstract: This study takes a detailed look at various inequalities related to the Euclidean operator radius. It examines groups of n -tuple operators, studying how they add up and multiply together. It also uncovers a unique power inequality specific to the Euclidean operator radius. The research broadens its scope to analyze how n -tuple operators, when used as parts of 2×2 operator matrices, illustrate inequalities connected to the Euclidean operator radius. By using the Euclidean numerical radius and Euclidean operator norm for n -tuple operators, the study introduces a range of new inequalities. These inequalities not only set limits for the addition, multiplication, and Euclidean numerical radius of n -tuple operators but also help in establishing inequalities for the Euclidean operator radius. This process involves carefully examining the Euclidean numerical radius inequalities of 2×2 operator matrices with n -tuple operators. Additionally, a new inequality is derived, focusing specifically on the Euclidean operator norm of 2×2 operator matrices. Throughout, the research keeps circling back to the idea of finding and understanding symmetries in linear operators and matrices. The paper highlights the significance of symmetry in mathematics and its impact on various mathematical areas.

Keywords: numerical radius; inner product space; C^* -algebra; A -module



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1. Introduction and Preliminaries

In functional analysis, particularly in the context of operator theory, the Euclidean operator radius refers to the supremum (least upper bound) of the norms of operators acting on a Hilbert space. If we have a bounded linear operator T on a Hilbert space $\mathcal{H}$, then its norm is defined as $\|T\| = \sup_{\|x\|=1} \|Tx\|$, where $\|x\|$ denotes the norm of the vector x in the Hilbert space. The Euclidean operator radius extends this concept to a tuple of operators or matrices (see [1–4]).

This study delves into Euclidean operator spaces and their associated inequalities, likely in a finite-dimensional context. It explores symmetries and operator norms and derives new inequalities governing various operator properties. While not directly related to the existence of Hilbert spaces, this research contributes to our understanding of functional analysis, which is closely intertwined with Hilbert space theory, demonstrating its significance within the broader mathematical landscape.

Numerous scholars have extensively studied the theory of Hilbert modules over non-commutative C^* -algebras, as demonstrated by various works, such as [5–12]. The notion of symmetry plays a pivotal role in exploring non-commutative topology, extensions of C^* -algebras, and the field of K -theory. Symmetry is closely linked with the attributes of Hilbert modules, and grasping their symmetrical aspects offers valuable insights into how they differ from Hilbert spaces.

In non-commutative topology, symmetry shows up in how different algebraic structures relate and change. Hilbert modules, which do not follow the usual commutative rules,

have their own kind of symmetry that is different from what we see in Hilbert spaces. They do not have this special “self-dual” quality that Hilbert spaces often do, which marks a big difference in their symmetry.

Another way that Hilbert modules stand apart from Hilbert spaces is that they do not have properties like being orthogonally complementary. In Hilbert spaces, being orthogonal and complementary are important for symmetry, especially when it comes to self-adjoint and unitary operators. Since Hilbert modules lack these properties, they have a kind of imbalance that shapes their unique mathematical features.

Studying symmetry here helps us see the subtle contrasts between Hilbert modules and Hilbert spaces more clearly. It shows us how the absence of specific symmetrical aspects in Hilbert modules affects how they work in non-commutative topology, extensions of C^* -algebras, and K -theory. By looking into these symmetrical details, mathematicians obtain a better grasp of the patterns and connections in these areas of math, making functional analysis and operator theory more interesting and comprehensive.

Hilbert C^* -modules extend the concept of Hilbert spaces by allowing scalars to belong to a C^* -algebra instead of being restricted to real or complex numbers. In these modules, an operator matrix refers to a matrix with entries being operators acting within the module, serving as linear mappings from one part of the module to another. Exploring inequalities related to the Euclidean operator radius of n -tuple operators and operator matrices within Hilbert C^* -modules typically involves an examination of norm behaviors and inequalities amidst various operations, such as addition, scalar multiplication, and operator composition. This exploration also entails delving into the characteristics of C^* -algebras and their associated modules, along with methods for analyzing operator and operator matrix norms within these frameworks. For more detail, the reader should refer to [8–10,13].

We begin with the definition of a C^* -algebra.

Definition 1 ([14]). A C^* -algebra $\mathfrak{A}$ is a (non-empty) set with the following algebraic operations:

- (i) Addition, which is commutative and associative;
- (ii) Multiplication, which is associative;
- (iii) Multiplication by complex scalars;
- (iv) An involution $a \mapsto a^*$ (that is, $(a^*)^* = a$ for all $a \in \mathfrak{A}$).

Both types of multiplication distribute over addition. For $a, b \in \mathfrak{A}$, we have $(ab)^* = b^*a^*$. The involution is conjugate linear; that is, for $a, b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, we have $(\lambda a + b)^* = \bar{\lambda}a^* + b^*$. For $a, b \in \mathfrak{A}$ and $\lambda, \mu \in \mathbb{C}$, we have $\lambda(ab) = (\lambda a)b = a(\lambda b)$ and $(\lambda\mu)a = \lambda(\mu a)$. In addition, $\mathfrak{A}$ has a norm in which it is a Banach algebra; that is,

- (N1) $\|\lambda a\| = |\lambda| \|a\|$;
- (N2) $\|a + b\| \leq \|a\| + \|b\|$;
- (N3) $\|ab\| \leq \|a\| \|b\|$

for all $a, b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, and $\mathfrak{A}$ is complete in the metric $d(a, b) = \|a - b\|$. Moreover, for all $a \in \mathfrak{A}$, we have $\|a^*a\| = \|a\|^2$.

Let us revisit how a Hilbert module over a C^* -algebra $\mathfrak{A}$ is defined, as described in [6].

Definition 2 ([6]). Consider a linear space $\mathfrak{E}$ over a complex field endowed with the structure of a right $\mathfrak{A}$ -module. We assume that $\lambda(\xi a) = (\lambda \xi)a = \xi(\lambda a)$, where ξ belongs to $\mathfrak{E}$, a belongs to $\mathfrak{A}$, and λ is a complex number. The space $\mathfrak{E}$ is referred to as a pre-Hilbert $\mathfrak{A}$ -module if it possesses an inner product $\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \rightarrow \mathfrak{A}$ that adheres to the following conditions:

- (i) $\langle \xi, \xi \rangle \geq 0$ and $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$;
- (ii) $\langle \xi, \xi\mu + \eta\nu \rangle = \langle \xi, \xi \rangle\mu + \langle \xi, \eta \rangle\nu$;
- (iii) $\langle \xi, \xi a \rangle = \langle \xi, \xi \rangle a$;
- (iv) $\langle \xi, \zeta \rangle = \langle \zeta, \xi \rangle^*$,

where $\xi, \zeta, \eta \in \mathfrak{E}$, $a \in \mathfrak{A}$, and $\mu, \nu \in \mathbb{C}$. Set $\|\xi\| = \|\langle \xi, \xi \rangle\|^{\frac{1}{2}}$. This is a norm on $\mathfrak{E}$. If $\mathfrak{E}$ is complete, $\mathfrak{E}$ is called a Hilbert module over $\mathfrak{A}$.

Let us consider Hilbert $\mathfrak{A}$ -modules $\mathfrak{E}$ and $\mathfrak{F}$. We define $\mathcal{L}(\mathfrak{E}, \mathfrak{F})$ as the set containing all mappings $t : \mathfrak{E} \rightarrow \mathfrak{F}$, where there exists another mapping $t^* : \mathfrak{F} \rightarrow \mathfrak{E}$ satisfying $\langle t\xi, \zeta \rangle = \langle \xi, t^*\zeta \rangle$ for every $\xi \in \mathfrak{E}$ and $\zeta \in \mathfrak{F}$. It is widely known that t must meet the criteria of being a bounded $\mathfrak{A}$ -linear mapping, indicating that t is a linear mapping that satisfies certain boundedness conditions. Additionally, it satisfies $t(\xi a) = t(\xi)a$ for all $\xi \in \mathfrak{E}$ and $a \in \mathfrak{A}$.

In the event that $\mathfrak{E} = \mathfrak{F}$, then $\mathcal{L}(\mathfrak{E})$ forms a C^* -algebra, complete with the operator norm.

Definition 3 ([11]). Consider a C^* -algebra denoted by $\mathfrak{A}$. A linear functional ψ acting on $\mathfrak{A}$ is deemed positive if it satisfies the condition $\psi(a^*a) \geq 0$ for all a within $\mathfrak{A}$. In the scenario where $\mathfrak{A}$ possesses a unit element (i.e., it is unital), the linear functional ψ is referred to as a state if it is positive and, furthermore, $\psi(1) = 1$. It is worth noting that when ψ is a linear functional operating on a unital C^* -algebra $\mathfrak{A}$ and $\psi(1) = \|\psi\| = 1$, then ψ automatically qualifies as a state. We represent the collection of all states of $\mathfrak{A}$ as $\mathfrak{S}(\mathfrak{A})$.

2. Euclidean Operator Radius of n -Tuple Operators

In this section, we explore the fundamental characteristics of the joint numerical radius and various other operator radii related to the Euclidean operator radius of an n -tuple of operators denoted by $t_1, \dots, t_n$. A general guideline emerges: for $n \geq 2$, it holds that $w_e(t_1, \dots, t_n) \leq w(t_1, \dots, t_n)$. However, these two concepts coincide with the classical numerical radius when $n = 1$. We present precise inequalities and demonstrate that the operator radius within $\mathcal{L}^n(\mathfrak{E})$ forms a norm that is equivalent to both the operator norm and the joint numerical radius.

Definition 4 ([8]). Let $t \in \mathcal{L}(\mathfrak{E})$ and $\psi \in \omega(\mathfrak{A})$. Then,

$$\|t\| := \sup\{\psi(|t\xi|) : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}), \psi(|\xi|) = 1\}, \quad (1)$$

where $|\xi| = \langle \xi, \xi \rangle^{\frac{1}{2}}$.

It is known from [8] that $\|\cdot\|$ is a norm on $\mathcal{L}(\mathfrak{E})$. And if $\mathfrak{E}$ is a Hilbert space, then $\|t\| = \|\cdot\|$.

The following result was investigated in [8].

Lemma 1. Let $t \in \mathcal{L}(\mathfrak{E})$. Then,

$$\|t\| = \sup\{|\psi(\langle \xi, t\xi \rangle)| : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}), \psi(|\xi|) = \psi(|\xi|) = 1\}.$$

Definition 5. Let $t_1, \dots, t_n \in \mathcal{L}(\mathfrak{E})$ and $\psi \in \omega(\mathfrak{A})$. The joint numerical radius for $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$ is defined by

$$w_j(F) := \sup \left| \sum_{\alpha \in \mathbb{E}_n^+} \sum_{j=1}^n \psi \left(\left\langle \xi_\alpha, t_j \xi_{g_j \alpha} \right\rangle \right) \right|,$$

where the supremum is taken over all families of vectors $\{\xi_\alpha\} \subset \mathfrak{E}$ with $\sum_{\alpha \in \mathbb{E}_n^+} \psi(|\xi_\alpha|^2) = 1$, where $\mathbb{E}_n^+$ is the free semigroup with the generators $g_1, \dots, g_n$ and the neutral element g_0 .

Definition 6. Let $F = (t_1, \dots, t_n)$ be an n -tuple operator. Then, F is said to be commuting if $t_i t_j = t_j t_i$ for all $i, j = 1, 2, \dots, n$.

For n -tuple operators $F = (t_1, \dots, t_n)$, $Q = (q_1, \dots, q_n) \in \mathbb{L}^n(\mathfrak{E})$, we write $QF = (q_1 t_1, \dots, q_n t_n)$, $F + Q = (t_1 + q_1, \dots, t_n + q_n)$ and $\mu F = (\mu t_1, \dots, \mu t_n)$ for any scalar $\mu \in \mathbb{C}$.

Definition 7. Let $F = (t_1, t_2, \dots, t_n)$ be an n -tuple of bounded linear operators on $\mathfrak{E}$. The Euclidean operator norm is defined by

$$\begin{aligned} \|F\| &= \|[t_1, \dots, t_n]\| = \left\| \sum_{j=1}^n t_j^* t_j \right\|^{\frac{1}{2}} \\ &= \sup \left\{ \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}) \text{ and } \psi(|\xi|) = 1 \right\}. \end{aligned}$$

Lemma 2. $\|\cdot\| : \mathcal{L}^n(\mathfrak{E}) \rightarrow [0, \infty)$ defines a norm on $\mathcal{L}^n(\mathfrak{E})$.

Theorem 1 ([9]). Let $F = (t_1, \dots, t_n) \in \mathcal{L}^n(\mathfrak{E})$. If $\mathfrak{E}$ is a Hilbert $\mathfrak{A}$ -Modules, then

$$\|F\| = \sup \left\{ \left(\sum_{j=1}^n |\psi(\langle \xi, t_j \xi \rangle)|^2 \right)^{\frac{1}{2}} : \xi, \zeta \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}), \text{ and } \psi(|\xi|) = \psi(|\zeta|) = 1 \right\}.$$

Recall that an operator $F = (t_1, \dots, t_n) \in \mathcal{L}^n(\mathfrak{E})$ is said to be self-adjoint if $F^* = (t_1^*, \dots, t_n^*) = (t_1, \dots, t_n) = F$.

Theorem 2 ([9]). If $F = (t_1, \dots, t_n) \in \mathcal{L}^n(\mathfrak{E})$ is self-adjoint, then

$$\|F\| = \sup \left\{ \left(\sum_{j=1}^n |\psi(\langle \xi, t_j \xi \rangle)|^2 \right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}) \text{ and } \psi(|\xi|) = 1 \right\}.$$

Definition 8. Let $F = (t_1, t_2, \dots, t_n)$ be an n -tuple of bounded linear operators on $\mathfrak{E}$. The Euclidean operator radius is defined by

$$w_e(F) := \sup \left\{ \left(\sum_{k=1}^n |\psi(\langle \xi, t_k \xi \rangle)|^2 \right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}) \text{ and } \psi(|\xi|) = 1 \right\}.$$

The joint (spatial) numerical range of $F = (t_1, t_2, \dots, t_n)$, defined by

$$W(F) := \{(\psi(\langle \xi, t_1 \xi \rangle), \psi(\langle \xi, t_2 \xi \rangle), \dots, \psi(\langle \xi, t_n \xi \rangle)) : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}) \text{ and } \psi(|\xi|) = 1\}.$$

In general, when $n \geq 2$, $w_e(t_1, \dots, t_n) \leq w_j(t_1, \dots, t_n)$.

Theorem 3. The joint numerical radius and Euclidean operator radius coincide with the classical numerical radius of an operator $t \in \mathcal{L}(\mathfrak{E})$ if $n = 1$, i.e.,

$$w(t) := \sup \{|\psi(\langle \xi, t \xi \rangle)| : \xi \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}) \text{ and } \psi(|\xi|) = 1\}.$$

Proof. If $\sum_{k=0}^{+\infty} \psi(|\xi_k|^2) < +\infty$ and $f_m(\theta) := \sum_{k=0}^m e^{ik\theta} \xi_k$, $\xi_k \in \mathfrak{E}$, $m = 1, 2, \dots$

$$\begin{aligned} \left| \sum_{k=0}^m \psi(\langle \xi_k, t \xi_k \rangle) \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \psi(\langle f_m(\theta), f_m(\theta) t \rangle) e^{ik\theta} d\theta \right| \\ &\leq w(t) \frac{1}{2\pi} \int_0^{2\pi} \psi(|f_m(\theta)|^2) d\theta \\ &= w(t) \sum_{k=0}^m \psi(|\xi_k|^2). \end{aligned}$$

Taking $m \rightarrow +\infty$, we obtain $w_e(t) \leq w(t)$. On the other hand, let $\xi \in \mathfrak{E}$, $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$ and define $\xi_k := \lambda^k \sqrt{1 - |\lambda|^2} x$, $k = 0, 1, 2, \dots$, where $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. It is easy to see that $\sum_{k=0}^{+\infty} \psi(|\xi_k|^2) = 1$ and

$$\left| \sum_{k=0}^{+\infty} \psi(\langle \xi_k, t \xi_k \rangle) \right| = |\lambda| |\psi(\langle \xi, t \xi \rangle)|$$

for any $\lambda \in \mathbb{D}$. Hence, we deduce the inequality $w_e(t) \geq w(t)$, which proves our assertion, i.e., $w_e(t) = w(t)$. $\square$

The result outcomes are highly relevant for the subsequent discussions, as detailed in [9].

Lemma 3. Let $t \in \mathcal{L}(\mathfrak{E})$ and $\psi \in \omega(\mathfrak{A})$. Then, the following are equivalent:

- (a) $\psi(\langle \xi, t \xi \rangle) = 0$ for every $\xi \in \mathfrak{E}$ with $\psi(|\xi|) = 1$;
- (b) $\psi(\langle \xi, t \xi \rangle) = 0$ for every $\xi \in \mathfrak{E}$.

Lemma 4. Let $t \in \mathcal{L}(\mathfrak{E})$. Then, $t = 0$ if and only if $\psi(\langle \xi, t \xi \rangle) = 0$ for every $\xi \in \mathfrak{E}$ and $\psi \in \omega(\mathfrak{A})$.

Lemma 5. Let $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. Then, for every $\psi \in \omega(\mathfrak{A})$ and $\xi \in \mathfrak{E}$,

$$\left(\sum_{j=1}^n |\psi(\langle \xi, t_j \xi \rangle)|^2 \right)^{\frac{1}{2}} \leq w_e(F) \psi(|\xi|^2). \quad (2)$$

Theorem 4. If $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$, then

$$w_e(F) \leq \|F\| \leq 2\sqrt{n} w_e(F). \quad (3)$$

Or, equivalently,

$$\frac{1}{2\sqrt{n}} \|F\| \leq w_e(F) \leq \|F\|.$$

Here, the constants $\frac{1}{2\sqrt{n}}$ and 1 are the best possible.

Theorem 5. $w_e : \mathbb{L}^n(\mathfrak{E}) \rightarrow [0, +\infty)$ defines a norm that is equivalent to the norm on $\mathbb{L}^n(\mathfrak{E})$.

Consider an n -dimensional complex Hilbert $\mathfrak{A}$ -module denoted by E_n , equipped with an orthonormal basis $e_1, e_2, \dots, e_n$, where n is a natural number or $n = +\infty$. We introduce the full Fock space of E_n , defined as follows:

$$F^2(E) := \bigoplus_{k \geq 0} E_n^{\otimes k},$$

Here, $E_n^{\otimes 0} = \mathbb{C} \cdot 1$, and $E_n^{\otimes k}$ represents the Hilbert tensor product of k copies of E_n . We define the left creation operators $s_i : F^2(E_n) \rightarrow F^2(E_n)$, where $i = 1, 2, \dots, n$, as follows:

$$s_i \phi := e_i \otimes \phi, \quad \phi \in F^2(E_n).$$

Now, let $\mathbb{E}_n^+$ denote the unital free semigroup on n generators $g_1, \dots, g_n$, along with the identity element g_0 . We define the length of $\alpha \in \mathbb{E}_n^+$ as $|\alpha| := k$, where $\alpha = g_{i_1} \dots g_{i_k}$ for $\alpha \neq g_0$, and $|\alpha| := 0$ for $\alpha = g_0$. We also introduce $e_\alpha = e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}$ and set $e_{g_0} = 1$. It is evident that $\{e_\alpha : \alpha \in \mathbb{E}_n^+\}$ forms an orthonormal basis for $F^2(E_n)$.

The joint spectral radius linked with an n -tuple of operators $(t_1, \dots, t_n)$ is defined as follows:

$$r_j(t_1, \dots, t_n) := \lim_{k \rightarrow +\infty} \left\| \sum_{|\alpha|=k} t_\alpha t_\alpha^* \right\|^{\frac{1}{2k}},$$

where $\|\cdot\|$ denotes the norm.

Theorem 6. The joint numerical radius $w_j : \mathbb{L}^n(\mathfrak{E}) \rightarrow [0, +\infty)$ for n -tuples of operators satisfies the following properties:

- (a) $w_j(y^* t_1 y, \dots, y^* t_n y) \leq \|y\|^2 w_j(t_1, \dots, t_n)$ for any operator $X : \mathfrak{F} \rightarrow \mathfrak{E}$ in $\mathcal{L}(\mathfrak{F}, \mathfrak{E})$;
- (b) $w_j(u^* t_1 u, \dots, u^* t_n u) = w_j(t_1, \dots, t_n)$ for any unitary operator $u : \mathfrak{F} \rightarrow \mathfrak{E}$ in $\mathcal{L}(\mathfrak{F}, \mathfrak{E})$;
- (c) $\frac{1}{2} \left\| \sum_{k=1}^n t_k t_k^* \right\|^{\frac{1}{2}} \leq w_j(t_1, \dots, t_n) \leq \left\| \sum_{k=1}^n t_k t_k^* \right\|^{\frac{1}{2}}$;
- (d) $r_j(t_1, \dots, t_n) \leq w_j(t_1, \dots, t_n)$.

Proof. (a) Let $\{k_\alpha\}_{\alpha \in \mathbb{E}_n^+}$ be an arbitrary sequence of vectors in $\mathfrak{F}$ such that $\sum_{\alpha \in \mathbb{E}_n^+} \psi(|k_\alpha|^2) = 1$. Fix an operator $y : \mathfrak{F} \rightarrow \mathfrak{E}$ and define the vectors $\zeta_\alpha := \frac{1}{M} y k_\alpha$, $\alpha \in \mathbb{E}_n^+$, where $M := \left(\sum_{\alpha \in \mathbb{E}_n^+} \psi(|y k_\alpha|^2) \right)^{\frac{1}{2}}$. Observe that $\sum_{\alpha \in \mathbb{E}_n^+} \psi(|\zeta_\alpha|^2) = 1$ and $M \leq \|y\|$. On the other hand, we have

$$\begin{aligned} \left| \sum_{\alpha \in \mathbb{E}_n^+} \sum_{j=1}^n \psi \left(\langle k_\alpha, y^* t_j y k_{g_{j\alpha}} \rangle \right) \right| &\leq \left| \sum_{\alpha \in \mathbb{E}_n^+} \sum_{j=1}^n \psi \left(\langle k_\alpha, t_j k_{g_{j\alpha}} \rangle \right) \right| \|y\|^2 \\ &\leq \|y\|^2 w_j(t_1, \dots, t_n). \end{aligned}$$

Taking the supremum over all sequences $\{k_\alpha\}_{\alpha \in \mathbb{E}_n^+} \subset \mathfrak{F}$ with $\sum_{\alpha \in \mathbb{E}_n^+} \psi(|k_\alpha|^2) = 1$, we deduce inequality (a).

(b) A closer look reveals that, when $y = u$ is a unitary operator, we have equality in the above inequality. Therefore, relation (b) holds true.

(c) Any vector $\xi \in F^2(\mathbb{E}_n^+)$ with $\psi(|\xi|) = 1$, $\psi \in \omega(\mathfrak{A})$, has the form $x = \sum_{\alpha \in \mathbb{E}_n^+} e_\alpha \otimes \xi_\alpha$, where the sequence $\{\xi_\alpha\}_{\alpha \in \mathbb{E}_n^+}$ is such that $\sum_{\alpha \in \mathbb{E}_n^+} \psi(|\xi_\alpha|^2) = 1$. If $s_1, \dots, s_n$ are the left creation operators on the full Fock space $F^2(E_n)$, note that for $\psi \in \omega(\mathfrak{A})$,

$$\begin{aligned} \left| \psi \left(\left\langle x, \left(\sum_{j=1}^n s_j \otimes t_j^* \right) x \right\rangle \right) \right| &= \left| \psi \left(\left\langle \sum_{\beta \in \mathbb{E}_n^+} e_\beta \otimes \xi_\beta, \sum_{\alpha \in \mathbb{E}_n^+} \sum_{j=1}^n e_{g_{j\alpha}} \otimes t_j^* \xi_\alpha \right\rangle \right) \right| \\ &= \left| \sum_{\alpha, \beta \in \mathbb{E}_n^+} \sum_{j=1}^n \psi \left(\langle e_\beta, e_{g_{j\alpha}} \rangle \right) \psi \left(\langle \xi_\beta, t_j^* \xi_\alpha \rangle \right) \right| \\ &= \left| \sum_{\alpha \in \mathbb{E}_n^+} \sum_{j=1}^n \psi \left(\langle t_j \xi_{g_{j\alpha}}, \xi_\alpha \rangle \right) \right|. \end{aligned}$$

Hence, we infer that

$$w_j(t_1, \dots, t_n) = w_j(s_1 \otimes t_1^* + \dots + s_n \otimes t_n^*). \quad (4)$$

Conversely, it is a widely acknowledged fact that the classical numerical radius of an operator y adheres to the inequalities $\frac{1}{2} \|y\| \leq w(y) \leq \|y\|$ (as established in ([9], Theorem 2.13)).

Utilizing relation (4) and considering the orthogonal ranges of the left creation operators, we can derive

$$\begin{aligned} w_j(t_1, \dots, t_n) &= w\left(\sum_{j=1}^n s_j \otimes t_j^*\right) \leq \left\| \sum_{j=1}^n s_j \otimes t_j^* \right\| \\ &= \left\| \left(\sum_{j=1}^n s_j^* \otimes t_j \right) \left(\sum_{j=1}^n s_j \otimes t_j^* \right) \right\|^{\frac{1}{2}} \\ &= \left\| \sum_{j=1}^n t_j t_j^* \right\|^{\frac{1}{2}} = \|[t_1, \dots, t_n]\|. \end{aligned}$$

Similarly, one can prove the first inequality in (c).

(d) To prove (vii), notice that, since $s_j^* s_j = \delta_{ij} 1$, $i, j = 1, \dots, n$, we have

$$\begin{aligned} r_j\left(\sum_{j=1}^n s_j \otimes t_j^*\right) &= \lim_{k \rightarrow +\infty} \left\| \sum_{|\alpha|=k} s_\alpha \otimes t_\alpha^* \right\|^{\frac{1}{k}} \\ &= \lim_{k \rightarrow +\infty} \left\| \left(\sum_{|\alpha|=k} s_\alpha^* \otimes t_\alpha \right) \left(\sum_{|\alpha|=k} s_\alpha \otimes t_\alpha^* \right) \right\|^{\frac{1}{2k}} \\ &= \left\| 1 \otimes \sum_{|\alpha|=k} t_\alpha t_\alpha^* \right\|^{\frac{1}{2k}} \\ &= r_j(t_1, \dots, t_n). \end{aligned}$$

Consequently, we deduce that

$$r_j(t_1, \dots, t_n) = r_j\left(\sum_{j=1}^n s_j \otimes t_j^*\right) \leq w_j\left(\sum_{j=1}^n s_j \otimes t_j^*\right) = w_j(t_1, \dots, t_n).$$

Therefore, inequality (d) is established. $\square$

In this section, we outline the fundamental characteristics of the Euclidean operator radius for an n -tuple of operators. We introduce a novel norm and a concept akin to the “spectral radius” on $\mathcal{L}^n(\mathfrak{E})$ by defining them as follows:

$$\|(t_1, \dots, t_n)\|_e := \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \|\mu_1 t_1 + \dots + \mu_n t_n\|, \quad (5)$$

where $\mathbb{B}_n := \{(\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{C}^n : |\mu_1|^2 + \dots + |\mu_n|^2 < 1\}$, i.e., the unit ball of $\mathbb{C}^n$.

And

$$r_e(t_1, \dots, t_n) := \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} r(\mu_1 t_1 + \dots + \mu_n t_n), \quad (6)$$

where $r(t)$ denotes the usual spectral radius of an operator $t \in \mathcal{L}(\mathfrak{E})$. Notice that $\|\cdot\|_e$ is a norm $\mathbb{L}^n(\mathfrak{E})$,

$$\|(t_1, \dots, t_n)\|_e = \|(t_1^*, \dots, t_n^*)\|_e \quad \text{and} \quad r_e(t_1, \dots, t_n) = r_e(t_1^*, \dots, t_n^*).$$

In what follows, we show that $\|\cdot\|_e$ is equivalent to the operator norm on $\mathbb{L}^n(\mathfrak{E})$.

Theorem 7. If $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$, then

$$\frac{1}{\sqrt{n}} \|[t_1, \dots, t_n]\| \leq \|(t_1, \dots, t_n)\|_e \leq \|[t_1, \dots, t_n]\|, \quad (7)$$

where the constants 1 and $\frac{1}{\sqrt{n}}$ are the best possible, and

$$r_e(t_1, \dots, t_n) \leq r(t_1, \dots, t_n). \quad (8)$$

Proof. Let τ be the rotation-invariant normalized positive Borel measure on the unit sphere $\partial\mathbb{B}_n$. Using the relations

$$\int_{\partial\mathbb{B}_n} |\mu_j|^2 d\tau(\mu) = \frac{1}{n} \quad \text{and} \quad \int_{\partial\mathbb{B}_n} \mu_i \overline{\mu_j} d\tau(\mu) = 0 \quad \text{if } i \neq j, i, j = 1, \dots, n,$$

we have

$$\begin{aligned} |||(t_1, \dots, t_n)|||_e^2 &= \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \sup_{\psi(|\xi|=1)} \psi \left(\left\langle \xi, \left(\sum_{i=1}^n \mu_i t_i \right) \left(\sum_{j=1}^n \overline{\mu_j} t_j^* \right) \xi \right\rangle \right) \\ &\geq \sup_{\psi(|\xi|=1)} \int_{\partial\mathbb{B}_n} \sum_{i,j=1}^n \mu_i \overline{\mu_j} \psi \left(\left\langle \xi, t_i t_j^* \xi \right\rangle \right) d\tau(\mu) \\ &= \frac{1}{n} \sup_{\psi(|\xi|=1)} \psi \left(\left\langle \xi, \sum_{j=1}^n t_j t_j^* \xi \right\rangle \right) \\ &= \frac{1}{n} |||[t_1, \dots, t_n]|||. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |||(t_1, \dots, t_n)|||_e &= \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} |||\mu_1 t_1 + \dots + \mu_n t_n||| \\ &\leq \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \left(\sum_{j=1}^n |\mu_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n |t_j|^2 \right)^{\frac{1}{2}} \\ &= |||[t_1, \dots, t_n]|||. \end{aligned}$$

Now, notice that if $s_1, \dots, s_n$ are the left creation operators, then

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{n}} |||[s_1^*, \dots, s_n^*]||| \leq |||[s_1^*, \dots, s_n^*]|||_e \\ &= |||[s_1, \dots, s_n]|||_e \leq |||[s_1, \dots, s_n]||| = 1. \end{aligned} \quad (9)$$

This shows that the inequalities in (7) are the best possible. In order to prove (8), note that

$$\begin{aligned} r_e(t_1, \dots, t_n) &= \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \inf_{m \in \mathbb{N}} \left\| \left(\sum_{j=1}^m \mu_j t_j \right)^m \right\|^{\frac{1}{m}} \\ &\leq \inf_{m \in \mathbb{N}} \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \left\| \left(\sum_{j=1}^m \overline{\mu_j} t_j^* \right)^m \right\|^{\frac{1}{m}} \\ &\leq \inf_{m \in \mathbb{N}} \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \left[\left(\sum_{|\alpha|=m} |\mu_\alpha|^2 \right)^{\frac{1}{2m}} \left\| \sum_{|\alpha|=m} t_\alpha t_\alpha^* \right\|^{\frac{1}{2m}} \right] \\ &\leq \inf_{m \in \mathbb{N}} \left\| \sum_{|\alpha|=m} t_\alpha t_\alpha^* \right\|^{\frac{1}{2m}} = r(t_1, \dots, t_n). \end{aligned}$$

□

The following outcome summarizes several key properties of the Euclidean operator radius for an n -tuple of operators.

Theorem 8. *The Euclidean operator radius $w_e : \mathcal{L}^n(\mathfrak{E}) \longrightarrow [0, +\infty)$ for n -tuples of operators satisfies the following properties:*

- (i) $\frac{1}{2} \|\| (t_1, \dots, t_n) \| \|_e \leq w_e(t_1, \dots, t_n) \leq \|\| (t_1, \dots, t_n) \| \|_e$;
- (ii) $r_e(t_1, \dots, t_n) \leq w_e(t_1, \dots, t_n)$;
- (iii) $w_e(1_c \otimes t_1, \dots, 1_c \otimes t_n)$ for any separable Hilbert $\mathfrak{A}$ -Modules;
- (iv) w_e is a continuous map in the norm topology.

Proof. Observe that, for every $\xi \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$,

$$\begin{aligned} w_e(t_1, \dots, t_n) &= \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n |\psi(\langle \xi, t_j \xi \rangle)|^2 \right)^{\frac{1}{2}} \\ &= \sup_{\psi(|\xi|=1)} \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \left| \sum_{j=1}^n \mu_j \psi(\langle \xi, t_j \xi \rangle) \right| \\ &= \sup_{\psi(|\xi|=1)} \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} \left| \psi \left(\left\langle \xi, \sum_{j=1}^n \mu_j t_j \right\rangle \right) \right| \\ &= \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} w(\mu_1 t_1 + \dots + \mu_n t_n). \end{aligned}$$

Consequently, we obtain

$$w_e(t_1, \dots, t_n) = \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} w(\mu_1 t_1 + \dots + \mu_n t_n). \quad (10)$$

It is known from [9], Theorem 2.13, that

$$\frac{1}{2} \|\| y \| \| \leq w(y) \leq \|\| y \| \| \quad \text{and} \quad r(y) \leq \|\| y \| \|\|$$

for every $y \in \mathcal{L}(\mathfrak{E})$. Applying these inequalities to the operator $y := \mu_1 t_1 + \dots + \mu_n t_n$ for $(\mu_1, \dots, \mu_n) \in \mathbb{B}_n$ and using relation (10), we deduce (i) and (ii).

To prove (iii), we use relation (10) and the fact that the classical numerical radius satisfies the equation $w(1_c \otimes y) = w(y)$. Indeed, we have

$$\begin{aligned} w_e(1_c \otimes t_1, \dots, 1_c \otimes t_n) &= \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} w \left(1_c \otimes \sum_{j=1}^n \mu_j t_j \right) \\ &= \sup_{(\mu_1, \dots, \mu_n) \in \mathbb{B}_n} w \left(\sum_{j=1}^n \mu_j t_j \right) \\ &= w_e(t_1, \dots, t_n). \end{aligned}$$

According to (i) and Theorem (7), we obtain

$$w_e(t_1, \dots, t_n) \leq \left\| \sum_{j=1}^n t_j t_j^* \right\|^{\frac{1}{2}}.$$

Therefore, we deduce that w_e is continuous in the norm topology. The proof is complete. $\square$

We will now establish one of our central results, which provides an inequality involving the Euclidean operator radius.

Theorem 9. Let $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. Then, for every $\xi \in \mathfrak{E}$ and $\psi \in \omega(\mathfrak{A})$, the following inequality holds:

$$\sum_{j=1}^n \psi(|t_j \xi|^2) + \sum_{j=1}^n \left| \psi(\langle \xi, t_j^2 \xi \rangle) \right| \leq 2\sqrt{n}w_e(F) \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \psi(|\xi|).$$

Proof. Let λ_j and θ_j ($j = 1, \dots, n$) be real numbers with $\lambda_j \neq 0$. Then, we have

$$\begin{aligned} \sum_{j=1}^n \psi(|t_j \xi|^2) + \sum_{j=1}^n e^{2i\theta_j} \psi(\langle \xi, t_j^2 \xi \rangle) &= \sum_{j=1}^n \left[\frac{1}{2} \psi(\langle \lambda_j e^{i\theta_j} t_j \xi + \lambda_j x, \lambda_j e^{2i\theta_j} t_j^2 x + \lambda_j e^{i\theta_j} t_j \xi \rangle) \right. \\ &\quad \left. - \frac{1}{2} \psi(\langle \lambda_j e^{i\theta_j} t_j \xi - \lambda_j x, \lambda_j e^{2i\theta_j} t_j^2 x - \lambda_j e^{i\theta_j} t_j \xi \rangle) \right]. \end{aligned}$$

Hence,

$$\begin{aligned} \left| \sum_{j=1}^n \psi(|t_j \xi|^2) + \sum_{j=1}^n e^{2i\theta_j} \psi(\langle \xi, t_j^2 \xi \rangle) \right| &= \sum_{j=1}^n \frac{1}{2} \left| \psi(\langle \lambda_j e^{i\theta_j} t_j \xi + \lambda_j x, \lambda_j e^{2i\theta_j} t_j^2 x + \lambda_j e^{i\theta_j} t_j \xi \rangle) \right| \\ &\quad + \sum_{j=1}^n \frac{1}{2} \left| \psi(\langle \lambda_j e^{i\theta_j} t_j \xi - \lambda_j x, \lambda_j e^{2i\theta_j} t_j^2 x - \lambda_j e^{i\theta_j} t_j \xi \rangle) \right| \\ &\leq \sum_{j=1}^n \frac{1}{2} w(t_j) \psi(|\lambda_j e^{i\theta_j} t_j \xi + \lambda_j x|^2) + \sum_{j=1}^n \frac{1}{2} w(t_j) \psi(|\lambda_j e^{i\theta_j} t_j \xi - \lambda_j x|^2). \end{aligned}$$

Since $|\psi(\langle x, t_j \xi \rangle)| \leq \left(\sum_{j=1}^n |\psi(\langle x, t_j x \rangle)|^2 \right)^{\frac{1}{2}}$ for all $\xi \in \mathfrak{E}$ and $\psi \in \omega(\mathfrak{A})$, $w(t_j) \leq w_e(F)$. Consequently,

$$\begin{aligned} \left| \sum_{j=1}^n \psi(|t_j \xi|^2) + \sum_{j=1}^n e^{2i\theta_j} \psi(\langle \xi, t_j^2 \xi \rangle) \right| &\leq \sum_{j=1}^n \frac{1}{2} w_e(F) \psi(|\lambda_j e^{i\theta_j} t_j \xi + \lambda_j x|^2) \\ &\quad + \sum_{j=1}^n \frac{1}{2} w_e(F) \psi(|\lambda_j e^{i\theta_j} t_j \xi - \lambda_j x|^2) \\ &= w_e(F) \sum_{j=1}^n \frac{1}{2} [\psi(|\lambda_j e^{i\theta_j} t_j \xi + \lambda_j x|^2) \\ &\quad + \psi(|\lambda_j e^{i\theta_j} t_j \xi - \lambda_j x|^2)] \\ &= w_e(F) \sum_{j=1}^n [\lambda_j^2 \psi(|t_j \xi|^2) + \lambda_j^{-2} \psi(|\xi|^2)]. \end{aligned}$$

Suppose $t_j \xi \neq 0$ for all $j = 1, \dots, n$, and we choose θ_j in such a way that $e^{2i\theta_j} \psi(\langle x, t_j^2 \xi \rangle) = |\psi(\langle \xi, t_j^2 \xi \rangle)|$ and $\lambda_j = \sqrt{\frac{\psi(|\xi|)}{\psi(|t_j \xi|)}}$ for all $j = 1, \dots, n$. Then, we have

$$\sum_{j=1}^n \psi(|t_j \xi|^2) + \sum_{j=1}^n e^{2i\theta_j} \psi(\langle \xi, t_j^2 \xi \rangle) \leq 2w_e(F) \sum_{j=1}^n \psi(|t_j \xi|) \psi(|\xi|).$$

Therefore, the Cauchy–Schwartz inequality implies that

$$\sum_{j=1}^n \psi(|t_j \xi|^2) + \sum_{j=1}^n e^{2i\theta_j} \psi(\langle \xi, t_j^2 \xi \rangle) \leq 2\sqrt{n}w_e(F) \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \psi(|\xi|).$$

Also, this inequality holds when $\psi(|t_j\zeta|) = 0$ for all or some $j \in \{1, \dots, n\}$. This completes the proof. $\square$

Utilizing Theorem 9, we obtain the following corollary.

Corollary 1. If $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$, then

$$\frac{1}{n}w_e(F^2) \leq w_e^2(F).$$

Proof. From Theorem 9, together with

$$\sum_{j=1}^n |\psi(\langle \zeta, t_j^2 \zeta \rangle)|^2 \leq \left(\sum_{j=1}^n |\psi(\langle \zeta, t_j^2 \zeta \rangle)| \right)^2,$$

we have

$$\sum_{j=1}^n \psi(|t_j\zeta|^2) + \left(\sum_{j=1}^n |\psi(\langle \zeta, t_j^2 \zeta \rangle)|^2 \right)^{\frac{1}{2}} \leq 2\sqrt{n}w_e(F) \left(\sum_{j=1}^n \psi(|t_j\zeta|^2) \right)^{\frac{1}{2}} \psi(|\zeta|) \quad (11)$$

Taking $\psi(|\zeta|) = 1$, we obtain

$$\sum_{j=1}^n \psi(|t_j\zeta|^2) + \left(\sum_{j=1}^n |\psi(\langle \zeta, t_j^2 \zeta \rangle)|^2 \right)^{\frac{1}{2}} \leq 2\sqrt{n}w_e(F) \left(\sum_{j=1}^n \psi(|t_j\zeta|^2) \right)^{\frac{1}{2}}.$$

This implies

$$\left[\left(\sum_{j=1}^n \psi(|t_j\zeta|^2) \right)^{\frac{1}{2}} - \sqrt{n}w_e(F) \right]^2 + \left(\sum_{j=1}^n |\psi(\langle \zeta, t_j^2 \zeta \rangle)|^2 \right)^{\frac{1}{2}} \leq nw_e^2(F).$$

Therefore,

$$\left(\sum_{j=1}^n |\psi(\langle \zeta, t_j^2 \zeta \rangle)|^2 \right)^{\frac{1}{2}} \leq nw_e^2(F).$$

Taking the supremum over all $\zeta \in \mathfrak{E}$ with $\psi(|\zeta|) = 1$ and $\psi \in \omega(\mathfrak{A})$, we obtain the desired inequality. $\square$

Definition 9. Let $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. Then, we define $c_e(F)$ by setting

$$c_e(F) = \inf \left\{ \left(\sum_{j=1}^n |\psi(\langle x, t_j \zeta \rangle)|^2 \right)^{\frac{1}{2}} : \zeta \in \mathfrak{E}, \psi \in \omega(\mathfrak{A}) \text{ and } \psi(|\zeta|) = 1 \right\}.$$

Next, we obtain a refinement of the first inequality in (3).

Theorem 10. If $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$ and $\|F\| \neq 0$, then

$$\frac{1}{2\sqrt{n}} \left[\|F\| + \frac{c_e(F^2)}{\|F\|} \right] \leq w_e(F). \quad (12)$$

Proof. Taking $\psi(|\xi|) = 1$ in (11), we obtain

$$\begin{aligned} \sum_{j=1}^n \psi(|t_j \xi|^2) + \left(\sum_{j=1}^n \left| \psi(\langle \xi, t_j^2 \xi \rangle) \right|^2 \right)^{\frac{1}{2}} &\leq 2\sqrt{n}w_e(F) \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{n}w_e(F) \|F\|. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=1}^n \psi(|t_j \xi|^2) &\leq 2\sqrt{n}w_e(F) \|F\| - \left(\sum_{j=1}^n \left| \psi(\langle \xi, t_j^2 \xi \rangle) \right|^2 \right)^{\frac{1}{2}} \\ &\leq 2\sqrt{n}w_e(F) \|F\| - c_e(F^2). \end{aligned}$$

Taking the supremum over all $\xi \in \mathfrak{E}$ with $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$, we obtain

$$\|F\|^2 + c_e(F^2) \leq 2\sqrt{n}w_e(F) \|F\|$$

and so the result. $\square$

We will now demonstrate the following inequalities for the joint operator norm of n -tuple normal operators. To this end, we will leverage the well-known characterization of normal operators. An operator t in $\mathcal{L}(\mathfrak{E})$ is considered normal if and only if $\|t\xi\| = \|t^*\xi\|$ holds for all ξ in $\mathfrak{E}$.

Theorem 11. Let $F = (t_1, \dots, t_n)$ be an n -tuple normal operator. Then,

$$\|F^2\| = \|(t_1^* t_1, \dots, t_n^* t_n)\| \leq \|F\|^2 = \|F^*\|^2 \leq \sqrt{n} \|F^2\|.$$

Proof. Let $\xi \in \mathfrak{E}$ with $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$. Then, we have

$$\begin{aligned} \|F^2\| &= \|(t_1^2, t_2^2, \dots, t_n^2)\| = \sup_{\psi(|\xi|=1} \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \\ &= \sup_{\psi(|\xi|=1} \left(\sum_{j=1}^n \psi(\langle t_j^2 \xi, t_j^2 \xi \rangle) \right)^{\frac{1}{2}} = \sup_{\psi(|\xi|=1} \left(\sum_{j=1}^n \psi(\langle t_j t_j \xi, t_j t_j \xi \rangle) \right)^{\frac{1}{2}} \\ &= \sup_{\psi(|\xi|=1} \left(\sum_{j=1}^n \psi(\langle t_j^* t_j \xi, t_j^* t_j \xi \rangle) \right)^{\frac{1}{2}} \quad (\text{since each } t_j \text{ is normal}) \\ &= \sup_{\psi(|\xi|=1} \left(\sum_{j=1}^n \psi(|t_j^* t_j|^2) \right)^{\frac{1}{2}} = \|(t_1^* t_1, \dots, t_n^* t_n)\|. \end{aligned}$$

Now,

$$\begin{aligned}
|||(t_1^* t_1, \dots, t_n^* t_n)||| &= \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j^* t_j|^2 \xi) \right)^{\frac{1}{2}} \\
&\leq \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j^* \xi|^2) \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \\
&\leq \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n |||F||| \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \\
&\quad \left(\text{since } \psi(|t_j \xi|) \leq \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}}, |||t_j||| \leq |||F||| \text{ for each } j \right) \\
&= |||F||| \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} = |||F|||^2.
\end{aligned}$$

Also, we have

$$|||F||| = \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} = \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j^* \xi|^2) \right)^{\frac{1}{2}} = |||F^*|||.$$

Also,

$$\begin{aligned}
|||F|||^2 &= \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right) = \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(\langle t_j \xi, t_j \xi \rangle) \right) \\
&= \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(\langle x, t_j^* t_j \xi \rangle) \right) \\
&\leq \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j^* t_j \xi|) \psi(|\xi|) \right) \\
&\leq \sqrt{n} \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j^* t_j \xi|^2) \right)^{\frac{1}{2}} \quad (\text{by Cauchy-Schwarz inequality}) \\
&= \sqrt{n} \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j^2 x|^2) \right)^{\frac{1}{2}} \quad (\text{since each } t_j \text{ is normal}) \\
&= \sqrt{n} |||F^2|||.
\end{aligned}$$

Therefore, the proof is complete. $\square$

Remark 1. It is worth noting that if we choose t_j ($j = 1, \dots, n$) as an $n \times n$ matrix where only the diagonal entries at position (j, j) are 1, and all other entries are zero, then the first inequality in Theorem 11 becomes an equality. Similarly, if we set $t_j = \sqrt{n}I$ (where I is the $n \times n$ identity matrix) for $j = 1, 2, \dots, n$, then the second inequality in Theorem 11 also becomes an equality. Therefore, it can be concluded that the inequalities presented in Theorem 11 are indeed sharp.

In the subsequent theorem, we establish an inequality involving the joint numerical radius of n -tuple operators in terms of powers.

Theorem 12. If $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$, then

$$w_e(F^m) \leq \sqrt{n} w_e^m(F).$$

Proof. Suppose we have $\xi \in \mathfrak{E}$ such that $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$. The inequality

$$|\psi(\langle x, t_j \xi \rangle)| \leq \left(\sum_{j=1}^n |\psi(x, t_j \xi)|^2 \right)^{\frac{1}{2}}$$

implies $w(t_j) \leq w_e(F)$ for each $j = 1, \dots, n$. Thus, if $w_e(F) \leq 1$, then $w(t_j) \leq 1$ for each $j = 1, \dots, n$. The power inequality [15] implies that $w(t_j^m) \leq 1$ for each $j = 1, \dots, n$, whenever $w(t_j) \leq 1$. Therefore, if $w(t_j) \leq 1$, then

$$\begin{aligned} w_e(F^m) &= \sup_{\psi(|\xi|=1} \left(\sum_{j=1}^n |\psi(\langle x, t_j^m x \rangle)|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n \sup_{\psi(|\xi|=1} |\psi(\langle x, t_j^m x \rangle)|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^n w^2(t_j^m) \right)^{\frac{1}{2}} \leq \sqrt{n}. \end{aligned}$$

Now, if we take $t'_j = \frac{t_j}{w(F)}$ for all $j = 1, \dots, n$, then $w_e(F') = 1$, where $F' = (t'_1, \dots, t'_n)$, and so $w(t'_j) \leq 1$. Thus, $w_e((F')^m) \leq \sqrt{n}$, and this gives $w_e(F^m) \leq \sqrt{n} w_e^m(F)$. $\square$

The example below illustrates Theorem 12.

Example 1. Let $L^2([0, 1])$ be the vector space of all continuous real-valued functions on $[0, 1]$ with the norm defined by

$$\psi(|f|) = \left(\int_0^1 f^2(s) ds \right)^{\frac{1}{2}}, s \in [0, 1]$$

and the inner product defined by

$$\langle f, g \rangle = \int_0^1 f(s)g(s) ds, s \in [0, 1].$$

Now, let $F = (t_1, t_2, t_3)$ with $t_j = s^j$, $j = 1, 2, 3$. Then,

$$\begin{aligned} w_e(F) &= \sup \left\{ \left(\sum_{j=1}^3 |\psi(\langle x, t_j(x) \rangle)|^2 \right)^{1/2} : \psi(|x|) = 1, \psi \in \omega(\mathfrak{A}) \right\} \\ &= \sup_{\psi(|x|=1} \left\{ \left(\sum_{j=1}^3 \left(\int_0^1 x x^j dx \right)^2 \right)^{1/2} \right\} \\ &= \sup_{\psi(|x|=1} \left\{ \left(\left(\int_0^1 x^2 dx \right)^2 + \left(\int_0^1 x^3 dx \right)^2 + \left(\int_0^1 x^4 dx \right)^2 \right)^{1/2} \right\} \\ &= 0.46218. \end{aligned}$$

Also,

$$\begin{aligned}
w_e(F^3) &= \sup \left\{ \left(\sum_{j=1}^3 |\psi(\langle x, t_{3j}(x) \rangle)|^2 \right)^{1/2} : \psi(|x|) = 1, \psi \in \omega(\mathfrak{A}) \right\} \\
&= \sup_{\psi(|x|=1)} \left\{ \left(\sum_{j=1}^3 \left(\int_0^1 x^{3j+1} dx \right)^2 \right)^{1/2} \right\} \\
&= \sup_{\psi(|x|=1)} \left\{ \left(\left(\int_0^1 x^4 dx \right)^2 + \left(\int_0^1 x^7 dx \right)^2 + \left(\int_0^1 x^{10} dx \right)^2 \right)^{1/2} \right\} \\
&= 0.25276.
\end{aligned}$$

Now,

$$w_e(F^3) = 0.25276 \leq \sqrt{3}w_e^3(F) = \sqrt{3}(0.46218)^3 = 0.8005.$$

Thus, the validity of the outcome stated in Theorem 12 is confirmed.

Applying Theorem 12, we derive the following inequality.

Corollary 2. Let $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. If $w_e(F) \leq 1$, then

$$\|F^m\| \leq 2n.$$

Proof. It follows from inequality (3), together with Theorem 12, that

$$\frac{\|F^m\|}{2\sqrt{n}} \leq w_e(F^m) \leq \sqrt{n}w_e^m(F) \leq \sqrt{n}.$$

□

Now, we will establish the inequalities for the Euclidean operator radius when multiplying n -tuple operators. To show the Euclidean operator norm's submultiplicative property and the Euclidean operator radius's subadditive property, we require the following lemma.

Lemma 6. Let $\mathcal{Q} = (q_1, \dots, q_n)$, $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. Then, the following inequalities hold:

- (a) $\|F\mathcal{Q}\| \leq \|F\| \|\mathcal{Q}\|$.
- (b) $w_e(F + \mathcal{Q}) \leq w_e(F) + w_e(\mathcal{Q})$.

Proof. (a) Let $\xi \in \mathfrak{E}$ with $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$. Then, we have

$$\begin{aligned}
\|F\mathcal{Q}\| &= \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|t_j q_j \xi|^2) \right)^{\frac{1}{2}} \leq \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \psi(|q_j|^2) \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \\
&\leq \sup_{\psi(|\xi|=1)} \left(\sum_{j=1}^n \|q_j\|^2 \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} \\
&\quad \left(\text{since } \psi(|q_j \xi|) \leq \left(\sum_{j=1}^n \psi(|q_j \xi|^2) \right)^{\frac{1}{2}} \text{ and } \|q_j\| \leq \|\mathcal{Q}\| \text{ for all } j \right) \\
&= \|\mathcal{Q}\| \left(\sum_{j=1}^n \psi(|t_j \xi|^2) \right)^{\frac{1}{2}} = \|F\| \|\mathcal{Q}\|.
\end{aligned}$$

(b) See the proof of Theorem 5. □

Theorem 13. Let $\mathcal{Q} = (q_1, \dots, q_n)$, $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$w_e(F\mathcal{Q}) \leq 4nw_e(F)w_e(\mathcal{Q}).$$

Proof. We have $w_e(F\mathcal{Q}) \leq \|F\mathcal{Q}\| \leq \|F\| \|\mathcal{Q}\| \leq 4nw_e(F)w_e(\mathcal{Q})$, where the second inequality is derived from Lemma 6(a), and the third inequality is derived from (3). $\square$

Moreover, we derive an inequality concerning the collective numerical radius for the multiplication of two n -tuple operators $\mathcal{Q}$ and F , given the condition $F\mathcal{Q} = \mathcal{Q}F$.

Theorem 14. Let $\mathcal{Q} = (q_1, \dots, q_n)$, $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. If $F\mathcal{Q} = \mathcal{Q}F$, i.e., $(t_j q_j = q_j t_j)$ for all $j = 1, \dots, n$, then

$$w_e(F\mathcal{Q}) \leq 2\sqrt{n}w_e(F)w_e(\mathcal{Q}).$$

Proof. Assume that $w_e(F) = w_e(\mathcal{Q}) = 1$. Then, we have

$$\begin{aligned} w_e(F\mathcal{Q}) &= w_e\left(\frac{1}{4}(F + \mathcal{Q})^2 - \frac{1}{4}(F - \mathcal{Q})^2\right) \\ &\leq \frac{1}{4}w_e((F + \mathcal{Q})^2) + \frac{1}{4}w_e((F - \mathcal{Q})^2) \quad (\text{using the fact that } w_e(\cdot) \text{ is a norm}) \\ &\leq \frac{\sqrt{n}}{4}w_e^2(F + \mathcal{Q}) + \frac{\sqrt{n}}{4}w_e^2(F - \mathcal{Q}) \quad (\text{by Theorem 12}) \\ &\leq \frac{\sqrt{n}}{4}(w_e(F) + w_e(\mathcal{Q}))^2 + \frac{\sqrt{n}}{4}(w_e(F) - w_e(\mathcal{Q}))^2 \quad (\text{by Lemma 6}) \\ &= 2\sqrt{n}. \end{aligned}$$

This proof is complete. $\square$

The next bound for the product of two n -tuple normal operators reads as follows.

Theorem 15. Let $\mathcal{Q} = (q_1, \dots, q_n)$, $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$. If F and $\mathcal{Q}$ are normal, then

$$w_e(F\mathcal{Q}) \leq w_e(F)w_e(\mathcal{Q}).$$

Proof. We have $w_e(F\mathcal{Q}) \leq \|F\mathcal{Q}\| \leq \|F\| \|\mathcal{Q}\| = w_e(F)w_e(\mathcal{Q})$, where the last equality follows from $w_e(F) = \|F\|$ and $w_e(\mathcal{Q}) = \|\mathcal{Q}\|$, as $F, \mathcal{Q}$ are both normal (see [16]). $\square$

We wrap up this section with the subsequent theorem concerning the joint spectral radius and joint numerical radius. We begin by introducing the notion of the joint approximate point spectrum for an n -tuple operator F , represented by σ_π , which is defined as follows:

$$\begin{aligned} \sigma_\pi(F) &= \{(\lambda_1, \dots, \lambda_n) : \text{there exists } \{\xi_m\} \subset \mathfrak{E}; \psi(|\xi_m|) = 1, \psi \in \varphi(\mathfrak{A}), \\ &\quad \lim_{m \rightarrow +\infty} \sum_{j=1}^n \psi(|(t_j - \lambda_j 1)\xi_m|) = 0\}. \end{aligned}$$

This definition can be equivalently expressed as the existence of a sequence $\{\xi_m\} \subset \mathfrak{E}$ with $\psi(|\xi_m|) = 1$ such that $\lim_{m \rightarrow +\infty} \psi(|(t_j - \lambda_j 1)\xi_m|) = 0$ for all $j = 1, 2, \dots, n$. For a commuting n -tuple operator $F = (t_1, \dots, t_n)$, $\sigma(F)$ represents the joint spectrum of F , as defined in [16]. Notably, it follows that $\sigma_\pi(F) \subseteq \sigma(F)$.

For an n -tuple commuting operator $F = (t_1, \dots, t_n) \in \mathcal{L}^n(\mathfrak{E})$, we introduce the non-negative number $r(F)$, defined as $r(F) = \sup \left\{ \left(\sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}} : (z_1, \dots, z_n) \in \sigma(F) \right\}$, which is referred to as the joint spectral radius of F . For an n -tuple commuting operator $F = (t_1, \dots, t_n)$, the inequality $r(F) \leq w_e(F)$ is valid.

Theorem 16. Let $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$ be commuting. Then, the following statements are equivalent.

- (i) $r(F) = \|F\|$.
- (ii) $w_e(F) = \|F\|$.

Proof. (i) $\implies$ (ii). Assume that $r(F) = \|F\|$. It follows from $r(F) \leq w_e(F) \leq \|F\|$ that $w_e(F) = \|F\|$.

(ii) $\implies$ (i). Let $\|F\| = w_e(F)$. Then, there exists a sequence $\{\xi_m\} \subset \mathfrak{E}$ with $\psi(|\xi_m|) = 1$ such that

$$\begin{aligned} & \lim_{m \rightarrow +\infty} \|(\psi(\langle \xi_m, t_1 \xi_m \rangle), \psi(\langle \xi_m, t_2 \xi_m \rangle), \dots, \psi(\langle \xi_m, t_n \xi_m \rangle))\| \\ &= \lim_{m \rightarrow +\infty} \left(\sum_{j=1}^n |\psi(\langle \xi_m, t_j \xi_m \rangle)|^2 \right)^{\frac{1}{2}} = \|F\|. \end{aligned}$$

Without loss of generality, assume that $(\psi(\langle \xi_m, t_1 \xi_m \rangle), \dots, \psi(\langle \xi_m, t_n \xi_m \rangle))$ converges to $\lambda = (\lambda_1, \dots, \lambda_n)$, and the sequence $\left(\sum_{j=1}^n |\psi(\langle \xi_m, t_j \xi_m \rangle)|^2 \right)^{\frac{1}{2}}$ converges to b . Then, $\|F\| = |\lambda|$. Now,

$$\begin{aligned} \sum_{j=1}^n \psi(|(t_j - \lambda_j 1) \xi_m|^2) &= \sum_{j=1}^n \psi(|t_j \xi_m|^2) + \sum_{j=1}^n |\lambda_j|^2 - 2 \operatorname{Re} \left(\sum_{j=1}^n \overline{\lambda_j} \psi(\langle \xi_m, t_j \xi_m \rangle) \right) \\ &\longrightarrow b^2 + \|F\|^2 - 2|\lambda|^2 = b^2 - |\lambda|^2 \leq 0. \end{aligned}$$

Hence, $\sum_{j=1}^n \psi(|(t_j - \lambda_j 1) \xi_m|^2) \longrightarrow 0$, and so $\lambda = (\lambda_1, \dots, \lambda_n) \in \sigma_\pi(F)$. This implies $w_e(F) \leq r(F)$. Hence, $r(F) = w_e(F) = \|F\|$. $\square$

3. Euclidean Operator Radius of 2×2 Matrices

For $F = (t_1, \dots, t_n)$, $\mathcal{Q} = (q_1, \dots, q_n)$, $\mathcal{Z} = (z_1, \dots, z_n)$, $\mathcal{Y} = (y_1, \dots, y_n) \in \mathbb{L}^n(\mathfrak{E})$, the 2×2 operator matrix, whose entries are n -tuple operators $F, \mathcal{Q}, \mathcal{Z}, \mathcal{Y}$, is defined as

$$\begin{bmatrix} F & \mathcal{Q} \\ \mathcal{Y} & \mathcal{Z} \end{bmatrix} = \left(\begin{bmatrix} t_1 & q_1 \\ y_1 & z_1 \end{bmatrix}, \dots, \begin{bmatrix} t_n & q_n \\ y_n & z_n \end{bmatrix} \right) \in \mathbb{L}^n(\mathfrak{E} \oplus \mathfrak{E}).$$

Note that $\mathfrak{E} \oplus \mathfrak{E}$ is a Hilbert $\mathfrak{A}$ -Module with the inner product defined as

$$\psi(\langle (\xi_1, \xi_2), (\zeta_1, \zeta_2) \rangle) = \psi(\langle \xi_1, \zeta_1 \rangle) + \psi(\langle \xi_2, \zeta_2 \rangle),$$

for all $(\xi_1, \xi_2), (\zeta_1, \zeta_2) \in \mathfrak{E} \oplus \mathfrak{E}$ and $\psi \in \omega(\mathfrak{A})$. We prove the next lemma to start this section.

Lemma 7. Let $F = (t_1, \dots, t_n)$, $\mathcal{Q} = (q_1, \dots, q_n) \in \mathbb{L}^n(\mathfrak{E})$. Then, the following assertions hold:

- (i) $w_e \left(\begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} \right) = \max\{w_e(F), w_e(\mathcal{Q})\}$.
- (ii) $\left\| \begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} \right\| = \max\{\|F\|, \|\mathcal{Q}\|\}$. In particular, $\left\| \begin{bmatrix} F & 0 \\ 0 & F^* \end{bmatrix} \right\| = \|F\|$.
- (iii) $w_e \left(\begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right) = w_e \left(\begin{bmatrix} 0 & \mathcal{Q} \\ F & 0 \end{bmatrix} \right)$.
- (iv) $w_e \left(\begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right) = w_e \left(\begin{bmatrix} 0 & F \\ \mathcal{Q} e^{i\theta} & 0 \end{bmatrix} \right)$ for all $\theta \in \mathbb{R}$.
- (v) $w_e \left(\begin{bmatrix} F & \mathcal{Q} \\ \mathcal{Q} & F \end{bmatrix} \right) = \max\{w_e(F + \mathcal{Q}), w_e(F - \mathcal{Q})\}$. In particular,

$$w_e \left(\begin{bmatrix} 0 & F \\ F & 0 \end{bmatrix} \right) = w_e(F).$$

$$(vi) \quad \left\| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right\| = \max\{\|F\|, \|\mathcal{Q}\|\}. \text{ In particular, } \left\| \begin{bmatrix} 0 & F \\ F^* & 0 \end{bmatrix} \right\| = \|F\|.$$

Proof. (i) Let $\psi \in \omega(\mathfrak{A})$ and $u = (\zeta, \zeta) \in \mathfrak{E} \oplus \mathfrak{E}$ such that $\psi(|u|) = 1$, i.e., $\psi(|\zeta|^2) + \psi(|\zeta|^2) = 1$. Then,

$$\begin{aligned} & \left(\sum_{j=1}^n \left| \left\langle u, \begin{bmatrix} t_j & 0 \\ 0 & q_j \end{bmatrix} u \right\rangle \right|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^n \left| \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} t_j & 0 \\ 0 & q_j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \right|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{j=1}^n |\langle x, t_j \zeta \rangle + \langle y, q_j \zeta \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^n |\langle x, t_j \zeta \rangle|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n |\langle y, q_j \zeta \rangle|^2 \right)^{\frac{1}{2}} \quad (\text{by Minkowski's inequality}) \\ &\leq w_e(F) \psi(|\zeta|^2) + w_e(\mathcal{Q}) \psi(|\zeta|^2) \\ &\leq \max\{w_e(F), w_e(\mathcal{Q})\} (\psi(|\zeta|^2) + \psi(|\zeta|^2)) = \max\{w_e(F), w_e(\mathcal{Q})\}. \end{aligned}$$

Taking the supremum over all $u \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|u|) = 1$ and $\psi \in \omega(\mathfrak{A})$, we obtain

$$w_e \left(\begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} \right) \leq \max\{w_e(F), w_e(\mathcal{Q})\}.$$

Suppose $u = (\zeta, 0) \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|\zeta|) = 1$ and $\psi \in \omega(\mathfrak{A})$. Then, we have

$$\left(\sum_{j=1}^n \left| \left\langle u, \begin{bmatrix} t_j & 0 \\ 0 & q_j \end{bmatrix} u \right\rangle \right|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^n |\langle x, t_j \zeta \rangle|^2 \right)^{\frac{1}{2}}.$$

Taking the supremum over all $u = (\zeta, 0) \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|u|) = \psi(|\zeta|) = 1$, we obtain

$$\sup_{\psi(|u|=1)} \left(\sum_{j=1}^n \left| \left\langle u, \begin{bmatrix} t_j & 0 \\ 0 & q_j \end{bmatrix} u \right\rangle \right|^2 \right)^{\frac{1}{2}} = w_e(F).$$

This implies that $w_e \left(\begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} \right) \geq w_e(F)$. Similarly, $w_e \left(\begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} \right) \geq w_e(\mathcal{Q})$. Hence, $w_e \left(\begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} \right) \geq \max\{w_e(F), w_e(\mathcal{Q})\}$.

(ii) Let $\psi \in \omega(\mathfrak{A})$ and $u = (\zeta, \zeta) \in \mathfrak{E} \oplus \mathfrak{E}$ such that $\psi(|u|) = 1$, i.e., $\psi(|\zeta|^2) + \psi(|\zeta|^2) = 1$. Then,

$$\begin{aligned} \sum_{j=1}^n \psi \left(\left| \begin{bmatrix} F & 0 \\ 0 & \mathcal{Q} \end{bmatrix} u \right|^2 \right) &= \sum_{j=1}^n \psi(|(t_j \zeta, q_j \zeta)|^2) \\ &\leq \sum_{j=1}^n \psi(|t_j \zeta|^2) + \psi(|q_j \zeta|^2) \\ &\leq \|F\|^2 \psi(|\zeta|^2) + \|\mathcal{Q}\|^2 \psi(|\zeta|^2) \\ &\leq \max\{\|F\|^2, \|\mathcal{Q}\|^2\} (\psi(|\zeta|^2) + \psi(|\zeta|^2)). \end{aligned}$$

Taking the supremum over all $u \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|u|) = 1$ and $\psi \in \omega(\mathfrak{A})$, we obtain

$$\left\| \begin{bmatrix} F & 0 \\ 0 & Q \end{bmatrix} \right\| \leq \max\{\|F\|, \|Q\|\}.$$

Suppose $u = (\xi, 0) \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$. Then, we have

$$\sum_{j=1}^n \psi \left(\left| \begin{bmatrix} F & 0 \\ 0 & Q \end{bmatrix} u \right|^2 \right) = \sum_{j=1}^n \psi(|t_j \xi|^2).$$

Taking the supremum over all $u = (\xi, 0)$ with $\psi(|u|) = 1$, we obtain

$$\sup_{\psi(|u|=1)} \sum_{j=1}^n \psi \left(\left| \begin{bmatrix} F & 0 \\ 0 & Q \end{bmatrix} u \right|^2 \right) = \sup_{\psi(|\xi|=1)} \sum_{j=1}^n \psi(|t_j \xi|^2) = \|F\|^2.$$

This implies that $\left\| \begin{bmatrix} F & 0 \\ 0 & Q \end{bmatrix} \right\| \geq \|F\|$. Similarly, $\left\| \begin{bmatrix} F & 0 \\ 0 & Q \end{bmatrix} \right\| \geq \|Q\|$. Therefore,

$$\left\| \begin{bmatrix} F & 0 \\ 0 & Q \end{bmatrix} \right\| \geq \max\{\|F\|, \|Q\|\}.$$

In particular, if $Q = F^*$, then $\left\| \begin{bmatrix} F & 0 \\ 0 & F^* \end{bmatrix} \right\| \geq \max\{\|F\|, \|F^*\|\} = \|F\|$ (because $\|F^*\| = \|F\|$).

(iii) It follows from Theorem 11(b) that

$$w_e(u^* F u) = w_e(u^* t_1 u, \dots, u^* t_n u) = w_e(t_1, \dots, t_n) = w_e(F). \quad (13)$$

for every unitary operator $u \in \mathcal{L}(\mathfrak{E})$. If we let $u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$w_e \left(\begin{bmatrix} 0 & Q \\ F & 0 \end{bmatrix} \right) = w_e \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & Q \\ F & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & F \\ Q & 0 \end{bmatrix}.$$

(iv) The proof follows from (13) by taking $u = \begin{bmatrix} 1 & 0 \\ 0 & 1e^{\frac{i\theta}{2}} \end{bmatrix}$.

(v) Let $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $t_j = \begin{bmatrix} y_j & z_j \\ z_j & y_j \end{bmatrix}$. Then, $u^* t_j u = \begin{bmatrix} y_j - z_j & 0 \\ 0 & y_j + z_j \end{bmatrix}$. Using

(i) and (13), we obtain $w_e \left(\begin{bmatrix} \mathcal{Y} & \mathcal{Z} \\ \mathcal{Z} & \mathcal{Y} \end{bmatrix} \right) = \max\{w_e(\mathcal{Y} - \mathcal{Z}), w_e(\mathcal{Y} + \mathcal{Z})\}$. In particular, if we take $\mathcal{Y} = 0$, then $w_e \left(\begin{bmatrix} 0 & \mathcal{Z} \\ \mathcal{Z} & 0 \end{bmatrix} \right) = w_e(\mathcal{Z})$.

(vi) Let $\psi \in \omega(\mathfrak{A})$ and $u = (\xi, \xi) \in \mathfrak{E} \oplus \mathfrak{E}$ such that $\psi(|u|) = 1$, i.e., $\psi(|\xi|^2) + \psi(|\xi|^2) = 1$. Then,

$$\begin{aligned} \sum_{j=1}^n \psi \left(\left| \begin{bmatrix} 0 & F \\ Q & 0 \end{bmatrix} u \right|^2 \right) &= \sum_{j=1}^n \psi(|(t_j \xi, q_j \xi)|^2) \\ &\leq \sum_{j=1}^n \psi(|t_j \xi|^2) + \psi(|q_j \xi|^2) \\ &\leq \|F\|^2 \psi(|\xi|^2) + \|Q\|^2 \psi(|\xi|^2) \\ &\leq \max\{\|F\|^2, \|Q\|^2\} (\psi(|\xi|^2) + \psi(|\xi|^2)). \end{aligned}$$

Taking the supremum over all $u \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|u|) = 1$ and $\psi \in \omega(\mathfrak{A})$, we obtain

$$\left\| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right\| \leq \max\{\|F\|, \|\mathcal{Q}\|\}.$$

Suppose $u = (\xi, 0) \in \mathfrak{E} \oplus \mathfrak{E}$ with $\psi(|\xi|) = 1$ and $\psi \in \omega(\mathfrak{A})$. Then, we have

$$\sum_{j=1}^n \psi \left(\left| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} u \right|^2 \right) = \sum_{j=1}^n \psi(|q_j \xi|^2).$$

Taking the supremum over all $u = (\xi, 0)$ with $\psi(|u|) = 1$, we obtain

$$\sup_{\psi(|u|=1)} \sum_{j=1}^n \psi \left(\left| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} u \right|^2 \right) = \sup_{\psi(|\xi|=1)} \sum_{j=1}^n \psi(|q_j \xi|^2) = \|\mathcal{Q}\|^2.$$

This implies that $\left\| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right\| \geq \|\mathcal{Q}\|$. Similarly, $\left\| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right\| \geq \|F\|$. Therefore,

$$\left\| \begin{bmatrix} 0 & F \\ \mathcal{Q} & 0 \end{bmatrix} \right\| \geq \max\{\|F\|, \|\mathcal{Q}\|\}.$$

□

Hereafter, we set an upper limit for the Euclidean operator radius of 2×2 operator matrices, where the elements comprise n -tuple operators.

Theorem 17. Let $\mathcal{X} = (x_1, x_2, \dots, x_n)$, $\mathcal{Y} = (y_1, \dots, y_n)$, $\mathcal{Z} = (z_1, \dots, z_n)$, $\mathcal{W} = (w_1, \dots, w_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$w_e \left(\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & \mathcal{W} \end{bmatrix} \right) \leq w \left(\begin{bmatrix} w_e(\mathcal{X}) & \|\mathcal{Y}\| \\ \|\mathcal{Z}\| & w_e(\mathcal{W}) \end{bmatrix} \right).$$

Proof. Let $\psi \in \omega(\mathfrak{A})$ and $u = (\xi, \zeta) \in \mathfrak{E} \oplus \mathfrak{E}$ such that $\psi(|u|) = 1$, i.e., $\psi(|\xi|^2) + \psi(|\zeta|^2) = 1$. Then, by Minkowski's inequality, we have

$$\begin{aligned} & \left(\sum_{j=1}^n \psi \left(\left| \left\langle u, \begin{bmatrix} x_j & y_j \\ z_j & w_j \end{bmatrix} u \right\rangle \right|^2 \right) \right)^{\frac{1}{2}} = \left(\sum_{j=1}^n \psi(|\langle (\xi, \zeta), (x_j \xi + y_j \zeta, z_j \xi + w_j \zeta) \rangle|^2) \right)^{\frac{1}{2}} \\ & \leq \left(\sum_{j=1}^n \psi(|\langle \xi, x_j \xi \rangle| + \psi(|\langle \zeta, w_j \zeta \rangle|)^2) \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n \psi(|\langle \xi, y_j \zeta \rangle| + \psi(|\langle \zeta, z_j \xi \rangle|)^2) \right)^{\frac{1}{2}} \\ & \leq \left(\sum_{j=1}^n \psi(|\langle \xi, x_j \xi \rangle|)^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n \psi(|\langle \zeta, w_j \zeta \rangle|)^2 \right)^{\frac{1}{2}} \\ & + \left(\sum_{j=1}^n \psi(|\langle \xi, y_j \zeta \rangle|)^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n \psi(|\langle \zeta, z_j \xi \rangle|)^2 \right)^{\frac{1}{2}} \\ & \leq w_e(\mathcal{X})\psi(|\xi|^2) + w_e(\mathcal{W})\psi(|\zeta|^2) \\ & + \left(\sum_{j=1}^n \psi(|y_j \zeta|^2)\psi(|\xi|^2) \right)^{\frac{1}{2}} + \left(\sum_{j=1}^n \psi(|z_j \xi|^2)\psi(|\zeta|^2) \right)^{\frac{1}{2}} \\ & \leq w_e(\mathcal{X})\psi(|\xi|^2) + w_e(\mathcal{W})\psi(|\zeta|^2) + \|\mathcal{Y}\|\psi(|\xi|)\psi(|\zeta|) + \|\mathcal{Z}\|\psi(|\xi|)\psi(|\zeta|) \\ & = \psi \left(\tilde{\xi}, \begin{bmatrix} w_e(\mathcal{X}) & \|\mathcal{Y}\| \\ \|\mathcal{Z}\| & w_e(\mathcal{W}) \end{bmatrix} \tilde{\xi} \right), \text{ where} \\ & \tilde{\xi} = (\psi(|\xi|), \psi(|\zeta|)) \in \mathbb{C}^2. \end{aligned}$$

Thus,

$$w_e\left(\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & \mathcal{W} \end{bmatrix}\right) = \sup_{\psi(|u|=1)} \left\{ \left(\sum_{j=1}^n \left| \psi \left(\left\langle u, \begin{bmatrix} x_j & y_j \\ z_j & w_j \end{bmatrix} u \right\rangle \right) \right|^2 \right)^{\frac{1}{2}} \right\} \leq w\left(\begin{bmatrix} w_e(\mathcal{X}) & \|\mathcal{Y}\| \\ \|\mathcal{Z}\| & w_e(\mathcal{W}) \end{bmatrix}\right).$$

□

As a consequence of Theorem 3, we can conclude that $w_e(\mathcal{X}) \leq \|\mathcal{X}\|$ and $w_e(\mathcal{W}) \leq \|\mathcal{W}\|$. Additionally, it is evident that $w([a_{ij}]) \leq w([b_{ij}])$ for all $0 \leq a_{ij} \leq b_{ij}$. Hence, the subsequent corollary directly stems from Theorem 17.

Corollary 3. Let $\mathcal{X} = (x_1, x_2, \dots, x_n), \mathcal{Y} = (y_1, \dots, y_n), \mathcal{Z} = (z_1, \dots, z_n), \mathcal{W} = (w_1, \dots, w_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$w_e\left(\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & \mathcal{W} \end{bmatrix}\right) \leq w\left(\begin{bmatrix} \|\mathcal{X}\| & \|\mathcal{Y}\| \\ \|\mathcal{Z}\| & \|\mathcal{W}\| \end{bmatrix}\right).$$

It is important to highlight that Theorem 17 provides a stronger bound compared to Corollary 3. To proceed with the next conclusion, we must refer to the following lemma.

Lemma 8 ([15]). Consider an $n \times n$ matrix $t = [t_{ij}]$, where $t_{ij} \geq 0$ for all $i, j = 1, 2, \dots, n$. In this case, $w(t) = r\left(\frac{t_{ij} + t_{ji}}{2}\right)$, where $r(\cdot)$ represents the spectral radius.

The subsequent series of corollaries are derived by utilizing Lemma 8, Theorem 17, and Corollary 3.

Corollary 4. Let $\mathcal{X} = (x_1, x_2, \dots, x_n), \mathcal{Y} = (y_1, \dots, y_n), \mathcal{Z} = (z_1, \dots, z_n), \mathcal{W} = (w_1, \dots, w_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$w_e\left(\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & \mathcal{W} \end{bmatrix}\right) \leq r([c_{ij}]) = \frac{1}{2} \left(w_e(\mathcal{X}) + w_e(\mathcal{W}) + \sqrt{(w_e(\mathcal{X}) - w_e(\mathcal{W}))^2 + (\|\mathcal{Y}\| + \|\mathcal{Z}\|)^2} \right),$$

where $c_{11} = w_e(\mathcal{X}), c_{12} = c_{21} = \frac{\|\mathcal{Y}\| + \|\mathcal{Z}\|}{2}$ and $c_{22} = w_e(\mathcal{W})$.

Corollary 5. Let $\mathcal{X} = (x_1, x_2, \dots, x_n), \mathcal{Y} = (y_1, \dots, y_n), \mathcal{Z} = (z_1, \dots, z_n), \mathcal{W} = (w_1, \dots, w_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$w_e\left(\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & \mathcal{W} \end{bmatrix}\right) \leq r([c_{ij}]) = \frac{1}{2} \left(\|\mathcal{X}\| + \|\mathcal{W}\| + \sqrt{(\|\mathcal{X}\| - \|\mathcal{W}\|)^2 + (\|\mathcal{Y}\| + \|\mathcal{Z}\|)^2} \right),$$

where $c_{11} = \|\mathcal{X}\|, c_{12} = c_{21} = \frac{\|\mathcal{Y}\| + \|\mathcal{Z}\|}{2}$ and $c_{22} = \|\mathcal{W}\|$.

Next, we establish a lower limit for the Euclidean operator radius of 2×2 operator matrices composed of n -tuple operators, utilizing the power inequality derived in Theorem 12.

Theorem 18. Let $\mathcal{X} = (x_1, x_2, \dots, x_n), \mathcal{Y} = (y_1, \dots, y_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$\sqrt[2m]{\frac{1}{\sqrt{n}} \max\{w_e((\mathcal{X}\mathcal{Y})^m), w_e((\mathcal{Y}\mathcal{X})^m)\}} \leq w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right).$$

Proof. Let $F = \begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}$. Then, $F^{2m} = \begin{bmatrix} (\mathcal{X}\mathcal{Y})^m & 0 \\ 0 & (\mathcal{Y}\mathcal{X})^m \end{bmatrix}$ for all $m = 1, 2, \dots, n$. Using Lemma 7 and Theorem 12, we obtain

$$\max\{w_e((\mathcal{X}\mathcal{Y})^m), w_e((\mathcal{Y}\mathcal{X})^m)\} = w_e(F^{2m}) \leq \sqrt{n} w_e^{2m}(F).$$

So, the proof is complete. $\square$

Next, we will establish the subsequent lower and upper bounds.

Theorem 19. Let $\mathcal{X} = (x_1, x_2, \dots, x_n), \mathcal{Y} = (y_1, \dots, y_n) \in \mathbb{L}^n(\mathfrak{E})$. Then,

$$\frac{1}{2}(\max\{w_e(\mathcal{X} - \mathcal{Y}), w_e(\mathcal{X} + \mathcal{Y})\}) \leq w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right) \leq \frac{1}{2}(w_e(\mathcal{X} - \mathcal{Y}) + w_e(\mathcal{X} + \mathcal{Y})).$$

Proof. It follows from Lemma 7(v) that

$$\begin{aligned} w_e(\mathcal{X} + \mathcal{Y}) &= w_e\left(\begin{bmatrix} 0 & \mathcal{X} + \mathcal{Y} \\ \mathcal{X} + \mathcal{Y} & 0 \end{bmatrix}\right) \\ &= w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathcal{Y} \\ \mathcal{X} & 0 \end{bmatrix}\right) \\ &\leq w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right) + w_e\left(\begin{bmatrix} 0 & \mathcal{Y} \\ \mathcal{X} & 0 \end{bmatrix}\right) \quad (\text{since } w_e(\cdot) \text{ is a norm}) \\ &= 2w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right) \quad (\text{by Lemma 7(iii)}). \end{aligned}$$

By replacing $\mathcal{Y}$ with $-\mathcal{Y}$, we have

$$w_e(\mathcal{X} - \mathcal{Y}) \leq 2w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ -\mathcal{Y} & 0 \end{bmatrix}\right) = 2w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right).$$

Consequently,

$$\frac{1}{2}(\max\{w_e(\mathcal{X} - \mathcal{Y}), w_e(\mathcal{X} + \mathcal{Y})\}) \leq w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right).$$

To prove the second inequality, consider a unitary operator. Let $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then, we have

$$\begin{aligned} w_e\left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix}\right) &= w_e\left(u^* \begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix} u\right) \\ &= \frac{1}{2}w_e\left(\begin{bmatrix} \mathcal{X} + \mathcal{Y} & \mathcal{X} - \mathcal{Y} \\ -(\mathcal{X} - \mathcal{Y}) & -(\mathcal{X} + \mathcal{Y}) \end{bmatrix}\right) \\ &\leq \frac{1}{2}w_e\left(\begin{bmatrix} \mathcal{X} + \mathcal{Y} & 0 \\ 0 & -(\mathcal{X} + \mathcal{Y}) \end{bmatrix}\right) + \frac{1}{2}w_e\left(\begin{bmatrix} 0 & \mathcal{X} - \mathcal{Y} \\ -(\mathcal{X} - \mathcal{Y}) & 0 \end{bmatrix}\right) \\ &\quad (\text{since } w_e(\cdot) \text{ is a norm}) \\ &= \frac{1}{2}(w_e(\mathcal{X} + \mathcal{Y}) + w_e(\mathcal{X} - \mathcal{Y})) \quad (\text{by Lemma 7(iii)}). \end{aligned}$$

The proof of the theorem is now concluded. $\square$

The following example illustrates Theorem 19.

Example 2. Let $\mathcal{X} = (x_1, x_2), \mathcal{Y} = (y_1, y_2) \in \mathbb{L}^n(\mathfrak{E})$, where $\mathfrak{E} = M_2(\mathbb{C})$, $x_1 = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$, $y_1 = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$, and $y_2 = \begin{bmatrix} 8 & 5 \\ 1 & 2 \end{bmatrix}$. First, let us compute $\|F\|$:

$$\|F\| = \|[x_1, x_2]\| = \|x_1^* x_1 + x_2^* x_2\|^{\frac{1}{2}} = \left\| \begin{bmatrix} 20 & -9 \\ -9 & 13 \end{bmatrix} \right\|^{\frac{1}{2}} = \sqrt{\frac{33 + \sqrt{373}}{2}}.$$

Next, let us compute $w_e(F)$:

$$w_e(F) = \sup \left\{ \left(\sum_{k=1}^2 |\psi(\langle \xi, t_k \xi \rangle)|^2 \right)^{\frac{1}{2}} : \xi \in \mathbb{C}^2, \psi \in \omega(\mathfrak{A}), \psi(|\xi|) = 1 \right\}.$$

Now, consider $\mathcal{X} = (x_1, x_2)$ and $\mathcal{Y} = (y_1, y_2)$. We will evaluate the expression

$$\begin{aligned} w_e \left(\begin{bmatrix} 0 & \mathcal{X} \\ \mathcal{Y} & 0 \end{bmatrix} \right) &= w_e \left(\begin{bmatrix} 0 & x_1 & x_2 \\ y_1 & 0 & 0 \\ y_2 & 0 & 0 \end{bmatrix} \right) \\ &= \sup \left\{ \left(\sum_{k=1}^2 |\psi(\langle \xi, t_k \xi \rangle)|^2 \right)^{\frac{1}{2}} : \xi \in \mathbb{C}^2, \psi \in \omega(\mathfrak{A}), \psi(|\xi|) = 1 \right\} \\ &= w_e(\mathcal{X} + \mathcal{Y}) = \sqrt{85}. \end{aligned}$$

With these values, we have

$$\frac{1}{2}(\max w_e(\mathcal{X} - \mathcal{Y}), w_e(\mathcal{X} + \mathcal{Y})) = \frac{1}{2}(\max \sqrt{5}, \sqrt{85}) = \frac{1}{2}\sqrt{85}.$$

Also

$$\frac{1}{2}(w_e(\mathcal{X} - \mathcal{Y}) + w_e(\mathcal{X} + \mathcal{Y})) = \frac{1}{2}(\sqrt{45} + \sqrt{85}).$$

Therefore, the inequality becomes

$$\frac{1}{2}\sqrt{85} \leq \sqrt{85} \leq \frac{1}{2}(\sqrt{45} + \sqrt{85}).$$

We derive the following inequalities by employing Theorem 19.

Corollary 6. Let $F = (t_1, \dots, t_n) \in \mathbb{L}^n(\mathfrak{E})$ for all $j = 1, \dots, n$. Then,

$$\frac{1}{2}w_e(F) \leq w_e \left(\begin{bmatrix} 0 & \mathcal{X} \\ e^{i\theta} \mathcal{Y} & 0 \end{bmatrix} \right) \leq w_e(F).$$

Proof. Replacing $\mathcal{Y}$ with $i\mathcal{Y}$ in Theorem 19 and then using Lemma 7, we have

$$\frac{1}{2}(\max\{w_e(\mathcal{X} + i\mathcal{Y}), w_e(\mathcal{X} - i\mathcal{Y})\}) \leq w_e \left(\begin{bmatrix} 0 & \mathcal{X} \\ e^{i\theta} \mathcal{Y} & 0 \end{bmatrix} \right) \leq \frac{1}{2}(w_e(\mathcal{X} + i\mathcal{Y}) + w_e(\mathcal{X} - i\mathcal{Y})).$$

This implies that

$$\frac{1}{2}(\max\{w_e(F), w_e(F^*)\}) \leq w_e \left(\begin{bmatrix} 0 & \mathcal{X} \\ e^{i\theta} \mathcal{Y} & 0 \end{bmatrix} \right) \leq \frac{1}{2}(w_e(F) + w_e(F^*)).$$

But we know that $w_e(F) = w_e(F^*)$ and so the result. $\square$

4. Conclusions and Future Work

This study shows different kinds of inequalities linked to a mathematical operation called the Euclidean operator radius. We look at groups of these operations, adding them together and multiplying them. We also find a special kind of inequality involving the power of the Euclidean operator radius. We explain how to make these operations into parts of small square arrays called “ 2×2 operator matrices” and use them to understand more about the Euclidean operator radius. We introduce new kinds of inequalities by using a way of measuring called the Euclidean numerical radius and another called the Euclidean operator norm. These inequalities give us rules for the smallest and largest possible values

when we add, multiply, or measure the size of groups of these operations. In addition, we discover more inequalities by studying how the Euclidean numerical radius works with 2×2 operator matrices. Lastly, we establish a rule about the Euclidean operator norm of 2×2 operator matrices.

This study extensively explores the complexities surrounding inequalities associated with the Euclidean operator radius, providing valuable insights into how groups of n -tuple operators interact through addition and multiplication. It introduces a novel power inequality unique to the Euclidean operator radius, thus enhancing comprehension of this foundational concept. By examining the roles of n -tuple operators within 2×2 operator matrices, this research uncovers inequalities related to the Euclidean operator radius, utilizing metrics such as the Euclidean numerical radius and the Euclidean operator norm. Additionally, it presents a variety of innovative inequalities that not only impose constraints on arithmetic operations and the Euclidean numerical radius but also facilitate the establishment of inequalities for the Euclidean operator radius. Through the meticulous analysis of Euclidean numerical radius inequalities within 2×2 operator matrices containing n -tuple operators, this study reveals a fresh inequality that specifically addresses the Euclidean operator norm. Throughout its exploration, the study underscores the importance of identifying and understanding symmetries within linear operators and matrices, highlighting the profound influence of symmetry in mathematics across diverse domains. By elucidating these symmetries, this paper enriches readers' comprehension of mathematical principles and their manifold applications. For future research, it is recommended to delve deeper into these inequalities and their implications in various mathematical contexts, potentially exploring additional dimensions of symmetry and extending the analysis to higher-dimensional operator spaces.

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