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Non-classical correlations beyond Bell's inequality violation in qubit-pairs with an intrinsic noise

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ABSTRACT

In this paper, we analytically explore two qubits interacting with a coherent harmonic oscillator field under intrinsic noise. The interaction of the uncorrelated two qubits with a harmonic oscillator initially in the superposition of coherent states leads to the generation of nonclassical correlations (NCCs). The NCCs are quantified by local quantum uncertainty (LQU), uncertainty-induced non-locality (UIN), concurrence and Bell's inequality. It is found that the generated NCCs are enhanced via the intensity coherence and the parameters of the initial harmonic oscillator state. For the even coherent state, the generated NCCs are more robust against the influence of intrinsic noise rate than that the ones of the coherent state. The appearance of the special type of skew-information correlation (in which, LQU and UIN are equal) depends on the superposition of the coherent states. The phenomena of sudden death and birth of entanglement depend not only on the intrinsic noise rate but also on the initial harmonic oscillator field. The generated NCCs could be controlled by adjusting these parameters.

Introduction

The coherent interactions between a harmonic oscillator and two-level systems (qubits) have potential applications in the development of photon-based quantum information technologies [1]. Tavis-Cummings (TC) model [2] is one of the ubiquitous and well-used models that describes these interactions. It describes the coherent interaction between a collection of identical qubits and a harmonic oscillator. There are various theoretical studies and experimental realizations of the TC-model for atom-photon interaction [3,4] and quantum dot-resonator [5]. Its physical realizations range from super-conducting qubits to quantum wells [6–10]. Multi-qubit properties such as measurement-induced chaos, quantum state discrimination and entanglement [11] can be explored with TC-model via various measures such as: concurrence [12] and negativity [13].

Non-classical correlations (NCCs) become the main investigations focus of quantum information science [14]. By using different methods, several definitions of NCCs were proposed, such as Hilbert-Schmidt norm [16], trace norm [17] and so on. However, the definition of the quantum correlation that capturing both good physical proprieties and computability is the local quantum uncertainty [18], which is based on the skew information [19]. Quantum entanglement (QE) is the most common type of the NCCs between quantum objects [20–23]. QE is not the only kind of NCCs, other types of NCCs beyond QE [24] offer some specific advantages as resources for quantum information processing as: measurement-induced disturbance [25], measurement-induced non-locality (MIN) [26], quantum discord [15], geometric quantum discord [16] and skew information [19].

GQD and MIN that based on 2-norm have been proved that they are not good measures of NCCs because of the increment under local operations on the unmeasured subsystem [27]. Skew-information [19] presents additional types of NCCs. This quantity is defined by Wigner and Yanase to quantify non-commutativity between a bipartite system and a local observable. It measures the uncertainty of the observable in the quantum state. "*local quantum uncertainty*" (LQU) [18] is an NCC based on the skew information used to quantify the minimum quantum uncertainty in a measurement performed on a system. For states that have vanishing discord, the LQU is zero [18]. So it has been introduced as a quantum discord-like measure. LQU is used not only to measure the quantum correlation in several systems [28], but also is applied to the field of quantum metrology [18,29].

Also, based on the skew information, another quantity, "*uncertainty-*

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induced quantum non-locality" (UIN) was presented to measure the quantum correlation [30]. It can be considered as the updated version of the original MIN preserving the good computability but eliminating the non-contractivity problem.

Real quantum systems unavoidably interact with its surrounding environment. There are different types of decoherence. One of them, the Milburn intrinsic decoherence (ID) [31], that leads intrinsically to a diagonalization of the density matrix when the quantum system evolves under a stochastic sequence of identical unitary transformation. Unlike others decoherence models, the ID effects are explored analytically for the interactions between harmonic oscillator (that are initially in a vacuum or a coherent state) and two-level systems [32]. ID model is one of the approaches that prevent or minimize the effect of environmental noise in the practical realization of quantum information [33]. Motivated by these, we would like to extend the previous studies of NCCs with ID model, for the harmonic oscillator field that is initially prepared in a superposition of coherent states.

In previous investigations, the nonclassical correlations in two two-level systems under intrinsic noise are studied for an interaction with a cavity field in vacuum or in Fock states of the harmonic oscillator [34–37] (not with general coherent cavity fields). The extension of the analysis to the superposition of coherent states leads to difficulties to find an analytical description. This is due to the appearance of the summation that runs over all the Fock states of the harmonic oscillator. The coherent states are used widely in the context of quantum information, where they are used to transfer entanglement between atomic state and two-mode cavity state [38], and to perform quantum teleportation [39].

Despite the complexity of the analysis with the superposition of coherent states, we introduce analytical representation based on the dressed states presentation for the model of two qubits interacting with a field and an intrinsic noise. The field is initially prepared in a superposition of coherent states. Our results show that the steady-state correlations can arise with intrinsic noise, and can be improved by the initial state of the harmonic oscillator and its coherence intensity.

The physical model and time evolution

The physical model consists of a qubit which interacts with a harmonic oscillator. This model describes various physical interactions such as: atoms with electromagnetic field [40], nuclear spins with magnetic field [41] and superconducting qubits with a resonator [42,43] or with LC circuit [45,46]. The Hamiltonian of the above system can be written as:

$$H_R = \omega\hat{\psi}^{\dagger}\hat{\psi} + \frac{\omega_0}{2}\left(\left|1\right\rangle\left\langle 1\right| - \left|0\right\rangle\left\langle 0\right|\right) + \lambda\left(\hat{\psi} + \hat{\psi}^{\dagger}\right)\left[\left|1\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right|\right], \tag{1}$$

where $\hat{\psi}^{\dagger}(\hat{\psi})$ represent respectively the creation (annihilation) operator of the harmonic oscillator mode with the frequency ω. Here, the constant λ is the coupling constant between the harmonic oscillator and the qubit (defined by the up $|1\rangle$ and down state $|0\rangle$ with the frequency ω_0). The Rabi Hamiltonian Eq. (1) was generalized to describe two qubits interacting with a harmonic oscillator [47]. If the counter rotating terms: $\hat{\psi}|0\rangle\langle 1|$ and $\hat{\psi}^{\dagger}|1\rangle\langle 0|$ are dropped, the Hamiltonian is given by

$$H = \omega\hat{\psi}^{\dagger}\hat{\psi} + \sum_{i=A,B}\left[\frac{\omega_{0,i}}{2}(|1\rangle_{ii}\langle 1| - |0\rangle_{ii}\langle 0|) + \lambda(\hat{\psi}|1\rangle_i\langle 0| + \hat{\psi}^{\dagger}|0\rangle_i\langle 1|)\right], \tag{2}$$

Here, we focus on the intrinsic noise effects; therefore, the dynamics of the density matrix of the system, $\hat{\rho}(t)$, are governed by the following Milburn equation [31],

$$\frac{d\hat{\rho}(t)}{dt} = \hat{\ell}\hat{\rho}(t), \tag{3}$$

where the super-operator $\hat{\ell}$ is: $\hat{\ell}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] - \gamma(\hat{H}^2\hat{\rho}(t) - 2\hat{H}\hat{\rho}(t)\hat{H} + \hat{\rho}(t)\hat{H}^2)$ and γ is the intrinsic decoherence coupling. More recent applications of coherent states and their superposition in quantum computation have been discussed as well [44]. Therefore, we assume the state of the harmonic oscillator is prepared initially in the state of the type generalized by superposing two "opposite" coherent states as: $\rho^H(0) = |\varphi\rangle\langle\varphi|$ with $|\varphi\rangle = (|\mu\rangle + \varkappa|-\mu\rangle)/\sqrt{\langle\varphi|\varphi\rangle}$ where $|\mu\rangle$ is the coherent state.

While the two qubits start with uncorrelated states (upper states), $|1_A 1_B\rangle\langle 1_A 1_B|$; therefore, the initial uncorrelated states of the entire system is given by

$$\rho(0) = |1_A 1_B\rangle\langle 1_A 1_B| \otimes |\varphi\rangle\langle\varphi|. \tag{4}$$

In the space states $\{|w_1\rangle = |1_A 1_B, n\rangle, |w_2\rangle = |1_A 0_B, n+1\rangle, |w_3\rangle = |0_A 1_B, n+1\rangle, |w_4\rangle = |0_A 0_B, n+2\rangle\}$, the used dressed states $|\Psi_i^m\rangle (i = 1-3)$ of the Hamiltonian (2) are given by

$$\begin{aligned}
|\Psi_1^n\rangle &= x_2^n\sqrt{2}|w_1\rangle - x_1^n\sqrt{2}|w_4\rangle,\\
|\Psi_2^n\rangle &= x_1^n|w_1\rangle - 0.5[|w_2\rangle + |w_3\rangle] + x_2^n|w_4\rangle,\\
|\Psi_3^n\rangle &= x_1^n|w_1\rangle + 0.5[|w_2\rangle + |w_3\rangle] + x_2^n|w_4\rangle,
\end{aligned} \tag{5}$$

where $x_i^n = \sqrt{(n+i)/(4n+6)}\,(i = 1, 2)$. By using the Eq. (3), the dynamical evolution for the dressed states $|\Psi_i^m\rangle\langle\Psi_j^n|$ are given by

$$|\Psi_i^m\rangle\langle\Psi_j^n| \longrightarrow d_{ij}^{mn}|\Psi_i^m\rangle\langle\Psi_j^n|, \tag{6}$$

where $d_{ij}^{mn} = e^{-\gamma(E_i^m - E_j^n)^2 t - i\lambda\left(E_i^m - E_j^n\right)t}$ is the evolution coefficient of the dressed states $|\Psi_i^m\rangle$. Using Eqs. (4) and (6), the expression of the density matrix is then

$$\begin{aligned}
\rho(t) = & \sum_{m,n=0}\beta_m\beta_n\{2x_1^m x_1^n d_{11}^{mn}|\Psi_1^m\rangle\langle\Psi_1^n| + x_2^m x_1^n\sqrt{2}\\
&\times(d_{21}^{mn}|\Psi_2^m\rangle + d_{31}^{mn}|\Psi_3^m\rangle)\langle\Psi_1^n| + x_2^m x_2^n[|\Psi_2^m\rangle(d_{22}^{mn}\langle\Psi_2^n|\\
&+ d_{23}^{mn}\langle\Psi_3^n|) + |\Psi_3^m\rangle(d_{32}^{mn}\langle\Psi_2^n| + d_{33}^{mn}\langle\Psi_3^n|)]\\
&+ x_1^m x_2^n\sqrt{2}[|\Psi_1^m\rangle(d_{12}^{mn}\langle\Psi_2^n| + d_{13}^{mn}\langle\Psi_3^n|)]\},
\end{aligned} \tag{7}$$

where $\beta_n = e^{-|\mu|^2/2}\frac{[1+\varkappa(-1)^n]\mu^n}{\sqrt{n!}}$ with $\varkappa = 0$ (for coherent state), $\varkappa = 1$ (for even coherent state).

To investigate NCCs for two qubits, we need the expression of the time evolution for the reduce density matrix of two qubits, $\rho^{AB}(t)$. By tracing over the coherent harmonic oscillator states, $|k\rangle$, we get

$$\rho^{AB}(t) = \mathrm{tr}_R\left\{\rho(t)\right\} = \sum_{k=0}^{\infty}\left\langle k\middle|\rho(t)\middle|k\right\rangle. \tag{8}$$

Non-classical correlation functions (NCCs)

In this section, we study the skew information, LQU, UIN and the Bell's inequality in order to explore the non-classical correlations. Also these measures will be compared with the concurrence entanglement of $\rho^{AB}(t)$, which is defined as: $\mathbf{C}(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where the quantities $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$are the square roots of the eigenvalues of the matrix: $R = \rho^{AB}(\sigma_y \otimes \sigma_y)\rho^{AB*}(\sigma_y \otimes \sigma_y)$.

Skew information functions

For a bipartite quantum state ρ^{AB} and a local observable K, the quantity of skew information (SI) is given by

$$I\left(\rho^{AB}, K\right) = -\frac{1}{2}Tr[\sqrt{\rho^{AB}}, K]^2, \tag{9}$$

where K is an observable of a quantum system. The SI was used to quantify not only the information in ρ^{AB} but also the uncertainty of the observable K of ρ^{AB}, due to its ability to measure the non-commutativity between ρ^{AB} and K [18].

I) – LQU:

The local quantum uncertainty (LQU) is a reliable quantifier of quantum correlation in bipartite quantum systems [18]. LQU is zero for classically correlated states and invariant under local unitary operations. However, such a quantity satisfies all necessary criteria to serve as measure of discord-like quantum correlation. It has been demonstrated that the LQU satisfies the full physical requirements for the measure of quantum correlations.

The function of LQU is derived using the quantity of skew information [18], it presents a new quantum correlation for bipartite quantum systems beyond the entanglement. It is defined as,

$$\mathscr{L}(\rho^{AB}) = \min_{K}\left\{I\left(\rho^{AB}, K\right)\right\}, \tag{10}$$

The minimization is operated over all local maximally informative observable. The properties of the function LQU are: (i) it vanishes for all states, ρ^{AB}, which have zero discord, so, LQU can be considered as measure of discord-like quantum correlation. (ii) Under local unitary operation, LQU is invariant and does not increase under local reversible operations on the subsystem B. (iii) For pure states, the LQU can be considered as a measure of the quantumness of bipartite systems. Where, for a two-qubit pure state, the local quantum uncertainty reduces to the linear entropy of entanglement [18]. (iv) The closed form of LQU is given by

$$\mathscr{L}(\rho^{AB}) = 1 - \lambda_{max}(W_{AB}), \tag{11}$$

where λ_{max} is the largest eigenvalue of the 3x3-matrix W_{AB}. The elements of the matrix W_{AB} are given by,

$$w_{ij} = \mathrm{Tr}\{\sqrt{\rho^{AB}}(\sigma_i \otimes I)\sqrt{\rho^{AB}}(\sigma_j \otimes I)\}, \tag{12}$$

where σ_i, $i = 1, 2, 3$ are standard Pauli matrices. Here, we use $\mathbf{L}(t) = 2\mathscr{L}(\rho^{AB})$.

II) – UIN:

For the density matrix ρ^{AB} of a bipartite state, the UIN is given by [30]

$$\mathscr{U}(\rho^{AB}) = \max_{K} I\left(\rho^{AB}, K\right), \tag{13}$$

For any bipartite state ρ^{AB}, the closed form of UIN is [30]:

$$\mathscr{U}(\rho^{AB}) = \begin{cases} 1 - \lambda_{\min}(W_{AB}), & \vec{\mathbf{r}} = 0; \\ 1 - \frac{1}{\|\vec{\mathbf{r}}\|^2}\vec{\mathbf{r}}\, W_{AB}\, \vec{\mathbf{r}}^{T}, & \vec{\mathbf{r}} \neq 0. \end{cases} \tag{14}$$

where $\vec{\mathbf{r}} = [r_i]$ is the Bloch vector of the density matrix ρ^{AB}, where $r_i = Tr\{\rho^{AB}(\sigma_i \otimes I)\}$ [32]. For a pure state, the closed form of UIN reduces to $[1 - Tr(\rho^i)^2]$, $i = A\mathrm{or}B$, i. e., UIN reduces to entanglement monotone. We use $\mathbf{U}(t) = 2\mathscr{U}(\rho^{AB})$ to investigate the NCC of UIN.

Maximum Bell function (MBF)

The maximal violation of the Bell's inequality, $\mathbf{B}_{\max}(t)$, is an indicator of quantum nonlocality. If $\mathbf{B}_{\max}(t)$ is larger than the classical threshold 2 ($\mathbf{B}_{\max}(t) > 2$), the Bell-Chsh inequality is violated, i. e., the NCC is nonlocal. The analytical expression of the $\mathbf{B}_{\max}(t)$ [48] for a two-qubit state ρ^{AB} is given by

$$\mathbf{B}_{\max}(t) = 2\sqrt{\sum_{i=1}^{2}\xi_i} \tag{15}$$

where $\xi_i (i = 1, 2)$ are the two largest eigenvalues of the matrix $\mathbf{T}^{\dagger}\mathbf{T}$, where $\mathbf{T}$ is the correlation matrix [48]. Here, we use $\mathbf{B}(t) = \mathbf{B}_{\max}(t) - 1$, therefore, the maximal violation of the Bell's inequality if $\mathbf{B}(t) > 1$.

Dynamics of NCC functions

Here, we quantify the NCCs by the parameters of the coherent states superposition, the amplitude of the coherent state, and the intrinsic noise, to explore the superposition effect of the coherent state on NCC. We study the cases: $\varkappa = 0$ (coherent state) and $\varkappa = 1$ (even coherent state). Physically, the combination of two or more coherent states generates more quantum coherence in the off-diagonal terms of the density matrix. Therefore, as we will show next that, the created NCCs with the even coherent state is more robust against the noise that those by the coherent state. It is known that if the amplitude of the coherent state $\mu \to 0$, then the coherent state reduces to a vacuum state $|\mu\rangle \to |0\rangle$. Therefore, we will take $\mu > 0$ to investigate the effect of the initial harmonic oscillator field state other than the vacuum state, that was studied perviously. Finally, the parameter of $\gamma/\lambda > 0$ is used to see the amount of the diagonalization of the two-qubit density matrix (and its effect on the NCCs) due to the intrinsic noise rate.

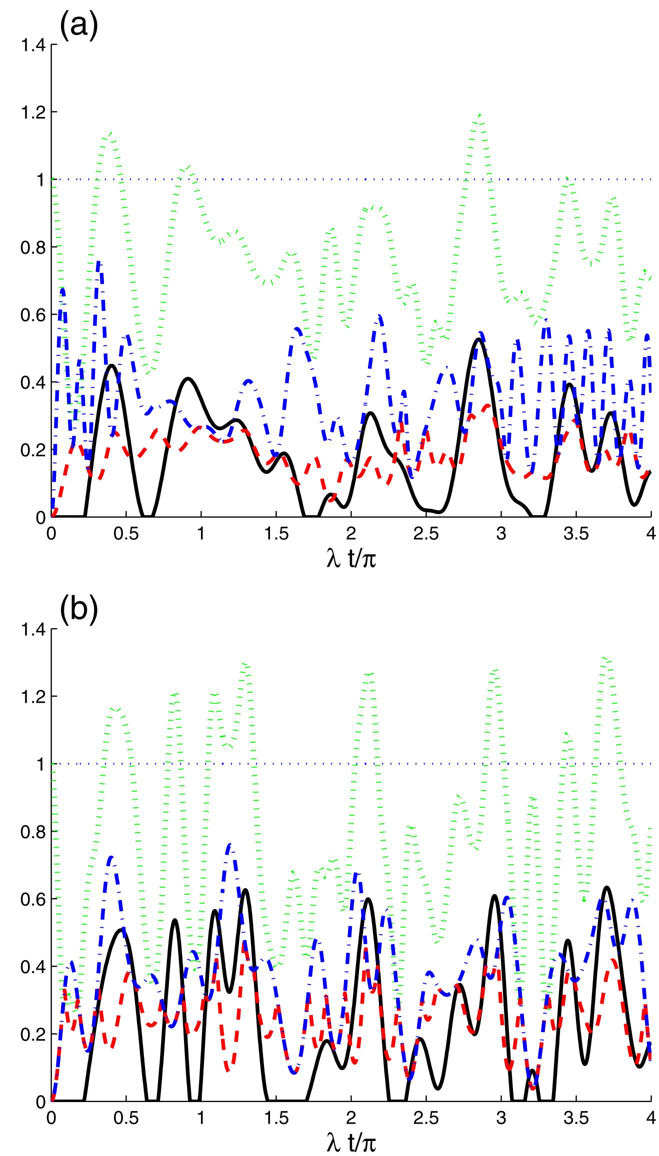

Fig. 1. Time evolutions of $L(t)$ (dashed plots), $U(t)$ (dashed dotted plots), $B(t)$ (dotted plots) and $C(t)$ (solid plots) when $\mu = \sqrt{2}$ and $\gamma = 0.0$. For coherent state in (a) and even coherent state in (b).

By using Eqs. (8), (12), (14) and (15), we can display the robustness and generation of the NCCs between the two qubits as function of the intrinsic noise rate via the NCC functions In Fig. 1, the functions $\mathbf{L}(t)$, $\mathbf{U}(t)$ and $\mathbf{C}(t)$ as well as $\mathbf{B}(t)$ are plotted at different values of the superposition parameter k. For the coherent state $k = 0$ in (a), and the even coherent state $k = 1$ in (b), with the small value of the intensity

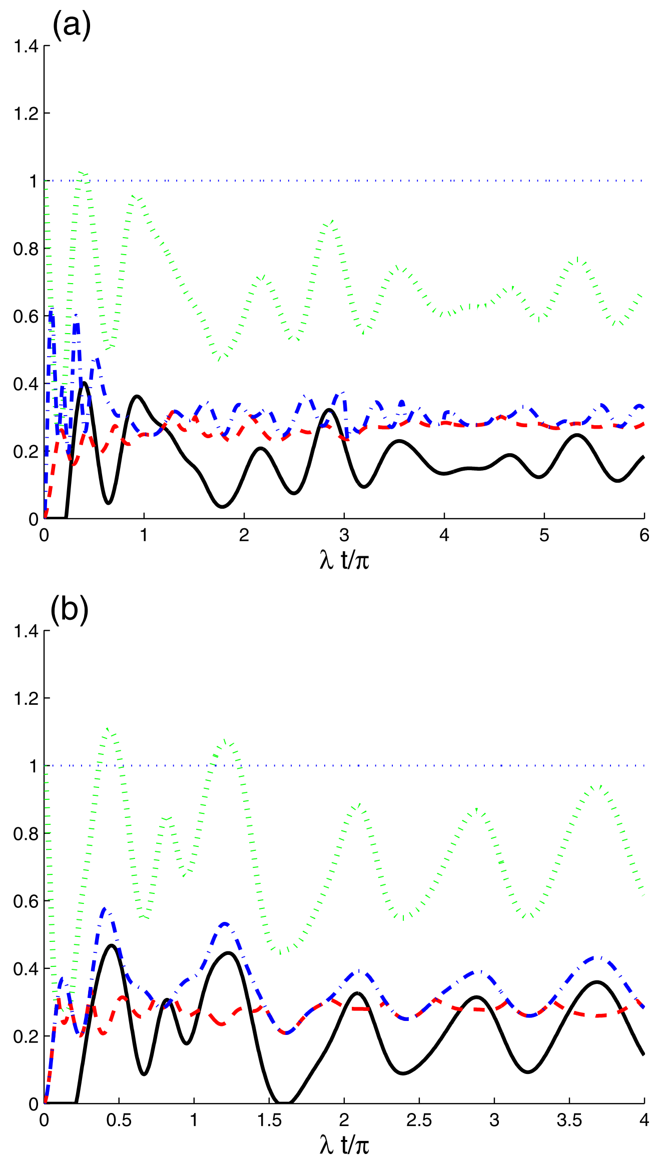

Fig. 2. As Fig. 1 but for $\gamma = 0.01\lambda$.

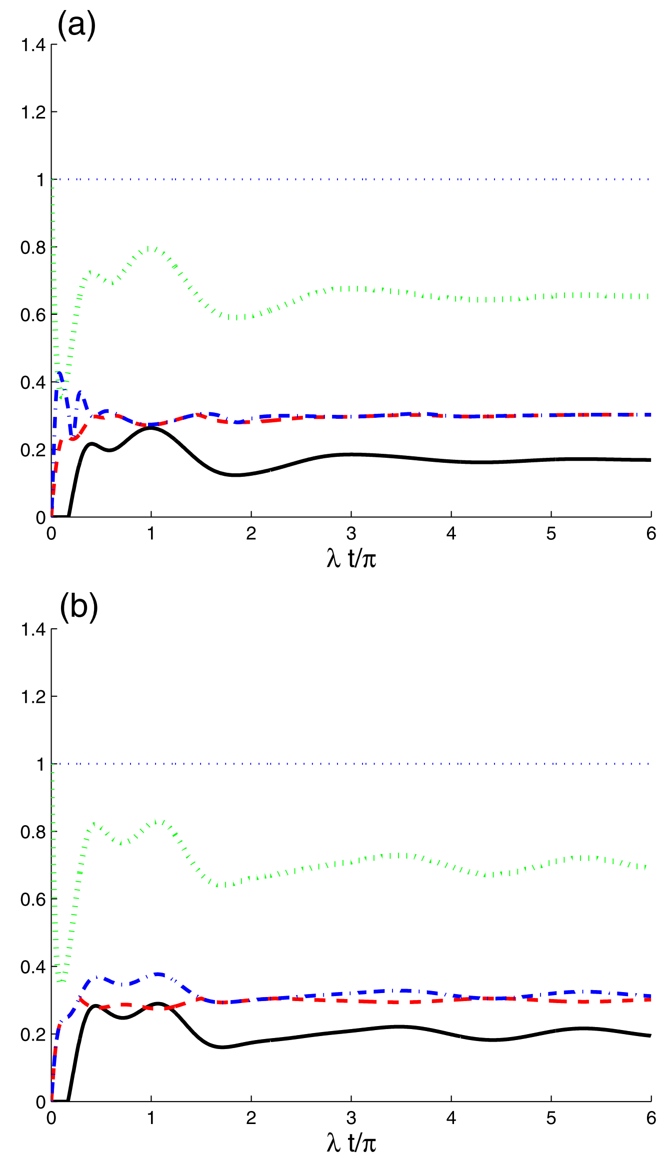

Fig. 3. As Fig. 1 but for $\gamma = 0.1\lambda$.

coherence, $\mu = \sqrt{2}$.

Since the initial density matrix of the uncorrected two qubits is: $\rho^{AB}(0) = |1_A 1_B\rangle\langle 1_A 1_B|$. The two largest eigenvalues of the initial matrix $\mathbf{T}^{\dagger}\mathbf{T}$ are 0 and 1 while the elements w_{ij} are zeros except the $w_{33} = 1$, and the Bloch vector is $\vec{\mathbf{r}} = (0, 0, 1)^t \neq 0$. Therefore, $\mathbf{L}(0) = \mathbf{U}(0) = 0$ and $\mathbf{B}(0) = 1$. These initial values of NCC functions indicate the uncorrected state.

From Fig. 1(a), for $\lambda t > 0$, the concurrence function $\mathbf{C}(t)$ is zero-constant function for a short time, until a particular time-point that is called **starting point**, at which the concurrence grows suddenly grows from 0 to its partial maximum values. After that the $\mathbf{C}(t)$ decreases and decays suddenly to completely vanishes. The processes of sudden growth and sudden decay indicate the phenomena of sudden birth and death of entanglement [49,50] which are repeated for several intervals of λt.

But, the skew information functions $\mathbf{L}(t)$ and $\mathbf{U}(t)$ grow from zero-value (i.e., their starting point is zero) to their partial maximum values. They grow early and oscillate between their upper and lower values with different behaviors. The $\mathbf{L}(t)$ and $\mathbf{U}(t)$ do not return to their initial zero-value and they show non-classical correlations at the intervals where $\mathbf{C}(t)$ vanishes. However, the generated NCC of uncertainty-induced non-locality is larger than that generated with the local uncertainty and the concurrence. We observe that the time evolution of the interaction of qubits with the coherent harmonic oscillator leads to: (i) generating NCCs. (ii) the Bell's inequality is violated at short intervals time, in which $\mathbf{B}(t) > 1$.

The effect of the superposition parameter of the coherent states, $\varkappa = 1$, on the behavior of NCC functions is display in Fig. 1(b) with the same data of Fig. 1(a). In this case $\varkappa = 1$, the coherence intensity of the initial harmonic oscillator state is more than that $\varkappa = 0$ and we observe that: (i) The NCC functions increase, i.e., their generated NCCs are stronger than those of the case of $\varkappa = 0$. (ii) The intervals of that is called *the skew-information correlation (SIC)*, this type of the correlation occurs at $\mathbf{L}(t) = \mathbf{U}(t)$ [51]. (iii) The increase of coherence intensity due to $\varkappa = 1$ leads to the increase of the intervals of the Bell's inequality maximal violation.

Fig. 2 shows the dependence of the NCC functions on the intrinsic noise rate, γ/λ, where, they are plotted with the same data as in Fig. 1 but for a small value of noise rate, $\gamma = 0.01\lambda$. From Fig. 2(a), the intrinsic noise term: $D^{mn}(t) = e^{-\gamma (E_i^m - E_j^n)^2 t}$, which leads to decrease the upper bounds and also increase the lower bounds of the NCC functions. Due to the noise rate, $\gamma = 0.01\lambda$, the generated correlations are partially protected. Neither the disappearance intervals of $\mathbf{C}(t)$ exist, nor does the maximal violation of Bell's inequality occur except at a very short

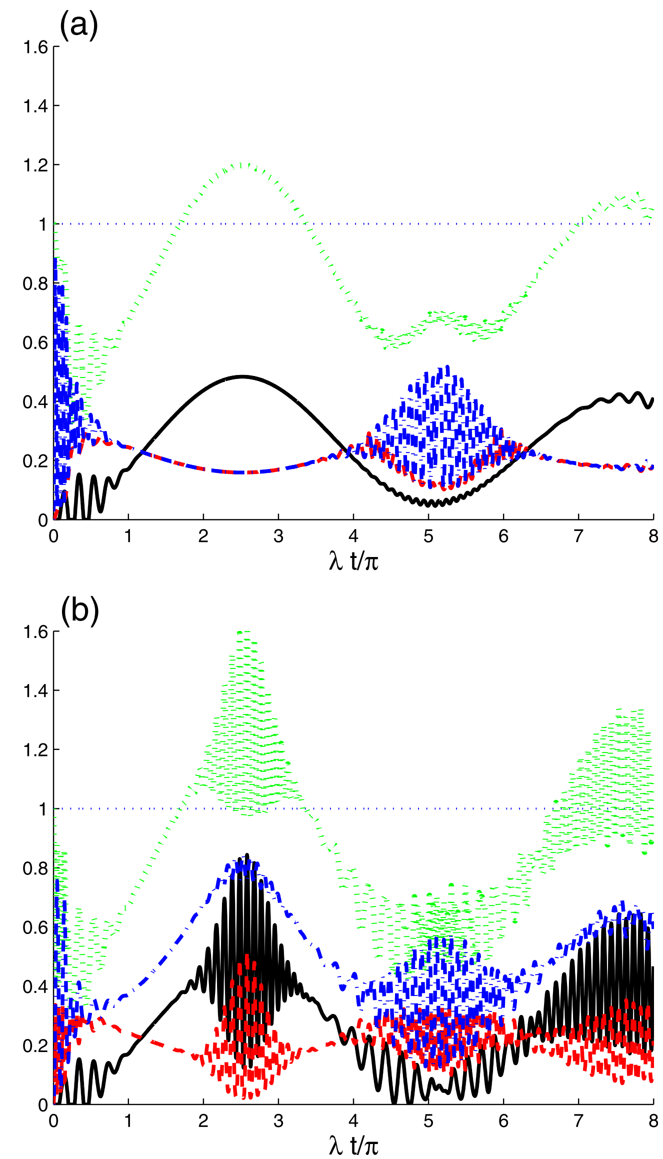

Fig. 4. Time evolutions of $\mathbf{L}(t)$ (dashed plots), $\mathbf{U}(t)$ (dashed dotted plots), $\mathbf{B}(t)$ (dotted plots) and $\mathbf{C}(t)$ (solid plots) when $\mu = 5$ and $\gamma = 0.0$. For $\varkappa = 0$ in (a) and $\varkappa = 1$ in (b).

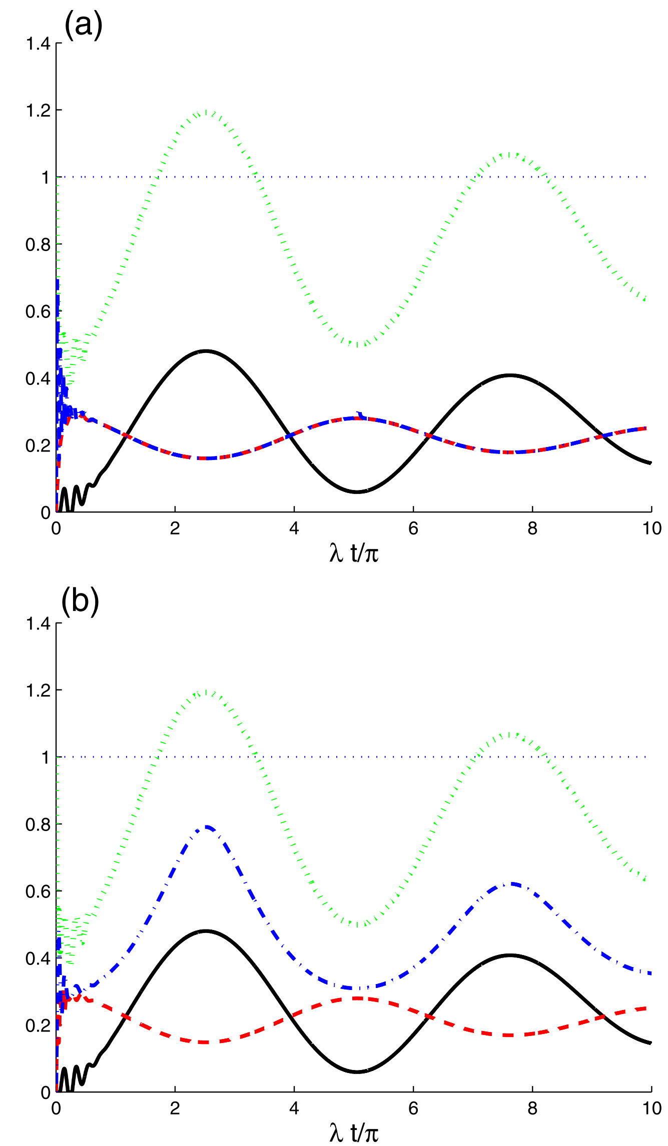

Fig. 5. As Fig. 4 but for $\gamma = 0.01\lambda$.

time. For the case of the even coherent state $\varkappa = 1$, the generated NCCs of the two qubits are more robust against the influence of intrinsic noise rate than that the case of the coherent state $\varkappa = 0$, the disappearance intervals of $\mathbf{C}(t)$ and the maximal violation of Bell's inequality exist.

In Fig. 3, the effect of the large value of intrinsic noise rate is shown, $\gamma = 0.1\lambda$, we observe that: (i) The starting point of $\mathbf{C}(t)$ decreases and is small compared with the cases $\gamma = 0$ and $\gamma = 0.01$. (ii) For the case $\varkappa = 0$, after a particular time, the NCC functions are time independent. The invariant correlations appear due to the intrinsic noise, these NCCs could be enhanced by increasing γ. The $\mathbf{L}(t)$ and $\mathbf{U}(t)$ present the same steady-state correlations, (stationary skew-correlation) which is larger than of $\mathbf{C}(t)$. (iii) For the case $\varkappa = 1$, the NCC functions have small oscillations and $\mathbf{L}(t)$ and $\mathbf{U}(t)$ present different stationary correlations. The starting times of the stationary correlations depend on $\varkappa$. In addition, the maximal violation of Bell's inequality does not occur by increasing intrinsic noise rate.

In Figs. 4–6, we exhibit the amount of the generated NCCs for larger value of the intensity coherence $|\mu|^2$. Where, NCC functions are plotted with the same data as in Fig. 1 but for the intensity coherence $\mu = 5$ at different values of the intrinsic noise rate γ/λ. From Fig. 4(a), $\gamma = 0$, the nonclassical correlations of $\mathbf{L}(t)$, $\mathbf{U}(t)$ and $\mathbf{C}(t)$ are generated strongly with more oscillations. The oscillatory evolution of the NCC functions is due to the larger values of the intensity coherence and the unitary evolution of the interaction. The phenomena of the sudden birth and death of entanglement are occurred only at short intervals and for $\lambda t < 0.5$. The SIC appears clearly at large time intervals. We note that the time intervals and the maximum value of the maximal violation of Bell's inequality are lager than these of small value of $\mu = \sqrt{2}$.

From Fig. 4(b), we can deduce that the NCCs depend on the parameter $\varkappa$ designing the type of the superposition for the initial state of the harmonic oscillator. The upper limits and oscillations of NCCs increase largely due to their interaction for the case of the even coherent harmonic oscillator. For $\varkappa = 1$ strong nonclassical correlations are observed. Also, $\mathbf{L}(t)$ and $\mathbf{U}(t)$ present different skew-information correlations unlike the case of $\varkappa = 0$. Furthermore, the time intervals of the maximal violation of Bell's inequality are very large, and the maximum value of $\mathbf{B}(t) \approx 2.62$ (i.e., $\mathbf{B}_{\max}(t) \approx 2.62$) is very large. This value is closer to the global maximum of the correlated states, $\mathbf{B}_{\max}(t) = 2.8$. We can deduce that the generated NCCs are enhanced via the intensity coherence parameters μ and $\varkappa$.

For a small value of $\gamma = 0.01$, the NCC functions are depicted in Fig. 5 with $\varkappa = 1$ and $\mu = 5$. We observe that the fluctuations of the

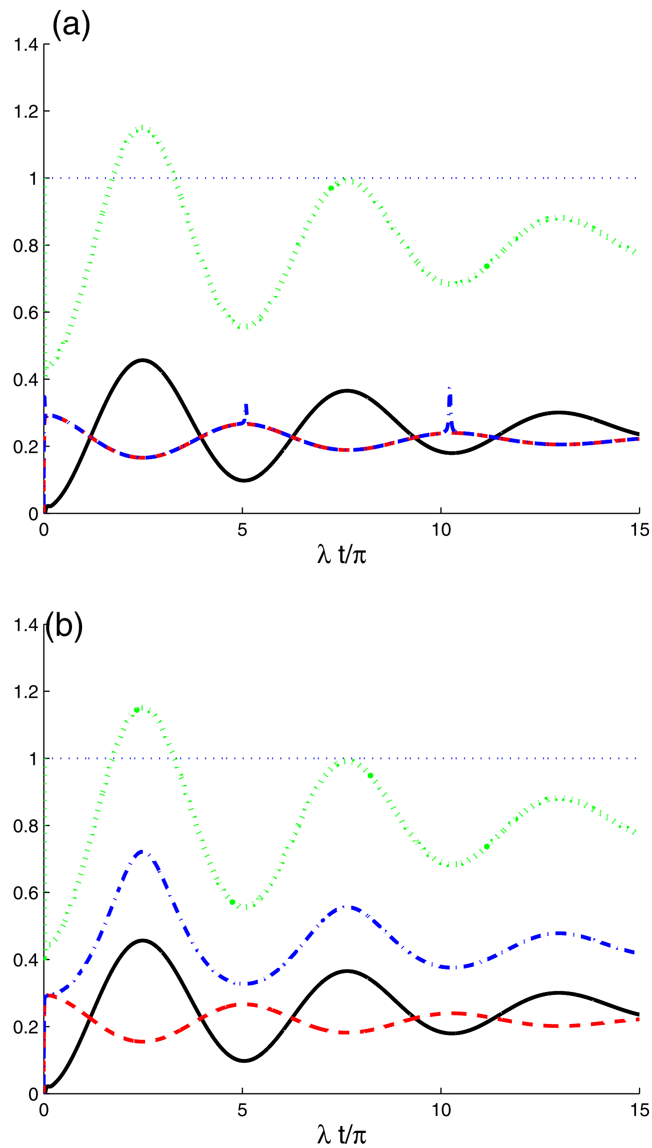

Fig. 6. As Fig. 4 but for $\gamma = 0.01\lambda$.

NCC functions disappear approximately and their lower bounds decrease. In the case $\kappa = 0$, the SIC appears only in the case $\kappa = 0$ at all $\lambda t \in [0, 10\pi]$. But with $\kappa = 1$, SIC disappears and the $\mathbf{L}(t)$ and $\mathbf{U}(t)$ have different dynamical behaviors. The violation of Bell's inequality is weakened by decreasing its time intervals and its maximum value.

In Fig. 6 we observe that as the intrinsic noise rate γ increases, the stationary correlations appear faster. For a large coherence intensity $\alpha = 5$, the NCC functions are more robust against the intrinsic noise rate and survive in the stationary steady states. Particularly, the stationary correlations and Bell's inequality violation can be controlled by adjusting the parameters of the initial coherence states and the intrinsic noise rate.

Conclusion

In this work, we have studied two qubits interacting with a coherent harmonic oscillator. The harmonic oscillator is initially prepared in a superposition of coherent states with intrinsic noise. The robustness of the generated nonclassical correlations between the qubits are investigated via various measures based on the skew information [including LQU and UIN] and the Bell's non-locality, as well as the concurrence entanglement. The unitary interaction leads to the generation of the LQU and UIN and the Bell's inequality violation beyond the concurrence entanglement. For the even coherent state, the generated NCCs are more robust against the influence of intrinsic noise rate than that the ones of the coherent state. The appearance of the special type of SIC depends on the parameter of the coherent states superposition. It is found that the occurrence and non-occurrence of the phenomena of sudden birth and death of entanglement depend not only on the intrinsic noise rate but also on the initial coherence intensity of the coherent states. The nonclassical correlations could be controlled by adjusting the parameters of the intrinsic noise, the initial state of the harmonic oscillator and its coherence intensity.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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