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Linear Diophantine Fuzzy Subspaces of a Vector Space

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Abstract: The notion of a linear diophantine fuzzy set as a generalization of a fuzzy set is a mathematical approach that deals with vagueness in decision-making problems. The use of reference parameters associated with validity and non-validity functions in linear diophantine fuzzy sets makes it more applicable to model vagueness in many real-life problems. On the other hand, subspaces of vector spaces are of great importance in many fields of science. The aim of this paper is to combine the two notions. In this regard, we consider the linear diophantine fuzzification of a vector space by introducing and studying the linear diophantine fuzzy subspaces of a vector space. First, we studied the behaviors of linear diophantine fuzzy subspaces of a vector space under a linear diophantine fuzzy set. Second, and by means of the level sets, we found a relationship between the linear diophantine fuzzy subspaces of a vector space and the subspaces of a vector space. Finally, we discuss the linear diophantine fuzzy subspaces of a quotient vector space.

Keywords: vector space; subspace; linear diophantine fuzzy set (LDFS); LDF subfield; LDF subspace; level subspace

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1. Introduction

Vagueness in many real-life problems cannot be approached by the use of classical sets. In 1965, Zadeh [1] generalized the classical set by introducing the fuzzy set. In a fuzzy set, an element's membership is a non-negative real number that can attain one as a maximum value. The new concept was studied by many scholars; in 1983, it was extended by Atanassov [2] to the intuitionistic fuzzy set (IFS). In an IFS, an element has membership and non-membership grades that are real numbers in the real unit interval where the sum is, again, in the real unit interval. Since then, many extensions of fuzzy sets have been proposed and applied to different real-life problems. For example, Yager [3] introduced Pythagorean fuzzy sets. In 2019, Riaz and Hashmi [4] generalized fuzzy sets into linear diophantine fuzzy sets (LDFS). Some related work can be found in [5,6].

In 1971, Rosenfeld [7] pointed to a link between fuzzy sets and algebraic structures. He introduced the concept of fuzzy subgroups of a group and studied some of their properties. This work was important in the field of mathematics as it introduced a new field of research: fuzzy algebraic structure. After that, fuzzification of almost all algebraic structures was introduced. In particular, fuzzy sets of fields and vector spaces were studied [8,9]. Fuzzy algebraic structures were generalized to linear diophantine fuzzy algebraic structures, and they were first studied in 2021 [10] by Kamaci. He studied LDFSs of different algebraic structures, such as groups, rings, and fields. Other scholars applied LDFSs to different algebraic structures [11–13]. For more details, we refer to [14–16]. Several different approaches to other fuzzy set extensions can be found in the literature. Molodtsov [17] dealt with uncertainties

occurring in natural and social sciences and initiated the soft set theory. In an attempt to study different existing mathematical structures in the context of the soft set theory, the notion of a soft point plays a significant role [18]. The results in the field of fuzzy structures are enormous and the literature related to this issue is extensive. For illustration, let's say for example [19–21].

This paper discusses LDFSs of fields and vector spaces, and the remaining part of it is organized as follows. In Section 2, we present some basic concepts related to LDFSs that are used throughout the paper. Our main results are presented in Sections 3 and 4. In Section 3, we introduce the notion of LDF subfields of a field; we investigate some of the properties that are essential in Section 4, where we introduce the concept of a LDF subspace of a vector space. Moreover, we present some examples and highlight some properties. Moreover, we discuss the relationship between LDF subspaces and subspaces through level subsets. Finally, by starting with a LDF subspace D of a vector space, we describe a LDF subspace of the quotient vector space $V/\text{supp}(D)$.

2. Preliminaries

In 1965, Zadeh [1] created a revolution in the theory of sets by introducing fuzzy sets. These sets deal with uncertain situations. Since then, many different sets were introduced to deal with vague situations [2,3,22].

Definition 1 ([1]). Let E be a universal set, $I = [0, 1]$, and $\mu : E \rightarrow I$ be a validity function. Then a fuzzy set (FS) of E is given as $A = \{(x, \mu(x)) : x \in E\}$.

Definition 2 ([3]). Let E be a universal set. Then a Pythagorean fuzzy set (PFS) on E is defined as follows: $P = \{(x, U(x), V(x)) : x \in E\}$, where U and V are mappings from E to $[0, 1]$ satisfying $0 \leq U^2(x) + V^2(x) \leq 1$ for all $x \in E$. Here, $U(x)$, $V(x)$ denote the degrees of membership and non-membership of an element $x \in E$, respectively.

Definition 3 ([4]). Let E be a universal set. Then a linear diophantine fuzzy set (LDFS) D on E is described as follows:

$$D = \{(x, \langle U(x), V(x) \rangle, \langle \alpha(x), \beta(x) \rangle) : x \in E\}.$$

Here, $U(x), V(x) \in [0, 1]$ are degrees of validity and non-validity of $x \in E$, respectively, $\alpha(x), \beta(x) \in [0, 1]$ are reference parameters. and $\alpha(x) + \beta(x) \in I$ and $\alpha(x)U(x) + \beta(x)V(x) \in [0, 1]$ for all $x \in E$.

Remark 1. FS and PFS are special cases of LDFS. In a FS μ on E , μ can be viewed as the LDFS D on E , defined as follows: $D = \{(\langle \mu(x), 0 \rangle, \langle 1, 0 \rangle) : x \in E\}$ and in a PFS P on E , P can be viewed as the LDFS D' on E , defined as follows: $D' = \{(\langle U(x), V(x) \rangle, \langle U(x), V(x) \rangle) : x \in E\}$.

Definition 4 ([4]). Let E be a universal set and D^*, D^{**} be LDFSs on E . Then

- (1) The union of D^* and D^{**} is denoted as $D^* \cup D^{**}$, where for all $x \in E$, $(D^* \cup D^{**})(x)$ is given as follows:
 $(\langle U^*(x) \vee U^{**}(x), V^*(x) \wedge V^{**}(x) \rangle, \langle \alpha^*(x) \vee \alpha^{**}(x), \beta^*(x) \wedge \beta^{**}(x) \rangle)$;
 - (2) The intersection of D^* and D^{**} is denoted as $D^* \cap D^{**}$, where for all $x \in E$, $(D^* \cap D^{**})(x)$ is given as follows:
 $(\langle U^*(x) \wedge U^{**}(x), V^*(x) \vee V^{**}(x) \rangle, \langle \alpha^*(x) \wedge \alpha^{**}(x), \beta^*(x) \vee \beta^{**}(x) \rangle)$.
- Here, " $r_1 \vee r_2 = \max\{r_1, r_2\}$ " and " $r_1 \wedge r_2 = \min\{r_1, r_2\}$ " for all $r_1, r_2 \in [0, 1]$.

Definition 5 ([11]). Let E be a universal set, $E_1, E_2 \subseteq E$, and D^*, D^{**} be LDFSs on E_1, E_2 , respectively. Then $D^* \times D^{**}$ is defined as follows:

$$\{(x, y), \langle U^*(x) \wedge U^{**}(y), V^*(x) \vee V^{**}(y) \rangle, \langle \alpha^*(x) \wedge \alpha^{**}(y), \beta^*(x) \vee \beta^{**}(y) \rangle\} : (x, y) \in E_1 \times E_2\}.$$

Notation 1. Let E be a universal set and D be a LDFS on E , given as follows.

$$D = \{(x, \langle U(x), V(x) \rangle, \langle \alpha(x), \beta(x) \rangle) : x \in E\},$$

where $U(X), V(x) \in [0, 1]$ are degrees of validity and non-validity, respectively, and $\alpha(x), \beta(x) \in [0, 1]$ are reference parameters. The degrees satisfy $0 \leq \alpha(x) + \beta(x) \leq 1$ and $0 \leq \alpha(x)U(x) + \beta(x)V(x) \leq 1$ for all $x \in E$. For $x, y \in E$,

1. $D(x) \wedge D(y) = (\langle u, v \rangle, \langle \alpha, \beta \rangle)$ where $u = U(x) \wedge U(y)$, $v = V(x) \vee V(y)$, $\alpha = \alpha(x) \wedge \alpha(y)$, $\beta = \beta(x) \vee \beta(y)$.
2. $D(x) \vee D(y) = (\langle u, v \rangle, \langle \alpha, \beta \rangle)$ where $u = U(x) \vee U(y)$, $v = V(x) \wedge V(y)$, $\alpha = \alpha(x) \vee \alpha(y)$, $\beta = \beta(x) \wedge \beta(y)$.
3. $D(x) \leq D(y)$ means $U(x) \leq U(y)$, $V(x) \geq V(y)$, $\alpha(x) \leq \alpha(y)$, $\beta(x) \geq \beta(y)$.

Proposition 1. Let E be a universal set and D be a LDFS on E , given as follows.

$$D = \{(x, \langle U(x), V(x) \rangle, \langle \alpha(x), \beta(x) \rangle) : x \in E\},$$

where $U(X), V(x) \in [0, 1]$ are degrees of validity and non-validity, respectively, and $\alpha(x), \beta(x) \in [0, 1]$ are reference parameters. The degrees satisfy $0 \leq \alpha(x) + \beta(x) \leq 1$ and $0 \leq \alpha(x)U(x) + \beta(x)V(x) \leq 1$ for all $x \in E$. If for $x, y \in E$, $D(x) \leq D(y)$ and $D(y) \leq D(x)$ then $D(x) = D(y)$.

3. LDF Subfields of a Field

The concept of LDF subfields of a field was introduced in [10]. In this section, we elaborate some properties of this concept that are used in Section 4.

Definition 6. Let K be a field and F a LDFS of K . Then F is a LDF subfield of K if the following conditions hold for all $a, b \in K$.

1. $F(a + b) \geq F(a) \wedge F(b)$;
2. $F(ab) \geq F(a) \wedge F(b)$;
3. $F(-a) \geq F(a)$;
4. $F(a^{-1}) \geq F(a)$.

Proposition 2. Let K be a field and F a LDF of K . Then the following statements are true.

1. $F(-a) = F(a)$.
2. $F(a^{-1}) = F(a)$.
3. $F(0) \geq F(a)$ for all $a \in K$.
4. $F(1) \geq F(a)$ for all $a \in K \setminus \{0\}$.
5. $F(0) \geq F(1)$.

Proof. The proof is straightforward. $\square$

Theorem 1. Let K be a field and F a LDFS of K . Then F is a LDF subfield of K if and only if the following conditions hold for all $a \in K, b \in K \setminus \{0\}$.

1. $F(a - b) \geq F(a) \wedge F(b)$ for all $a, b \in K$;
2. $F(ab^{-1}) \geq F(a) \wedge F(b)$ for all $a \in K, b \in K \setminus \{0\}$.

Example 1. Let $\mathbb{R}$ be the field of real numbers and D be the LDFS of $\mathbb{R}$ defined as follows.

$$D(a) = \begin{cases} (\langle 0.9, 0.15 \rangle, \langle 0.76, 0.2 \rangle) & \text{if } a \in \mathbb{Q}; \\ (\langle 0.7, 0.34 \rangle, \langle 0.56, 0.34 \rangle) & \text{otherwise.} \end{cases}$$

One can easily see that D is a LDF subfield of $\mathbb{R}$.

Theorem 2. Let K be a field and F a LDFS of K . Then F is a LDF subfield of K if and only if every non-void level set $F_t = \{k \in K : F(k) \geq t\}$ of F is either $\{0\}$ or a subfield of K .

Proof. Let F be a LDF subfield of K and $a, b \in F_t \neq \{0\}$. Having $F(a+b) \geq F(a) \wedge F(b) \geq t$ implies that $a+b \in F_t$ and having $F(ab) \geq F(a) \wedge F(b) \geq t$ implies that $ab \in F_t$. Let $a \neq 0 \in K$, then $a^{-1} \in F_t$ as $F(a^{-1}) \geq F(a) \geq t$.

Let $a, b \in K$. We show that the conditions of Definition 6 hold. If $a = b = 0$, we are done. Without loss of generality, suppose that $a \neq 0$ and that $t_1 = F(a)$, $t_2 = F(a) \wedge F(b)$ having $a, b \in F_{t_2}$ implies that $a+b, ab \in F_{t_2}$ and, hence, $F(a+b) \geq F(a) \wedge F(b)$ and $F(ab) \geq F(a) \wedge F(b)$. Moreover, having $a \in F_{t_1}$ and F_{t_1} , a subfield of K implies that $-a, a^{-1} \in F_{t_1}$. Thus, $F(a) \geq F(a)$ and $F(a^{-1}) \geq F(a)$. $\square$

Corollary 1. Let K be a field and F a LDFS of K with $F(1) = F(0)$. Then F is a LDF subfield of K if and only if every non-void level set F_t of F is a subfield of K .

Proof. The proof follows from Theorem 2. $\square$

Corollary 2. Let p be a prime number and $\mathbb{Z}_p$ be the field of integers modulo p . Then F is a LDF subfield of $\mathbb{Z}_p$ if and only if it has one of the following forms, F_1 or F_2 , where $F_1(k) = F_1(0)$ for all $k \in$

K and $F_2(k) = \begin{cases} r_1 & \text{if } k = 0; \\ r_2 & \text{otherwise.} \end{cases}$ for some $r_2 = \langle U'(x), V'(x) \rangle, \langle \alpha'(x), \beta'(x) \rangle < r_1 = \langle U(x), V(x) \rangle, \langle \alpha(x), \beta(x) \rangle$. Here, $U(x), V(x), U'(x), V', \alpha(x), \beta(x), \alpha'(x), \beta'(x) \in [0, 1]$ satisfying $0 \leq \alpha(x) + \beta(x) \leq 1, 0 \leq \alpha'(x) + \beta'(x) \leq 1$ and $0 \leq \alpha(x)U(x) + \beta(x)V(x) \leq 1, 0 \leq \alpha'(x)U'(x) + \beta'(x)V'(x) \leq 1$ for all $x \in \mathbb{Z}_p$.

Proof. The proof follows from Theorem 2 and $\mathbb{Z}_p$ with no proper subfields. $\square$

4. LDF Subspaces of a Vector Space

In this section, we introduce and study the concept of the LDF subspaces of a vector space.

Definition 7. Let $\mathbb{V}$ be a vector space over a field K , D a LDFS of $\mathbb{V}$, and F a LDF subfield of K . Then D is a LDF subspace of $\mathbb{V}$ if the following conditions hold for all $x, y \in \mathbb{V}, a \in K$.

1. $D(x+y) \geq D(x) \wedge D(y)$;
2. $D(ax) \geq F(a) \wedge D(x)$.

Example 2. Let $\mathbb{V}$ be a vector space over a field K and let F, D be any constant LDFSs of $K, \mathbb{V}$, respectively. Then D is a LDF subspace of $\mathbb{V}$.

Example 3. Let K be a field. By considering K as a vector space over K , then every LDF subfield of K is a LDF subspace of K .

Proposition 3. Let $\mathbb{V}$ be a vector space over a field K , D a LDF subspace of $\mathbb{V}$, and F a LDF subfield of K . If $x \in \mathbb{V}$ and $F(1) \geq D(x)$, then $D(-x) = D(x)$.

Proof. We have that $D(-x) = D((-1)x) \geq F(-1) \wedge D(x)$. Since $F(-1) = F(1) \geq D(x)$, it follows that $D(-x) \geq D(x)$. In a similar manner, we can prove that $D(x) = D((-1)(-x)) \geq D(-x)$. Proposition 1 completes the proof. $\square$

Example 4. Let $K = \mathbb{R}$ be the field of real numbers and let $\mathbb{V} = \mathbb{R}^2$ be the vector space of all couples of real numbers over K . Define the LDFSs F, D of $K, \mathbb{V}$ as follows.

$$F(a) = \begin{cases} (< 0.8, 0.3 >, < 0.7, 0.25 >) & \text{if } a \in \mathbb{Q}; \\ (< 0.5, 0.6 >, < 0.6, 0.38 >) & \text{otherwise.} \end{cases}$$

$$D((x, y)) = \begin{cases} (< 0.7, 0.3 >, < 0.7, 0.25 >) & \text{if } x, y \in \mathbb{Q}; \\ (< 0.5, 0.6 >, < 0.6, 0.38 >) & \text{otherwise.} \end{cases}$$

One can easily see that F is a LDF subfield of K and D is a LDF subspace of $\mathbb{V}$.

Proposition 4. Let $\mathbb{V}$ be a vector space over a field K , D a LDFS of $\mathbb{V}$, and F a LDF subfield of K . If D is a LDF subspace of $\mathbb{V}$, then the following condition holds for all $x, y \in \mathbb{V}, a, b \in K$.

$$D(ax + by) \geq F(a) \wedge F(b) \wedge D(x) \wedge D(y).$$

Proof. The proof is straightforward. $\square$

Theorem 3. Let $\mathbb{V}$ be a vector space over a field K , D_1, D_2 LDFSs of $\mathbb{V}$, and F a LDF subfield of K . Then $D = D_1 \cap D_2$ is a LDF subspace of $\mathbb{V}$.

Proof. The proof is straightforward. $\square$

Remark 2. Let $\mathbb{V}$ be a vector space over a field K , D_1, D_2 LDFSs of $\mathbb{V}$, and F a LDF subfield of K . Then $D = D_1 \cup D_2$ is not necessarily a LDF subspace of $\mathbb{V}$. We present an illustrative example.

Example 5. Let F_2 be the field of integers modulo 2 and let F_2^2 be the vector space over F_2 consisting of all couples with entries from F_2 . Define the LDFS F of F_2 as $F(a) = (< 1, 0 >, < 1, 0 >)$ and the LDFSs D_1, D_2 of $\mathbb{V}$ are defined as follows:

$$D_1(x) = \begin{cases} (< 0.7, 0.3 >, < 0.7, 0.25 >) & \text{if } x \in \{(0, 0), (0, 1)\}; \\ (< 0.5, 0.6 >, < 0.6, 0.38 >) & \text{otherwise.} \end{cases}$$

$$D_2(x) = \begin{cases} (< 0.7, 0.3 >, < 0.7, 0.25 >) & \text{if } x \in \{(0, 0), (1, 0)\}; \\ (< 0.5, 0.6 >, < 0.6, 0.38 >) & \text{otherwise.} \end{cases}$$

It is clear that D_1, D_2 are LDF subspaces of F_2^2 . We show that $D_1 \cup D_2$ is not a LDF subspace of F_2^2 . This is obvious as $(1, 1) = (0, 1) + (1, 0)$ but $(D_1 \cup D_2)(1, 1) = (< 0.5, 0.6 >, < 0.6, 0.38 >) \not\geq (< 0.7, 0.3 >, < 0.7, 0.25 >) = (D_1 \cup D_2)(0, 1) \wedge (D_1 \cup D_2)(1, 0)$.

In [9], Kumar proved that for a fuzzy set μ on a non-empty set X with $|Im(\mu)| = 1$, if $\mu = \mu_1 \cup \mu_2$ then $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.

Example 6. Let F_2 be the field of integers modulo 2 and let F_2^2 be the vector space over F_2 consisting of all couples with entries from F_2 . Define the LDFS F of F_2 as $F(a) = (< 1, 0 >, < 1, 0 >)$ and the LDFSs D_1, D_2 of $\mathbb{V}$ is defined as follows. For all $x \in F_2^2$,

$$D_1(x) = (< 0.71, 0.6 >, < 0.5, 0.5 >), D_2(x) = (< 0.7, 0.56 >, < 0.5, 0.5 >).$$

Then $(D_1 \cup D_2)(x) = (< 0.71, 0.56 >, < 0.5, 0.5 >)$ for all $x \in F_2^2$. Thus, $D_1 \cup D_2$ is a LDF subspace of F_2^2 but $D_1 \not\subseteq D_2$ and $D_2 \not\subseteq D_1$.

Theorem 4. Let $\mathbb{V}$ be a vector space over K , F a LDF subfield of K with $F(0) = F(1)$, and D a LDF subspace of $\mathbb{V}$. For all $t \in Im(D), l \in Im(F)$ with $t \leq l$, D_t is a subspace of $\mathbb{V}$ over the subfield F_l .

Proof. We have that $D_t \neq \emptyset$ and $F_l \neq \emptyset$ as $t \in \text{Im}(D)$ and $l \in \text{Im}(F)$. Let $x, y \in D_t$ and $a, b \in F_l$. Having D as a LDF subspace of $\mathbb{V}$ implies that $D(ax + by) \geq F(a) \wedge F(b) \wedge D(x) \wedge D(y) \geq l \wedge t = t$. The latter implies that $ax + by \in D_t$. $\square$

Corollary 3. Let $\mathbb{V}$ be a vector space over K , F the LDF subfield of K with $F(a) = (\langle 1, 0 \rangle, \langle 1, 0 \rangle)$ for all $a \in K$, and D a LDF subspace of $\mathbb{V}$. For all $t \in \text{Im}(D)$, D_t is a subspace of $\mathbb{V}$.

Proposition 5. Let $\mathbb{V}$ be a vector space over K , F a LDF subfield of K , and D a LDFS of $\mathbb{V}$. If $F(a) \geq D(x)$ for all $a \in F, x \in \mathbb{V}$ then D is a LDF subspace of $\mathbb{V}$ if and only if $D(ax + by) \geq D(x) \wedge D(y)$ for all $a, b \in F, x, y \in \mathbb{V}$.

Proof. Let D be a LDFS of $\mathbb{V}$. Proposition 4 asserts that $D(ax + by) \geq F(a) \wedge F(b) \wedge D(x) \wedge D(y) \geq D(x) \wedge D(y)$ as $F(a) \geq D(x)$ for all $a \in F, x \in \mathbb{V}$.

Let $D(ax + by) \geq D(x) \wedge D(y)$ for all $a, b \in F, x, y \in \mathbb{V}$. By setting $x = y = 1$, we have $D(x + y) \geq D(x) \wedge D(y)$. Having $D(0_{\mathbb{V}}) = D(0x + 0x) \geq F(0) \wedge D(x) = D(x)$ implies that $D(ax) = D(ax + 0x) \geq D(x) \geq F(a) \wedge D(x)$. $\square$

In what follows, our results are based on the condition of Proposition 5.

Proposition 6. Let $\mathbb{V}$ be a vector space over K and D a LDF subspace of $\mathbb{V}$. Then $D(0_{\mathbb{V}}) \geq D(x)$ for all $x \in \mathbb{V}$.

Theorem 5. Let $\mathbb{U}, \mathbb{V}$ be vector spaces over a field K and D, D' be LDF subspaces of $\mathbb{U}, \mathbb{V}$, respectively. Then $D \times D'$ is a LDF subspace of $\mathbb{U} \times \mathbb{V}$.

Proof. The proof is straightforward. $\square$

Definition 8. Let $\mathbb{V}$ be a vector space over a field K and A a subspace of $\mathbb{V}$. We define the characteristic function D of A as follows:

$$D(x) = \begin{cases} (\langle 1, 0 \rangle, \langle 1, 0 \rangle) & \text{if } x \in A; \\ (\langle 0, 1 \rangle, \langle 0, 1 \rangle) & \text{otherwise.} \end{cases}$$

Theorem 6. Let $\mathbb{V}$ be a vector space over a field K and A a subset of $\mathbb{V}$. Then A is a subspace of $\mathbb{V}$ if and only if its characteristic function is a LDF subspace of $\mathbb{V}$.

Proof. Let A be a subspace of $\mathbb{V}$, $x, y \in \mathbb{V}$, and $a, b \in K$. We consider the following cases for $ax + by \in A$ and $ax + by \notin A$. If $ax + by \in A$ then $D(ax + by) = (\langle 1, 0 \rangle, \langle 1, 0 \rangle) \geq D(x) \wedge D(y)$. If $ax + by \notin A$ then $x \notin A$ or $y \notin A$. The latter implies that $D(ax + by) = (\langle 0, 1 \rangle, \langle 0, 1 \rangle) \geq D(x) \wedge D(y)$ and, hence, the characteristic function D is a LDF subspace of $\mathbb{V}$.

Let $A \subseteq \mathbb{V}$ with characteristic function D , $x, y \in A$, and $a, b \in K$. Then $D(x) = D(y) = (\langle 1, 0 \rangle, \langle 1, 0 \rangle)$. Since A is a subspace of $\mathbb{V}$, it follows that $D(ax + by) \geq D(x) \wedge D(y) = (\langle 1, 0 \rangle, \langle 1, 0 \rangle)$. The latter implies that $ax + by \in A$ and, hence, A is a subspace of $\mathbb{V}$. $\square$

Next, we present some results related to level subspaces.

Proposition 7. Let $\mathbb{V}$ be a vector space over a field K , D a LDFS of $\mathbb{V}$ and $t_1 \leq t_2 \in \text{Im}(D)$. Then $D_{t_2} \subseteq D_{t_1}$.

Proof. The proof is straightforward. $\square$

Theorem 7. Let $\mathbb{V}$ be a vector space over a field K and D a LDFS of $\mathbb{V}$. Then D is a LDF subspace of $\mathbb{V}$ if and only if D_t is a subspace of $\mathbb{V}$ for all $t \in \text{Im}(D)$.

Proof. Let D be a LDF subspace of $\mathbb{V}$, $t \in \text{Im}(D)$, and $x, y \in D_t \neq \emptyset$. Then $D(ax + by) \geq D(x) \wedge D(y) \geq t$ and, hence, D_t is a subspace of $\mathbb{V}$ as $ax + by \in D_t$.

Let $x, y \in \mathbb{V}$ with $D(x) \wedge D(y) = t$. Then $x, y \in D_t$. Having D_t a subspace of $\mathbb{V}$ implies that $ax + by \in D_t$ and, hence, $D(ax + by) \geq t = D(x) \wedge D(y)$. $\square$

Proposition 8. Let $\mathbb{V}$ be a vector space over a field K and A a subspace of $\mathbb{V}$. Then A is a level subspace of a LDF subspace of $\mathbb{V}$.

Proof. Let A be a subspace of $\mathbb{V}$ and D be the LDFS of $\mathbb{V}$, defined as follows.

$$D(x) = \begin{cases} (\langle r_1, r_2 \rangle, \langle a_1, a_2 \rangle) & \text{if } x \in A; \\ (\langle r_3, r_4 \rangle, \langle a_3, a_4 \rangle) & \text{otherwise,} \end{cases}$$

where $r_1, r_2, r_3, r_4, a_1, a_2, a_3, a_4 \in [0, 1]$, $a_1 + a_2, a_3 + a_4 \in [0, 1]$, $r_1 a_1 + r_2 a_2, r_3 a_3 + r_4 a_4 \in [0, 1]$, $r_1 > r_3, r_2 < r_4, a_1 > a_3, a_2 < a_4$. One can easily see that D is a LDF subspace of $\mathbb{V}$ and that $D(\langle r_1, r_2 \rangle, \langle a_1, a_2 \rangle) = A$. $\square$

Next, we discuss the LDF subspaces of the quotient vector space $\mathbb{V}/\text{Supp}(D)$.

Definition 9. Let $\mathbb{V}$ be a vector space over a field K and D a LDFS of $\mathbb{V}$. Then $\text{Supp}(D) = \{x \in \mathbb{V} : D(x) = D(0_{\mathbb{V}})\}$.

Lemma 1. Let $\mathbb{V}$ be a vector space over a field K and D a LDF subspace of $\mathbb{V}$. For $x, y \in \mathbb{V}$, if $D(x - y) = D(0_{\mathbb{V}})$ then $D(x) = D(y)$.

Proof. Let $D(x - y) = D(0_{\mathbb{V}})$. Then $D(x) = D((x - y) + y) \geq D(x - y) \wedge D(y) = D(0_{\mathbb{V}}) \wedge D(y) = D(y)$ by Proposition 6. On the other hand, having $D(y - x) = D(-(x - y)) \geq D(x - y) = D(0_{\mathbb{V}})$ implies that $D(y) = D((y - x) + x) \geq D(y - x) \wedge D(x) = D(x)$. Thus, $D(x) = D(y)$. $\square$

Lemma 2. Let $\mathbb{V}$ be a vector space over a field K and D a LDF subspace of $\mathbb{V}$. Then $\text{Supp}(D)$ is a subspace of $\mathbb{V}$.

Proof. Let $x, y \in \text{Supp}(D)$ and $a, b \in K$. Then $D(ax + by) \geq D(x) \wedge D(y) = D(0_{\mathbb{V}})$. The latter implies that $ax + by \in \text{Supp}(D)$ and, hence, $\text{Supp}(D)$ is a subspace of $\mathbb{V}$. $\square$

Corollary 4. Let $\mathbb{V}$ be a vector space over a field K and D a LDF subspace of $\mathbb{V}$. Then $\mathbb{V}/\text{Supp}(D) = \{x + \text{Supp}(D) : x \in \mathbb{V}\}$ is a vector space over $\mathbb{K}$.

Proof. The proof follows from Lemma 2. $\square$

Proposition 9. Let $\mathbb{V}$ be a vector space over a field K , $t \in \text{Im}(D)$, and D a LDF subspace of $\mathbb{V}$. Then $\text{Supp}(D) \subseteq D_t$.

Proof. The proof follows from having $D(0_{\mathbb{V}}) \geq D(x)$ for all $x \in \mathbb{V}$. $\square$

Theorem 8. Let $\mathbb{V}$ be a vector space over a field K and D a LDF subspace of $\mathbb{V}$. Then D^* is a LDF subspace of $\mathbb{V}/\text{Supp}(D)$. Here, D^* is the LDFS of $\mathbb{V}/\text{Supp}(D)$ defined as follows.

$$D^*(x + \text{Supp}(D)) = D(x).$$

Proof. Let D be a LDF subspace of $\mathbb{V}$. First, we show that D^* is well-defined on $\mathbb{V}/\text{Supp}(D)$. Let $x + \text{Supp}(D) = y + \text{Supp}(D)$. Then $x - y \in \text{Supp}(D)$ and hence, $D(x - y) = D(0_{\mathbb{V}})$. Lemma 1 asserts that $D^*(x) = D^*(y)$. Thus, D^* is well-defined. Let $x + \text{Supp}(D), y + \text{Supp}(D) \in \mathbb{V}/\text{Supp}(D)$ and $a, b \in K$. Then $D^*(a(x + \text{Supp}(D)) + b(y + \text{Supp}(D))) =$

$D^*(ax + by + \text{Supp}(D)) = D(ax + by)$. Having D , a LDF subspace of $\mathbb{V}$ implies that $\geq D(x) \wedge D(y)$. The latter implies that $D^*(a(x + \text{Supp}(D)) + b(y + \text{Supp}(D))) \geq D^*(x + \text{Supp}(D)) \wedge D^*(y + \text{Supp}(D))$ and, hence, D^* is a LDF subspace of $\mathbb{V}/\text{Supp}(D)$. $\square$

Example 7. Let F_2 be the field of integers modulo 2 and let F_2^3 be the vector space over F_2 consisting of all triples with entries from F_2 . Define the LDFS D of F_2^3 as follows. For $(x, y, z) \in F_2^3$,

$$D((x, y, z)) = \begin{cases} (< 0.7, 0.3 >, < 0.7, 0.25 >) & \text{if } x = y \text{ and } z = 0; \\ (< 0.5, 0.6 >, < 0.6, 0.38 >) & \text{if } x = y \text{ and } z \neq 0; \\ (< 0.4, 0.6 >, < 0.5, 0.38 >) & \text{otherwise.} \end{cases}$$

One can easily see that D is a LDF subspace of F_2^3 . Having $\text{Supp}(D) = \{(x, y, 0) : x, y \in F_2\}$ and using Theorem 8, then D^* is a LDF subspace of $F_2^3/\text{Supp}(D)$. Here, D^* is defined as follows.

$$D^*((x, y, z) + \text{Supp}(D)) = \begin{cases} (< 0.7, 0.3 >, < 0.7, 0.25 >) & \text{if } (x, y, z) \in \text{Supp}(D); \\ (< 0.5, 0.6 >, < 0.6, 0.38 >) & \text{if } x = y \text{ and } z \neq 0; \\ (< 0.4, 0.6 >, < 0.5, 0.38 >) & \text{otherwise.} \end{cases}$$

Proposition 10. Let $\mathbb{V}$ be a vector space over a field K , D a LDF subspace of $\mathbb{V}$, and $t \in \text{Im}(D)$. Then $D_t^* = D_t/\text{Supp}(D)$.

Proof. The proof is straightforward. $\square$

Proposition 11. Let $\mathbb{V}$ be a vector space over a field K and D a LDF subspace of $\mathbb{V}$. Then D is the constant LDFS of $\mathbb{V}$ if and only if D^* is the constant LDFS of $\mathbb{V}/\text{Supp}(D)$.

Proof. If D is the constant LDFS of $\mathbb{V}$ then it is clear that D^* is the constant LDFS of $\mathbb{V}/\text{Supp}(D)$.

Let D^* be the constant LDFS of $\mathbb{V}/\text{Supp}(D)$. Then $D^*(x + \text{Supp}(D)) = D^*(0_{\mathbb{V}} + \text{Supp}(D))$ for all $x + \text{Supp}(D) \in \mathbb{V}/\text{Supp}(D)$. The latter implies that $D(x) = D(0_{\mathbb{V}})$ for all $x \in \mathbb{V}$ and, hence, D is the constant LDFS of $\mathbb{V}$. $\square$

5. Conclusions

This paper studied the LDF subspaces of a vector space as a generalization of fuzzy subspaces of a vector space. Different illustrative examples were presented and various properties were investigated. Moreover, a relationship between LDF subspaces of a vector space and level subspaces was found. Furthermore, LDF subspaces of the quotient vector space were discussed. As linear diophantine fuzzy sets are a generalization of fuzzy sets, the results of this paper generalize some results related to fuzzy subspaces of a vector space [8,9].

The results of this paper were approached from a theoretical point of view. For future work, it would be interesting to discuss some related real-life examples.

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