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The q-derivative and differential equation

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Abstract. The q-calculus appeared as a connection between mathematics and physics. It has several applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, and electronics. Recently, a great interest to its applications in differential transform methods,in order to get analytical approximate solutions to the ordinary as well as partial differential equations. In this paper, we present some of the interesting definitions of q-calculus and q-derivatives. By using q-calculus, solutions of some differential equations could be generated.

1. Introduction and preliminaries

The quantum calculus is an ordinary calculus without taking the limit. More precisely, in the classic calculus the derivative of a function $f(x)$ is defined as

$$f'(x) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}$$

subject to the existence of the limit.
The p -derivative is defined by

$$\frac{f(x^p)-f(x)}{x^p-x}$$

where p is a fixed scalar different from 1. Note that this type of derivatives does not use the limit. This is a special type of quantum calculus. Quantum derivative can be defined in many different ways. For instance, we can define different types of quantum derivatives $f'(x)$, where we do not take the limit. By considering $y = qx,\ y = x + h$,and $y = x^p$, the derivatives are then called respectively, q-derivative, h-derivative, and p-derivative of $f(x)$ where $p,\ q$ are a fixed number different from $1,$ and h is a fixed number different from 0.
These quantum derivatives (h-derivative, q-derivative and p-derivative) for the function $f(x) = x^3$ are respectively,

$$\frac{(x+h)^3-x^3}{h} = 3x^2+3hx+h^2$$

$$\frac{x^{3p}-x^3}{x^p-x} = \frac{x^{3(p-1)}-1}{x^{p-1}-1}x^2,$$

$$\frac{(qx)^3-x^3}{(q-1)x} = (q^2+q+1)x^2$$

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The history of q-calculus goes back to Euler and Jacobi in the eighteenth century and F.H. Jackson, in the early twentieth century, revisited their related work. In the second half of the twentieth century, a remarkable research interest in the area of the q-calculus were observed due to its applications in several areas of mathematics and physics, $[1-18]$. Recently, a tremendous interest was driven by high demands for mathematical models in quantum computing. The q-calculus bridges a connection between mathematics and physics. It has a lot of applications in research areas such as number theory, combinatorics, orthogonal polynomials, hypergeometric functions, quantum theory, and electronics. For more detailed material on quantum calculus, see the monographs $[7],[17]$ and $[11]$ and references therein. Following the notations and terminology from the references $[9],[11]$ and $[13]$, for an arbitrary function $f(x)$ the q-differential is defined by

$$(d_q f)(x) = f(qx) - f(x).$$

In particular, let

$$d_q x = (q-1)x,$$

then the definition of the q-derivative in the sense of Jackson is as follows:

$$D_q f(x) = \frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

where $x \neq 0, 0 < q < 1$ and for $x = 0$,

$$\frac{d_q}{d_q x} f(0) = \lim_{x \to 0} \frac{d_q f(x)}{dx}$$

Clearly, if $f(x)$ is differentiable, then $\lim_{q \to 1}(\frac{d_q}{d_q x} f)(x) = \frac{df(x)}{dx}$. For $f(x)$ an arbitrary function, the p-differential of $f(x)$ is defined by

$$d_p f(x) = f(x^p) - f(x)$$

and in particular denoting $d_p x = x^p - x,$ then p-derivative of a function $f(x)$ is defined as

$$D_p f(x) = \frac{f(x^p) - f(x)}{x^p - x}, if \quad x \neq 0,1$$
$$D_p f(0) = \lim_{x \to 0^+} D_p f(x), \tag{1}$$
$$D_p f(1) = \lim_{x \to 1} D_p f(x)$$

For any real number $\alpha,$ let us define

$$[\alpha] = \frac{q^{\alpha} - 1}{q - 1},$$

in particular, if $n \in \mathbb{Z}^+,$

$$[n] = \frac{q^n - 1}{q - 1} = q^{n-1} + \ldots + q + 1,$$

and

$$[n]! = \begin{cases} 1, & n = 0, \\ n(n-1)\ldots 1, & n \in \mathbb{Z}^+ \end{cases} \tag{2}$$

$$(x-a)_q^n = \begin{cases} 1, n = 0 \\ (x-a)\ldots(x - q^{n-1}a), \ n \in \mathbb{Z}^+ \end{cases} \tag{3}$$

The following formulae and definitions $[3],[9[,[11],[17]$ are very useful:

$$R_{q,r,f}(c,x) = f(x) - \sum_{j=0}^{r} (D_q^j f)(c) \frac{(x-c)_q^j}{[j]!}$$

where $D_q^j f$ is defined by $D_q^0 f = f$ and $D_q^k f = D_q(D_q^{k-1} f),\ k = 1,2,\ldots,$ and $R_{q,r,f}(c,x)$ is called q-Taylor's remainder of the order r with center at c.

$$D_q(x-a)_q^n = [n](x-a)_q^{n-1}, \qquad n \in \mathbb{Z}^+$$

$$(c-qx)_q^n = \frac{-1}{[n+1]} D_q (c-x)_q^{n+1}, \qquad n \in \mathbb{Z}^+$$

The p-derivative of higher order of function $f(x)$ is defined by

$$(D_p^0 f)(x) = f(x), \tag{4}$$

$$(D_p^n f)(x) = D_p (D_p^{n-1} f)(x),\ n \in \mathbb{N} \tag{5}$$

The p-derivative is a linear operator, i.e., for any constants a and b and arbitrary functions $f(x)$ and $g(x)$ satisfying:

$$D_p(af(x)+bg(x)) = aD_p f(x) + bD_p g(x).$$

Therefore, the p-derivative of the product and quotient of $f(x)$ and $g(x)$ are defined as follows:

$$D_p(f(x)g(x)) = \frac{f(x^p)g(x^p) - f(x)g(x)}{x^p - x}$$

$$= \frac{f(x^p)g(x^p) - f(x)g(x^p) + f(x)g(x^p) - f(x)g(x)}{x^p - x}$$

$$= \frac{(f(x^p) - f(x))g(x^p) + f(x)(g(x^p) - g(x))}{x^p - x} \tag{6}$$

This result is simply presented as

$$D_p(f(x)g(x)) = g(x^p)D_p f(x) + f(x)D_p g(x) \tag{7}$$

By interchanging f and g, we get

$$D_p(f(x)g(x)) = g(x)D_p f(x) + f(x^p)D_p g(x) \tag{8}$$

Equation (7) and (8) are valid and equivalent to each other. For the quotient rule, following [9], by replacing $f(x)$ with $\frac{f(x)}{g(x)}$ and using the product rule, we get

$$D_p(f(x)g(x)) = D_p[\frac{f(x)}{g(x)} g(x)]$$

$$= g(x^p)D_p[\frac{f(x)}{g(x)}] + \frac{f(x)}{g(x} D_p g(x)$$

and from this expression

$$D_p[\frac{f(x)}{g(x)}] = \frac{g(x)D_p f(x) - f(x)D_p g(x)}{g(x)g(x^p)} \tag{9}$$

Similarly, using the functions $\frac{f(x)}{g(x)}$ and $g(x)$,we obtain easily

$$D_p\left[\frac{f(x)}{g(x)}\right] = \frac{g(x^p)D_p f(x) - f(x^p)D_p g(x)}{g(x)g(x^p)}. \tag{10}$$

Thus, equations (9) and (10)are equivalent. The p-derivative has its own peculiar chain rule

$$D_p[f(u(x))] = D_p u(x) D \frac{h(x)}{\ln(u(x))} f(u(x)),$$

where $h(x)$ depends on the function $u(x)$, see [13].

The function $F(x)$ is called a p-antiderivative of $f(x)$ if $D_p F(x) = f(x)$ and it is denoted by

$$F(x) = \int f(x) d_p x$$

In contrast to the ordinary calculus, the p-antiderivative of a function might not be unique. To guarantee the uniqueness, some restrictions and conditions must be imposed. In 1910, Jackson [10] defined the general q-integral in the form:

$$\int_a^b f(t) d_q(t) = \int_0^b f(t) d_q(t) - \int_0^a f(t) d_q(t)$$

and

$$\int_0^a f(t)d_q(t) = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n,$$

where $0 < |q| < 1$, and following Jackson's idea

$$\int_0^\infty f(t)d_q(t) = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n,$$

provided that the sum of the infinity series converges.
Introducing an operator $\widehat{M}_p$ defined by $\widehat{M}_p(F(x)) = F(x^p)$,see more details in [13] then

$$\frac{1}{x^p - x}(\widehat{M}_p - 1)F(x) = \frac{F(x^p) - F(x)}{x^p - x} D_pF(x) = f(x).$$

Knowing that $\widehat{M}_p^j(F(x)) = F(x^{p^j})$ for $j \in \{0,1,2,3,\dots\}$, and using properties of the geometric series, we obtain the definition of the p-integral of $f(x)$,

$$\begin{aligned} F(x) &= \frac{1}{1-\widehat{M}_p}((x - x^p)f(x)) \\ &= \sum_{j=0}^{\infty} \widehat{M}_p^i((x - x^p)f(x) \\ &= \sum_{j=0}^{\infty} (x^{p^j} - x^{p^{j+1}})f(x^{p^j}) \end{aligned}$$

Thus,

$$F(x) = \int f(x)d_px = \sum_{j=0}^{\infty} (x^{p^j} - x^{p^{j+1}})f(x^{p^j}).$$

The definite p-integral of the function $f(x)$ can be written in different forms. Assume that $1 < a < b$, where $a, b \in \mathbb{R}^+$, $p \in (0,1)$, and the function f is defined on the interval $(1, b]$. Then

$$\int_1^b f(x)d_px = \lim_{N\to\infty} \sum_{j=0}^{N} (b^{p^j} - b^{p^{p+1}})f(b^{p^j}) = \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{p+1}})f(b^{p^j}) \tag{11}$$

where for any $j \in \{0,1,2,\dots\}$, $b^{p^j} \in (1, b]$,

$$\int_a^b f(x)d_px = \int_1^b f(x)d_px - \int_1^a f(x)d_px \tag{12}$$

Geometrical interpretation of Equation (12) is the area of the union of an infinite number of rectangles on the interval $[1+\varepsilon, b]$, where ε is a small positive arbitrary number and the summation consists of infinitely many terms as $p \to 1$; if f a continous function over the interval $[1, b]$,the norm of the partition approaches zero, and the sum becomes a Riemann integral sum on the interval $[1+\varepsilon, b]$, therefore

$$\lim_{p\to 1}\int_1^b f(x)d_px = \int_1^b f(x)dx$$

In general, the p-integral does not always converge to a p-antiderivative [9], [13]. We can guarantee the convergence of the series by imposing necessary and sufficient conditions.
We now define the first order partial q-derivatives of multivariable functions, for example, if we consider the function u in two variables

$$u = u(t, x): \mathbb{T} \times \mathbb{T} \to \mathbb{R},$$

where $q > 1$,and $\mathbb{T} = \{q^k : k \in \mathbb{N}_0\}$, the first order partial q-derivatives [3]

$$u_t(t, x) = \frac{u(qt, x) - u(t, x)}{(q-1)t},$$

$$u_x(t, x) = \frac{u(t, qx) - u(t, x)}{(q-1)x}$$

and the second order partial q-derivative of the function $u(x,y)$ with respect to x, u_{xx} is given by

$$u_{xx}(x,y)=\frac{u_x(qx,y)-u_x(x,y)}{(q-1)x}$$

and expending this closed form, we get

$$u_{xx}(x,y)=\frac{\frac{u(q^2x,y)-u(qx,y)}{(q-1)qx}-\frac{u(qx,y)-u(x,y)}{(q-1)x}}{(q-1)x} \tag{13}$$

The second order partial q-derivative of the function u_{yy} is defined in a similar way.

2. q-transform on differential equations

Example 1:

Consider the 3-dimensional Laplace equation

$$u_{xx}+u_{yy}+u_{zz}=0 \tag{14}$$

Following the notation and definition from Chu, we have respectively,

$$\begin{gathered}u_{xx}(x,y,z)=\frac{\frac{u(q^2x,y,z)-u(qx,y,z)}{(q-1)qx}-\frac{u(qx,y,z)-u(x,y,z)}{(q-1)x}}{(q-1)x}\\ u_{yy}(x,y,z)=\frac{\frac{u(x,q^2y,z)-u(x,qy,z)}{(q-1)qy}-\frac{u(x,qy,z)-u(x,y,z)}{(q-1)y}}{(q-1)y}\\ u_{zz}(x,y,z)=\frac{\frac{u(x,y,q^2z)-u(x,y,qz)}{(q-1)qz}-\frac{u(x,y,qz)-u(x,y,z)}{(q-1)z}}{(q-1)z}\end{gathered} \tag{15}$$

Then the second order q-difference equivalent form of the equation (14) is in the form

$$\begin{gathered}u(q^2x,y,z)-(q+1)u(qx,y,z)\\ +qu(x,y,z)+u(x,q^2y,z)\\ -(q+1)u(x,qy,z)+qu(x,y,z)\\ +u(x,y,q^2z)-(q+1)u(x,y,qz)\\ +qu(x,y,z)\\ =0\end{gathered}$$

by rearranging and simplifying, we get

$$\begin{gathered}u(q^2x,y,z)+u(x,q^2y,z)+u(x,y,q^2z)\\ -(q+1)[u(qx,y,z)+u(x,qy,z)+u(x,y,qz)]\\ +3qu(x,y,y,z)=0\end{gathered} \tag{16}$$

Using the method of separation of variables, we will assume that

$$\begin{gathered}u(x,y,z)=f(x)g(y)h(z),\\ u(qx,y,z)=f(qx)g(y)h(z)\\ u(x,qy,z)=f(x)g(qy)h(z),\\ u(x,y,qz)=f(x)g(y)h(qz)\end{gathered}$$

and similarly for the terms

$$\begin{gathered}u(q^2x,y,z)=f(q^2x)g(y)h(z),\\ u(x,q^2y,z)=f(x)g(q^2y)h(z)\\ u(x,y,q^2z)=f(x)g(y)h(q^2z)\end{gathered}$$

We substitute these values into the partial q-difference equation (16), we obtain

$$\begin{gathered}f(q^2x)g(y)h(z)+f(x)g(q^2y)h(z)\\ +f(x)g(y)h(q^2z)\end{gathered} \tag{17}$$

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$$-(q+1)[f(qx)g(y)h(z)+f(x)g(qy)h(z)$$
$$+f(x)g(y)h(qz)]+3qf(x)g(y)h(z)=0$$

Now divide each term by $f(x)g(y)h(z)$, the equation takes the form

$$\frac{f(q^2x)}{f(x)}+\frac{g(q^2y)}{g(y)}+\frac{h(q^2z)}{h(z)}$$
$$-(q+1)[\frac{f(qx)}{f(x)}+\frac{g(qy)}{g(y)}+\frac{h(qz)}{h(z)}]$$
$$+3q=0. \tag{18}$$

Following the notation from [3], and letting

$$f(x)=\alpha^{\log_q x},$$
$$f(qx)=\alpha_q^{\log qx}=\alpha f(x),$$
$$f(q^2x)=f(q(qx))$$
$$=\alpha f(qx)=\alpha^2 f(x)$$
$$g(y)=\alpha^{\log_q y},$$
$$g(qy)=\alpha_q^{\log qy}=\alpha g(y),$$
$$g(q^2y)=g(q(qy))$$
$$=\alpha g(qy)=\alpha^2 g(y)$$
$$h(z)=\alpha^{\log_q z},$$
$$h(qz)=\alpha_q^{\log qz}=\alpha h(z),$$
$$h(q^2z)=h(q(qz))$$
$$=\alpha z(qz)=\alpha^2 h(z)$$

and substituting into (18), we obtain the characteristic equation $\alpha^2-(q+1)\alpha+q=0$. Roots of the equation are: $\alpha_1=q$ and $\alpha_2=1$. Then the solutions are respectively,

$$\Phi(x,y,z)=C_1q^{\log_q x}+C_2q^{\log_q y}+C_3q^{\log_q z}$$

$$\Psi(x,y,z)=C_1\log_q x+C_2\log_q y+C_3\log_q z$$

Example 2:
Consider the partial differential equation

$$t\frac{\partial u}{\partial t}=c^2x^2\frac{\partial^2 u}{\partial x^2} \tag{19}$$

where c is a constant. The partial first order q-derivatives with respect to t and second order partial q-derivative with respect to x are given by

$$u_t(t,x)=\frac{u(qt,x)-u(t,x)}{(q-1)t}, \tag{20}$$

$$u_{xx}(t,x)=\frac{\frac{u(t,q^2x)-u(t,qx)}{(q-1)qx}-\frac{u(t,qx)-u(t,x)}{(q-1)x}}{(q-1)x}. \tag{21}$$

Substituting into equation (19) and rearranging, we get

$$\frac{u(qt,x)-u(t,x)}{t}=c^2\frac{\big(u(t,q^2x)-u(t,qx)-qu(t,qx)+qu(t,x)\big)}{(q^2-q)x^2} \tag{22}$$

For the solution of the heat equation, we apply the method of separation of variables by assuming that

$$u(t,x)=f(t)g(x),$$
$$u(qt,x)=f(qt)g(x),$$
$$u(t,qx)=f(t)g(qx),$$
$$u(t,q^2x)=f(t)g(q^2x)$$

and substituting these values into the partial q-difference equation (21). Dividing each side by the term $f(t)g(x)$ and collecting like terms, then setting both sides equal to a constant λ, and without loss of generality, letting $c=1$, we obtain

$$\frac{(\beta-1)}{(q-1)}=\lambda, \tag{23}$$

then using the relation $f(t)=\beta^{\log_q t}$,

$f(qt)=\beta f(t),\ g(x)=\alpha^{\log_q x},\ \ g(qx)=\alpha g(x)$,and $g(q^2x)=\alpha^2 g(x)$ we get

$$x^2u_{xx}=c^2\left[\frac{g(q^2x)}{g(x)}-\frac{(1+q)g(qx)}{g(x)}+q\right]$$
$$=\lambda q(q-1) \tag{24}$$

Now make these substitutions into the equation (23) and (24), then the corresponding Euler type equations and characteristic equations are

$$\beta=\lambda(q-1)+1 \tag{25}$$
$$\alpha^2-\alpha(1+q)+\left(q-\lambda(q^2-q)\right)=0 \tag{26}$$

From the first characteristic equations (25), we have the corresponding solution
$f(t)=(\lambda(q-1)+1)^{\log_q t}$ and from the second characteristic equation (26), the roots of α are

$$\alpha_{1,2}=\frac{1+q\mp\sqrt{(1-q)(1-q-4\lambda q)}}{2}$$

The explicit form of the solution depends on the discriminant $\Delta=(1-q)(1-q-4\lambda q)$. If $\Delta>0,\ \lambda<\frac{1-q}{4q},\ 0<q<1$, then $g(x)=c_1\alpha_1^{\log_q x}+c_2\alpha_2^{\log_q x}$ and therefore, the solution to the partial differential (19) is $u(t,x)=(\lambda(q-1)+1)^{\log_q t}\alpha^{\log_q x}$. If $\Delta=0,\ \alpha=\frac{1-q}{4q},\ 0<q<1$, then $g(x)=[\frac{1-q}{4q}]^{\log_q x}$, and the solution is $u(t,x)=f(t)g(x)=(\lambda(q-1)+1)^{\log_q t}[\frac{1-q}{4q}]^{\log_q x}$.
If $\Delta<0,\ \lambda>\frac{1-q}{4q},\ 0<q<1$, then $g(x)$ is in the form
$g(x)=|\alpha|^{\log_q x}(c_3\cos(\theta\log_q x)+c_4\sin(\theta\log_q x))$, where $c_1,c_2\in\mathbb{R}$ and $\theta=\arccos(\frac{Re\alpha}{|\alpha|})$. The solution is therefore, $u(t,x)=(\lambda(q-1)+1)^{\log_q t}(|\alpha|^{\log_q x}(c_4\cos(\theta\log_q x)+c_4\sin(\theta\log_q x)))$

3. References

[1] Avancini S. S. & Eiras A. & Galetti D. & Pimentel B. M.& Lima C. L. Phase transition in a q-deformed Lipkin model. J. Phys. A 28 (1995), no. 17, 4915(4923).
[2] Babinec P. On the q-analogue of a black body radiation. Acta Phys. Polon. A 82 (1992), no. 6, 957(60).
[3] Bohner M. and Hudson T. Euler-type Boundary value problems in quantum calculus, International Journal of Applied Mathematics and Statistics (IJAMAS), Vol. 9, No.J07, 19-23.
[4] Chu Wenchang. Basic almost-poised hypergeometric series. Mem. Amer.Math. Soc. 135 (1998), no. 642, x+99 pp.
[5] Chung K.S. & Chung W.S. & Nam S.T. & Kang H.J. New q-derivative and q-logarithm. International Journal of theoretical physics, 33, 10 (1994), 2019-2029.
[6] Dolgov A. D. & Novikov I. D. Superluminal propagation of light in gravitational field and non-causal signals. Phys. Lett. B 442 (1998), no. 1-4, 82(89).
[7] Ernst, T. The History of q-Calculus and A New Method, Thesis, Uppsala University, 2001.
[8] Ezeafulukwe U. A. & Darus M. (2015). A Note on q–Calculus, Fasciculi Mathematici, 55(1), 53-63.
[9] Gauchman H. Integral inequalities in q-calculus, Computers and Mathematics with Applications, 47(2004), 281–300.
[10] Jackson F.G. On q-definite integrals, Quart. J. Pure and Appl. Math. 41 (1910), 193–203.
[11] Kac V. and Cheung P. Quantum Calculus, Universitext, Springer-Verlag, New York, 2002.

[12] Monteiro M.R. & Rodrigues L.M.C.S. Quantum algebraic nature of the phonon spectrum in 4He. Physical review letters 76, 7 (1996), 1098-1100.
[13] Neamaty A. and Tourani M. The presentation of a new type of quantum calculus, Tiblisi Mathematical Journal 10(2) 2017, 15–28.
[14] Finkelstein R. J. Gauge theory. Internat. J. Modern Phys. A 11 (1996), no. 4, 733(746).
[15] Grammaticos B & Nijho F.H. & Papageorgiou V. & Ramani A & Satsuma J. Linearization and solutions of the discrete Painleve III equation. Physics letters A 185 (1994), 446-452.
[16] Kaniadakis G. & Lavagno A.& Quarati P. Kinetic model for q-deformed bosons and fermions. Phys. Lett. A 227 (1997), no. 3-4, 227(231).
[17] Siegel W. Introduction to string field theory. Advanced Series in Mathematical Physics, 8. World Scientific Publishing Co., Inc., Teaneck, NJ, 1988.
[18] Timoteo V. S. & Lima C. L. Effect of q-deformation in the NJL gap equation. Phys. Lett. B 448 (1999), no. 1-2, 1(5).