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ON A MILD SOLUTION OF A SEMILINEAR FUNCTIONAL-DIFFERENTIAL EVOLUTION NONLOCAL PROBLEM[1]

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The existence, uniqueness, and continuous dependence of a mild solution of a nonlocal Cauchy problem for a semilinear functional-differential evolution equation in a general Banach space are studied. Methods of a C_0 semigroup of operators and the Banach contraction theorem are applied. The result obtained herein is a generalization and continuation of those reported in references [2-8].

Key words: Abstract Cauchy Problem, Evolution Equation, Functional-Differential Equation, Nonlocal Condition, Mild Solution, Existence and Uniqueness of the Solution, Continuous Dependence of the Solution, a C_0 Semigroup, the Banach Contraction Theorem.

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1. Introduction

In this paper we study the existence, uniqueness, and continuous dependence of a mild solution of a nonlocal Cauchy problem for a semilinear functional-differential evolution equation. Methods of functional analysis concerning a C_0 semigroup of operators and the Banach theorem about the fixed point are applied. The nonlocal Cauchy problem considered here is of the form:

$$u'(t) + Au(t) = f(t, u_t), \quad t \in [0, a], \tag{1.1}$$

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$$u(s)+(g(u_{t_1},\ldots,u_{t_p}))(s)=\phi(s), \quad s\in[-r,0], \tag{1.2}$$

where $0<t_1<\ldots<t_p\leq a \quad (p\in\mathbb{N})$; $-A$ is the infinitesimal generator of a C_0 semigroup of operators on a general Banach space; f, g and ϕ are given functions satisfying some assumptions, and $u_t(s){:}=u(t+s)$ for $t\in[0,a]$, $s\in[-r,0]$.

Theorems about the existence, uniqueness, and stability of solutions of differential and functional-differential abstract evolution Cauchy problems were studied previously by Byszewski and Lakshmikantham [2], by Byszewski [3-8], and by Lin and Liu [10]. The result obtained herein is a generalization and continuation of those reported in references [2-8].

If the case of the nonlocal condition considered in this paper is reduced to the classical initial condition, the result of the paper is reduced to previous results of Hale [9], Thompson [11], and Akca, Shakhmurow and Arslan [1] on the existence, uniqueness, and continuous dependence of the functional-differential evolution Cauchy problem.

2. Preliminaries

We assume that E is a Banach space with norm $\|\cdot\|$; $-A$ is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on E, $D(A)$ is the domain of A;

$$0<t_1<\ldots<t_p\leq a \quad (p\in\mathbb{N})$$

and

$$M{:}=\sup_{t\in[0,a]}\|T(t)\|_{BL(E,E)}. \tag{2.1}$$

In the sequel the operator norm $\|\cdot\|_{BL(E,E)}$ will be denoted by $\|\cdot\|$.

For a continuous function $w{:}[-r,a]\to E$, we denote by w_t a function belonging to $C([-r,0],E)$ given by the formula

$$w_t(s){:}=w(t+s) \text{ for } t\in[0,a], \quad s\in[-r,0].$$

Let $f{:}[0,a]\times C([-r,0],E)\to E$. We require the following assumptions:

Assumption (A_1): For every $w\in C([-r,a],E)$ and $t\in[0,a]$,

$$f(\cdot,w_t)\in C([0,a],E).$$

Assumption (A_2): There exists a constant $L>0$ such that:

$$\|f(t,w_t)-f(t,\widetilde{w}_t)\|\leq L\|w-\widetilde{w}\|_{C([-r,t],E)}$$

$$\text{for } w,\widetilde{w}\in C([-r,a],E), \quad t\in[0,a].$$

Let $g{:}[C([-r,0],E)]^p\to C([-r,0],E)$. We apply the assumption:

Assumption (A_3): There exists a constant $K>0$ such that:

$$\|(g(w_{t_1},\ldots,w_{t_p}))(s)-(g(\widetilde{w}_{t_1},\ldots,\widetilde{w}_{t_p}))(s)\|\leq K\|w-\widetilde{w}\|_{C([-r,a],E)}$$

$$\text{for } w,\widetilde{w}\in C([-r,a],E), \quad s\in[-r,0].$$

Moreover, we require the assumption:

Assumption (A_4): $\phi \in C([-r,0],E)$.

A function $u \in C([-r,a],E)$ satisfying the conditions:

$$(i) \qquad u(t) = T(t)\phi(0) - T(t)\left[(g(u_{t_1},\ldots,u_{t_p}))(0)\right] + \int_0^t T(t-s)f(s,u_s)ds, \quad t \in [0,a],$$

$$(ii) \qquad u(s) + (g(u_{t_1},\ldots,u_{t_p}))(s) = \phi(s), \;\; s \in [-r,0),$$

is said to be a *mild solution* of the nonlocal Cauchy problem (1.1)-(1.2).

3. Existence and Uniqueness of a Mild Solution

Theorem 3.1: *Assume that the functions f, g, and ϕ satisfy Assumptions (A_1)-(A_4). Additionally, suppose that:*

$$M(aL+K) < 1. \tag{3.1}$$

Then the nonlocal Cauchy problem (1.1)-(1.2) *has a unique mild solution.*

Proof: Introduce an operator F on the Banach space $C([-r,a],E)$ by the formula:

$$(Fw)(t) := \begin{cases} \phi(t) - (g(w_{t_1},\ldots,w_{t_p}))(t), & t \in [-r,0), \\ T(t)\phi(0) - T(t)\left[(g(w_{t_1},\ldots,w_{t_p}))(0)\right] & \\ \quad + \int_0^t T(t-s)f(s,w_s)ds, & t \in [0,a], \end{cases}$$

where $w \in C([-r,a],E)$.

It is easy to see that

$$F: C([-r,a],E) \rightarrow C([-r,a],E). \tag{3.2}$$

Now, we will show that F is a contraction on $C([-r,a],E)$. For this purpose consider two differences:

$$(Fw)(t) - (F\widetilde{w})(t) = (g(\widetilde{w}_{t_1},\ldots,\widetilde{w}_{t_p}))(t) - (g(w_{t_1},\ldots,w_{t_p}))(t) \quad \text{for } w,\widetilde{w} \in C([-r,a],E), \;\; t \in [-r,0) \tag{3.3}$$

and

$$(Fw)(t) - (F\widetilde{w})(t) = T(t)\left[(g(\widetilde{w}_{t_1},\ldots,\widetilde{w}_{t_p}))(0) - (g(w_{t_1},\ldots,w_{t_p}))(0)\right] + \int_0^t T(t-s)\left[f(s,w_s) - f(s,\widetilde{w}_s)\right]ds \tag{3.4}$$

$$\text{for } w,\widetilde{w} \in C([-r,a],E), \quad t \in [0,a].$$

From (3.3) and Assumption (A_3):

$$\begin{aligned} &\| (Fw)(t) - (F\widetilde{w})(t) \| \le K \| w - \widetilde{w} \|_{C([-r,a],E)} \\ &\text{for } w, \widetilde{w} \in C([-r,a],E), \quad t \in [-r,0). \end{aligned} \tag{3.5}$$

Moreover, by (3.4), (2.1), Assumption (A_2), and Assumption (A_3):

$$\begin{aligned} \| (Fw)(t) - (F\widetilde{w})(t) \| &\le \| T(t) \| \, \| (g(w_{t_1}, \dots, w_{t_p}))(0) - (g(\widetilde{w}_{t_1}, \dots, \widetilde{w}_{t_p}))(0) \| \\ &+ \int_0^t \| T(t-s) \| \, \| f(s, w_s) - f(s, \widetilde{w}_s) \| \, ds \\ &\le MK \| w - \widetilde{w} \|_{C([-r,a],E)} + ML \int_0^t \| w - \widetilde{w} \|_{C([-r,s],E)} ds \\ &\le M(aL + K) \| w - \widetilde{w} \|_{C([-r,a],E)} \\ &\text{for } w, \widetilde{w} \in C([-r,a],E), \quad t \in [0,a]. \end{aligned} \tag{3.6}$$

Formulas (3.5) and (3.6) imply the inequality

$$\begin{aligned} &\| Fw - F\widetilde{w} \|_{C([-r,a],E)} \le q \| w - \widetilde{w} \|_{C([-r,a],E)} \\ &\text{for } w, \widetilde{w} \in C([-r,a],E), \end{aligned} \tag{3.7}$$

where $q := M(aL + K)$.

Since, from (3.1), $q \in (0,1)$, then (3.7) shows that F is a contraction on $C([-r,a],E)$. Consequently, by (3.2) and (3.7), operator F satisfies all the assumptions of the Banach contraction theorem. Therefore, in space $C([-r,a],E)$ there is only one fixed point of F and this point is the mild solution of the nonlocal Cauchy problem (1.1)-(1.2).

The proof of Theorem 3.1 is complete.

4. Continuous Dependence of a Mild Solution

Theorem 4.1: *Suppose that the functions f and g satisfy Assumptions (A_1)-(A_3) and $M(aL+K) < 1$. Then, for each $\phi_1, \phi_2 \in C([-r,0],E)$, and for the corresponding mild solutions u_1, u_2 of the problems*

$$\left\{ \begin{array}{l} u'(t) + Au(t) = f(t, u_t), \quad t \in [0,a], \\ u(s) + (g(u_{t_1}, \dots, u_{t_p}))(s) = \phi_i(s), \ s \in [-r,0] \quad (i = 1,2), \end{array} \right. \tag{4.1}$$

the inequality

$$\begin{aligned} &\| u_1 - u_2 \|_{C([-r,a]E)} \\ &\le Me^{aML} \Big(\| \phi_1 - \phi_2 \|_{C([-r,0],E)} + K \| u_1 - u_2 \|_{C([-r,a],E)} \Big) \end{aligned} \tag{4.2}$$

is true.

Additionally, if $K < \frac{1}{Me^{aML}}$, *then*

$$\| u_1 - u_2 \|_{C([-r,a],E)} \leq \frac{Me^{aML}}{1 - KMe^{aML}} \| \phi_1 - \phi_2 \|_{C([-r,0],E)}. \tag{4.3}$$

Proof: Let ϕ_i $(i = 1,2)$ be arbitrary functions belonging to $C([-r,0],E)$, and let u_i $(i = 1,2)$ be the mild solutions of problems (4.1).

Consequently,

$$u_1(t) - u_2(t) = T(t)[\phi_1(0) - \phi_2(0)] \tag{4.4}$$

$$- T(t)\left[(g((u_1)_{t_1}, \ldots, (u_1)_{t_p}))(0) - (g((u_2)_{t_1}, \ldots, (u_2)_{t_p}))(0)\right]$$

$$+ \int_0^t T(t-s)\big[f(s,(u_1)_s) - f(s,(u_2)_s)\big]ds \text{ for } t \in [0,a],$$

and

$$\begin{gathered} u_1(t) - u_2(t) = \phi_1(t) - \phi_2(t) \\ + (g((u_2)_{t_1}, \ldots, (u_2)_{t_p}))(t) - (g((u_1)_{t_1}, \ldots, (u_1)_{t_p}))(t) \\ \text{for } t \in [-r,0). \end{gathered} \tag{4.5}$$

From (4.4), (2.1), Assumption (A_2) and Assumption (A_3):

$$\| u_1(\tau) - u_2(\tau) \| \leq M \| \phi_1 - \phi_2 \|_{C([-r,0],E)} + MK \| u_1 - u_2 \|_{C([-r,a],E)}$$

$$+ ML \int_0^\tau \| u_1 - u_2 \|_{C([-r,s],E)} ds$$

$$\leq M \| \phi_1 - \phi_2 \|_{C([-r,0],E)} + MK \| u_1 - u_2 \|_{C([-r,a],E)}$$

$$+ ML \int_0^t \| u_1 - u_2 \|_{C([-r,s],E)} ds \quad \text{for } 0 \leq \tau \leq t \leq a.$$

Therefore,

$$\sup_{\tau \in [0,t]} \| u_1(\tau) - u_2(\tau) \|$$

$$\leq M \| \phi_1 - \phi_2 \|_{C([-r,0],E)} + MK \| u_1 - u_2 \|_{C([-r,a],E)} \tag{4.6}$$

$$+ ML \int_0^t \| u_1 - u_2 \|_{C([-r,s],E)} ds \text{ for } t \in [0,a].$$

Simultaneously, by (4.5) and Assumption (A_3):

$$\| u_1(t) - u_2(t) \| \leq \| \phi_1 - \phi_2 \|_{C([-r,0],E)} + K \| u_1 - u_2 \|_{C([-r,a],E)} \quad \text{for } t \in [-r, 0). \tag{4.7}$$

Since $M \geq 1$, formulas (4.6) and (4.7) imply:

$$\begin{aligned} &\| u_1 - u_2 \|_{C([-r,t],E)} \\ &\leq M \| \phi_1 - \phi_2 \|_{C([-r,0],E)} + MK \| u_1 - u_2 \|_{C([-r,a],E)} \\ &+ ML \int_0^t \| u_1 - u_2 \|_{C([-r,s],E)} ds \quad \text{for } t \in [0, a]. \end{aligned} \tag{4.8}$$

From (4.8) and Gronwall's inequality:

$$\begin{aligned} &\| u_1 - u_2 \|_{C([-r,a],E)} \\ &\leq \Big[M \| \phi_1 - \phi_2 \|_{C([-r,0],E)} + MK \| u_1 - u_2 \|_{C([-r,a],E)} \Big] e^{aML}. \end{aligned}$$

Therefore, (4.2) holds. Finally, inequality (4.3) is a consequence of inequality (4.2). The proof of Theorem 4.1 is complete.

Remark 4.1: If $K = 0$, inequality (4.2) is reduced to the classical inequality

$$\| u_1 - u_2 \|_{C([-r,a],E)} \leq M e^{aML} \| \phi_1 - \phi_2 \|_{C([-r,0],E)},$$

which is characteristic for the continuous dependence of the semilinear functional-differential evolution Cauchy problem with the classical initial condition.

5. Remarks

1. Let

$$0 < t_1 < \ldots < t_p \leq a \quad (p \in \mathbb{N}).$$

Theorems 3.1 and 4.1 can be applied for g defined by the formula:

$$(g(w_{t_1}, \ldots, w_{t_p}))(s) = \sum_{k=1}^{p} c_k w(t_k + s) \quad \text{for } w \in C([-r,a],E), \ \ s \in [-r, 0],$$

where $c_k (k = 1, \ldots, p)$ are given constants such that

$$M \left(aL + \sum_{k=1}^{p} | c_k | \right) < 1. \tag{5.1}$$

2. Let

$$0 < t_1 < \ldots < t_p \leq a \quad (p \in \mathbb{N})$$

and let $\epsilon_k (k = 1, \ldots, p)$ be given positive constants such that:

$$0 < t_1 - \epsilon_1 \text{ and } t_{k-1} < t_k - \epsilon_k \quad (k = 2, \ldots, p).$$

Theorems 3.1 and 4.1 can be applied for g defined by the formula:

$$(g(w_{t_1},\ldots,w_{t_p}))(s)=\sum_{k=1}^{p}\frac{c_k}{\epsilon_k}\int_{t_k-\epsilon_k}^{t_k} w(\tau+s)d\tau$$

$$\text{for } w\in C([-r,a],E),\quad s\in[-r,0],$$

where c_k $(k=1,\ldots,p)$ are given constants satisfying condition (5.1). Indeed,

$$\|(g(w_{t_1},\ldots,w_{t_p}))(s)-(g(\widetilde{w}_{t_1},\ldots,\widetilde{w}_{t_p}))(s)\|$$

$$=\|\sum_{k=1}^{p}\frac{c_k}{\epsilon_k}\int_{t_k-\epsilon_k}^{t_k}[w(\tau+s)-\widetilde{w}(\tau+s)]d\tau\|$$

$$\leq\left(\sum_{k=1}^{p}|c_k|\right)\|w-\widetilde{w}\|_{C([-r,a],E)}\ \text{ for } s\in[-r,0].$$

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