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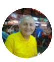

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Nonlinear Analysis 34 (1998) 65–72

Nonlinear Analysis

Existence of solutions of a semilinear functional-differential evolution nonlocal problem[1]

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Received 21 February 1997; received in revised form 18 June 1997

Keywords: Abstract Cauchy problem; Evolution equation; Functional-differential equation; Nonlocal condition; Mild and classical solutions; Existence of the solutions; A compact C_0 semigroup; Schauder's fixed point theorem

1. Introduction

In this paper we study the existence of mild and classical solutions of a nonlocal Cauchy problem for a semilinear functional-differential evolution equation. Methods of the functional analysis concerning to a compact C_0 semigroup of operators and Schauder's fixed point theorem are applied. The nonlocal Cauchy problem considered here is of the following form:

$$u'(t) + Au(t) = f\,(t, u(t), u(b_1(t)), \ldots, u(b_m(t))), \quad t \in (t_0, t_0 + a], \;\; m \in \mathbb{N}, \tag{1.1}$$

$$u(t_0) + g(u) = u_0, \tag{1.2}$$

where $t_0 \geq 0$, $a > 0$, $-A$ is the infinitesimal generator of a compact C_0 semigroup of operators on a Banach space, f, g, b_i $(i = 1, \ldots, m)$ are given functions satisfying some assumptions and u_0 is an element of the Banach space.

Theorems about the existence, uniqueness and stability of solutions of differential and functional-differential abstract evolution Cauchy problems were studied previously by Byszewski and Lakshmikantham [3], by Byszewski [4–9], by Balachandran and Chandrasekaran [2], and by Lin and Liu [10]. The result obtained is a generalization

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[1] The paper was supported by NATO grant: SA. 11-1-05-OUTREACH (CRG. 960586)1189/96/473.

PII: S0362-546X(97)00693-7

and a continuation of some results reported in publications [2–9]. This result consists, among other things, in this that we apply Schauder's fixed point theorem to obtain the solutions of the nonlocal problem (1.1) and (1.2). Consequently, the assumptions (see Assumptions (A_1)–(A_3)) concerning to functions f and g are weaker than in papers [3–9], where the Banach contraction theorem and the theory of approximation were applied for nonlocal evolution problems. The result of the paper consists, also, in this that, applying Schauder's theorem, we obtain a solution of problem (1.1) and (1.2) of much stronger reguality than in [2].

Functional-differential evolution Cauchy problems together with classical initial conditions were also considered by Akca et al. [1].

In the case if the functional-differential nonlocal evolution Cauchy problem, studied in the paper, is reduced to the differential evolution Cauchy problem together with the classical initial condition then the result of the paper is reduced to the result of Pazy [11] on the existence of solutions of the classical differential evolution Cauchy problem.

2. Preliminaries

Let E be a Banach space with norm $\|\cdot\|$ and let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup of operators on E. A C_0 semigroup $\{T(t)\}_{t\geq 0}$ is said to be *a compact C_0 semigroup* of operators on E if $T(t)$ is a compact operator for every $t>0$. A compact C_0 semigroup of operators on E will be denoted by $\{T(t)\}$.

In this paper we assume that $-A$ is the infinitesimal generator of a compact C_0 semigroup $\{T(t)\}$ of operators on E, $D(A)$ is the domain of A, $t_0\geq 0, a>0$,

$$I := [t_0, t_0 + a],$$

$$M := \sup_{t\in[0,a]} \|T(t)\|_{\mathrm{BL}(E,E)} \tag{2.1}$$

and

$$X := C(I, E).$$

In the sequel, the operator norm $\|\cdot\|_{\mathrm{BL}(E,E)}$ will be denoted by $\|\cdot\|$. We will need the following sets:

$$E_\rho := \{z\in E, \|z\|\leq\rho\} \quad \text{and} \quad X_\rho := \{w\in X, \|w\|_X\leq\rho\},$$

where $\rho>0$.

Let $f: I\times E^{m+1}\to E,\quad g: X\to E,\quad b_i: I\to I\quad (i=1,\dots,m)$ and $u_0\in E$.

We will need the following assumptions:

Assumption (A_1)**.** $f\in C(I\times E^{m+1}, E), g\in C(X,E)$ and $b_i\in C(I,I) (i=1,\dots,m)$. Moreover, there are $C_i>0\quad (i=1,2)$ such that

$$\|f(s,z_0,z_1,\dots,z_m)\|\leq C_1 \quad \text{for } s\in I,\ z_i\in E_r\ (i=0,1,\dots,m) \tag{2.2}$$

and

$$\|g(w)\| \le C_2 \quad \text{for } w \in X_r, \tag{2.3}$$

where

$$r := M\,(aC_1 + \|u_0\| + C_2)\,. \tag{2.4}$$

Assumption (A_2).

$$g\,(\lambda w_1 + (1-\lambda)w_2) = \lambda g(w_1) + (1-\lambda)g(w_2)$$

$$\text{for } w_i \in X_r \quad (i = 1,2), \quad \lambda \in (0,1),$$

where r is given by Eq. (2.4).

Assumption (A_3). The set

$$\{w(t_0):\ w \in X_r,\ w(t_0) = u_0 - g(w)\}\,,$$

where r is given by Eq. (2.4), is precompact in E.

3. Existence of a mild solution

A function $u \in X$ satisfying the integral equation

$$u(t) = T(t-t_0)\,u_0 - T(t-t_0)\,g(u) + \int_{t_0}^{t} T(t-s)f\,(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))\,\mathrm{d}s, \quad t \in I$$

is said to be *a mild solution* of the nonlocal Cauchy problem (1.1) and (1.2).

Theorem 3.1. *Suppose that the functions f, g and $b_i (i = 1, \ldots, m)$ satisfy Assumptions* (A_1) *and* (A_2)*, where $u_0 \in E$. Then in the class of all the functions w, for which Assumption* (A_3) *holds, the nonlocal Cauchy problem* (1.1) *and* (1.2) *has a mild solution.*

Proof. Let

$$Y := \{w \in X:\ w \in X_r,\ w(t_0) + g(w) = u_0\}\,, \tag{3.1}$$

where r is given by Eq. (2.4).

It is easy to see that Y is a bounded closed convex subset of X.

Define a mapping F on Y by the formula

$$(Fw)(t) := T(t-t_0)\,u_0 - T(t-t_0)\,g(w) + \int_{t_0}^{t} T(t-s)f\,(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\,\mathrm{d}s, \quad w \in Y,\ t \in I. \tag{3.2}$$

Since, by Eqs. (3.2), (3.1) and (2.1)–(2.4),

$$\|(Fw)(t)\| \le M\|u_0\| + MC_2 + aMC_1 = r, \quad w \in Y, \ t \in I,$$

then

$$F : Y \to Y.$$

Moreover, the first part of Assumption (A_1) imply that $F \in C(Y, Y)$.

Now, we will show that F maps Y into a precompact subset $F(Y)$ of Y. For this purpose we will show that

$$Y(t) := \{(Fw)(t)\colon w \in Y\}, \quad t \in I$$

is precompact in E and $F(Y)$ is an uniformly equicontinuous family of functions.

Observe that

$$\begin{aligned} Y(t_0) &= \{(Fw)(t_0)\colon w \in Y\} \\ &= \{u_0 - g(w)\colon w \in X_r, w(t_0) + g(w) = u_0\}. \end{aligned}$$

Therefore, according to Assumption (A_3), $Y(t_0)$ is precompact in E.

Let $t > t_0$ be fixed. For an arbitrary $\varepsilon \in (t_0, t)$, define a mapping F_ε on Y by the formula

$$\begin{aligned} (F_\varepsilon w)(t) &:= T(t - t_0)\, u_0 - T(t - t_0)\, g(w) \\ &\quad + \int_{t_0}^{t-\varepsilon} T(t-s) f(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\, \mathrm{d}s \\ &= T(t - t_0)\, u_0 - T(t - t_0)\, g(w) \\ &\quad + T(\varepsilon) \int_{t_0}^{t-\varepsilon} T(t - s - \varepsilon) f(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\, \mathrm{d}s, \quad w \in Y. \end{aligned} \tag{3.3}$$

Since $T(\tau)$ is compact for every $\tau > t_0$ then the set

$$Y_\varepsilon(t) := \{(F_\varepsilon w)(t)\colon w \in Y\}$$

is precompact in E for every $\varepsilon \in (t_0, t)$. Moreover, from Eqs. (3.2), (3.3), (2.1) and (2.2),

$$\begin{aligned} &\|(Fw)(t) - (F_\varepsilon w)(t)\| \\ &\quad = \left\| \int_{t-\varepsilon}^{t} T(t - s) f(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\, \mathrm{d}s \right\| \le \varepsilon M C_1, \quad w \in Y. \end{aligned}$$

Consequently, the set $Y(t)$, where $t > t_0$, is precompact in E.

To show that $F(Y)$ is an uniformly equicontinuous family of functions, observe that, by Eqs. (3.2), (2.3), (2.2) and (2.1),

$$\begin{aligned} &\|(Fw)(t_1) - (Fw)(t_2)\| \\ &\quad \le \|\,(T(t_1 - t_0) - T(t_2 - t_0))\, u_0\| \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&+\|(T(t_1-t_0)-T(t_2-t_0))\,g(w)\| \\
&+\left\|\int_{t_0}^{t_1}(T(t_1-s)-T(t_2-s))\,f\,(s,w(s),w(b_1(s)),\dots,w(b_m(s)))\,\mathrm{d}s\right\| \\
&+\left\|\int_{t_1}^{t_2}T(t_2-s)f\,(s,w(s),w(b_1(s)),\dots,w(b_m(s)))\,\mathrm{d}s\right\| \\
\le{}&(\|u_0\|+C_2)\,\|T(t_1-t_0)-T(t_2-t_0)\| \\
&+C_1\int_{t_0}^{t_1}\|T(t_1-s)-T(t_2-s)\|\,\mathrm{d}s+MC_1(t_2-t_1) \\
&\quad\text{for}\quad w\in Y,\ t_0<t_1<t_2.
\end{aligned}
$$

The last three terms in Eq. (3.4) are independent of $w \in Y$ and tend to zero, when $t_2 \to t_1$, as a consequence of the continuity of $T(t)$ in the uniform operator topology for $t > 0$, which follows from the compactness of $T(t)$ for $t > 0$. Therefore, $F(Y)$ is equicontinuous family of functions.

Since all the assumptions of Arzela–Ascoli's theorem are satisfied then $F(Y)$ is precompact subset of Y.

Finally, applying Schauder's fixed point theorem to X, Y and F, it follows that F has a fixed point in Y and any fixed point of F is a mild solution of the nonlocal Cauchy problem (1.1) and (1.2).

The proof of Theorem 3.1 is complete. □

4. Existence of a classical solution

A function $u : I \to E$ is said to be *a classical solution* of the nonlocal Cauchy problem (1.1) and (1.2) if

(1) u is continuous on I and continuously differentiable on $I\backslash\{t_0\}$,

(2) $u'(t) + Au(t) = f\,(t, u(t), u(b_1(t)), \dots, u(b_m(t)))$, $t \in I\backslash\{t_0\}$

and

$$u(t_0) + g(u) = u_0.$$

Theorem 4.1. *Suppose that the functions f, g and $b_i (i = 1, \dots, m)$ satisfy Assumptions* (A$_1$) *and* (A$_2$), *where $u_0 \in E$. Then in the class of all the functions w, for which Assumption* (A$_3$) *holds, the nonlocal Cauchy problem* (1.1) *and* (1.2) *has a mild solution u. Assume, additionally, that*

(i) *E is a reflexive Banach space,*

(ii) *there exists a constant $L > 0$ such that*

$$
\|f(s,z_0,z_1,\dots,z_m)-f(s,\tilde z_0,\tilde z_1,\dots,\tilde z_m)\|\le L\left(|s-\tilde s|+\sum_{k=0}^{m}\|z_k-\tilde z_k\|\right) \tag{4.1}
$$

$$\text{for}\quad s,\tilde s\in I,\ z_i,\tilde z_i\in E_r\quad (i=0,1,\dots,m),$$

(iii) *u is the unique mild solution of problem* (1.1) *and* (1.2) *and there is a constant* $\mathscr{H} > 0$ *such that*

$$\|u(b_i(s)) - u(b_i(\tilde{s}))\| \le \mathscr{H}\|u(s) - u(\tilde{s})\| \quad \text{for} \quad s, \tilde{s} \in I, \tag{4.2}$$

(iv) $u_0 \in D(A)$ *and* $g(u) \in D(A)$.

Then u is the unique classical solution of the nonlocal Cauchy problem (1.1) *and* (1.2).

Proof. Since all the assumptions of Theorem 3.1 are satisfied then the nonlocal Cauchy problem (1.1) and (1.2) possesses a mild solution u which, according to assumption (iii), is the unique mild solution of problem (1.1) and (1.2).

Now, we will show that u is the unique classical solution of problem (1.1) and (1.2). To this end, observe that

$$\begin{aligned}
&u(t+h) - u(t)\\
&\quad = [T(t+h-t_0)\,u_0 - T(t-t_0)\,u_0]\\
&\qquad - [T(t+h-t_0)\,g(u) - T(t-t_0)\,g(u)]\\
&\qquad + \int_{t_0}^{t_0+h} T(t+h-s) f\,(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))\,\mathrm{d}s\\
&\qquad + \int_{t_0+h}^{t+h} T(t+h-s) f\,(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))\,\mathrm{d}s\\
&\qquad - \int_{t_0}^{t} T(t-s) f\,(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))\,\mathrm{d}s\\
&\quad = T(t-t_0)\,[T(h) - I]\;u_0 - T(t-t_0)\,[T(h)-I]\;g(u)\\
&\qquad + \int_{t_0}^{t_0+h} T(t+h-s) f\,(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))\,\mathrm{d}s\\
&\qquad + \int_{t_0}^{t} T(t-s)\,[f\,(s+h, u(s+h), u(b_1(s+h)), \ldots, u(b_m(s+h)))\\
&\qquad - f\,(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))]\,\mathrm{d}s\\
&\qquad\qquad \text{for} \quad t \in [t_0, t_0+a),\; h > 0 \text{ and } t+h \in (t_0, t_0+a].
\end{aligned} \tag{4.3}$$

Consequently, by Eqs. (4.3), (2.1), (2.2), (4.1) and (4.2),

$$\begin{aligned}
&\|u(t+h) - u(t)\|\\
&\quad \le hM\|Au_0\| + hM\|Ag(u)\| + hMC_1 + MLah\\
&\qquad + ML\int_{t_0}^{t}\left(\|u(s+h) - u(s)\| + \sum_{k=1}^{m}\|u(b_k(s+h)) - u(b_k(s))\|\right)\mathrm{d}s\\
&\quad = C_* h + ML(1 + m\mathscr{H})\int_{t_0}^{t}\|u(s+h) - u(s)\|\,\mathrm{d}s\\
&\qquad\qquad \text{for} \quad t \in [t_0, t_0+a),\; h > 0 \text{ and } t+h \in (t_0, t_0+a],
\end{aligned} \tag{4.4}$$

where

$$C_* := M\left[\|Au_0\| + \|Ag(u)\| + C_1 + aL\right].$$

From Eq. (4.4) and Gronwall's inequality,

$$\|u(t+h) - u(t)\| \le C_* e^{aML(1+m\mathscr{H})} h$$

$$\text{for} \quad t \in [t_0, t_0 + a),\ h > 0 \text{ and } t + h \in (t_0, t_0 + a].$$

Hence, u is Lipschitz continuous on I.

The Lipschitz continuity of u on I combined with the Lipschitz continuity of f on $I \times E^{m+1}$ imply that $t \to f(t, u(t), u(b_1(t)), \ldots, u(b_m(t)))$ is Lipschitz continuous on I. This property of $t \to f(t, u(t), u(b_1(t)), \ldots, u(b_m(t)))$ together with the assumptions of Theorem 4.1 imply, by Theorem 1 from [12] and by Theorem 3.1 from this paper, that the linear Cauchy problem

$$v'(t) + Av(t) = f\left(t, u(t), u(b_1(t)), \ldots, u(b_m(t))\right), \quad t \in I \setminus \{t_0\},$$

$$v(t_0) = u_0 - g(u)$$

has a unique classical solution v such that

$$v(t) = T(t - t_0)\, u_0 + T(t - t_0)\, g(u)$$

$$+ \int_{t_0}^{t} T(t - s) f\left(s, u(s), u(b_1(s)), \ldots, u(b_m(s))\right) \mathrm{d}s = u(t), \quad t \in I.$$

Consequently, u is the unique classical solution of the nonlocal Cauchy problem (1.1) and (1.2) and, therefore, the proof of Theorem 4.1 is complete. □

5. Examples and remarks

I. Let $p \in \mathbb{N}$ and $a_1, \ldots, a_p$ be given real numbers such that $t_0 < a_1 < \ldots < a_p \le t_0 + a$. Theorems 3.1 and 4.1 can be applied for g defined by the formula

$$g(w) = \sum_{i=1}^{p} c_i w(a_i) \quad \text{for } w \in X,$$

$$\left[g(w) = \sum_{i=1}^{p} \frac{C_i}{\varepsilon_i} \int_{a_i - \varepsilon_i}^{a_i} w(s)\, \mathrm{d}s \text{ for } w \in X \right],$$

where c_i $(i = 1, \ldots, p)$ are given constants [and ε_i $(i = 1, \ldots, p)$ are given positive constants such that $t_0 < a_1 - \varepsilon_1$ and $a_{i-1} < a_i - \varepsilon_i$ $(i = 2, \ldots, p)$, respectively].

II. Theorem 3.1 can be applied for b_i $(i = 1, \ldots, m)$ defined by the following formulas:

1. $b_i(t) := k_i t^n$ $(i = 1, \ldots, m)$ for $t \in I$, where $t_0 = 0$, $a = 1$, $k_i \in (0, 1]$ $(i = 1, \ldots, m)$, $n \in \mathbb{N}$;

2. $b_i(t) := k_i \sin t \, (i = 1,\dots,m)$ for $t \in I$, where $t_0 = 0$, $a = \frac{\pi}{2}$, $k_i \in (0,1]$ $(i = 1,\dots,m)$;

3. $$b_i(t) := \begin{cases} -\left(t - \dfrac{a}{2}\right)^2 + \left(\dfrac{a}{2}\right)^2 & \text{for } t \in \left[t_0, t_0 + \dfrac{a}{2}\right), \\ k_i\left(t - \dfrac{a}{2}\right)^2 + \left(\dfrac{a}{2}\right)^2 & \text{for } t \in \left[t_0 + \dfrac{a}{2}, t_0 + a\right], \end{cases}$$

where $t_0 = 0$, $a = 1$, $k_i \in (0,3]$ $(i = 1,\dots,m)$.

Indeed, $b_i : I \to I (i = 1,\dots,m)$ and $b_i (i = 1,\dots,m)$ are continuous on I.

III. Let $t_0 = 0$, $a = 1$, $E = \mathbb{R}$; $u(t) = \alpha t^n$ for $t \in I$, where $\alpha \in (0,1], n \in \mathbb{N}$; and $b_i(t) := k_i t$ $(i = 1,\dots,m)$ for $t \in I$, where $k_i \in (0,1]$ $(i = 1,\dots,m)$. Then Theorem 3.1 can be applied for the above b_i $(i = 1,\dots,m)$, and condition (4.2) holds, where $\mathscr{K} := \max\{k_1^n,\dots,k_m^n\}$.

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