

Elastic and inelastic scattering of flat-top solitons

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We investigate the interaction between two flat-top solitons within the cubic-quintic nonlinear Schrödinger equation framework. Our study results point towards a significant departure of flat-top solitons' collisional characteristics from the conventional behaviors exhibited in the scattering dynamics of two bright solitons. Our investigation outlines specific regimes corresponding to the dual flat-top solitons' elastic and inelastic collisions. We determine regimes in the parameter space where exchange in the widths of the interacting flat-top solitons is possible, even within the elastic collision domain. We find a periodic occurrence of completely elastic scattering of flat-top solitons in terms of the solitons' parameters. Investigating the internal energy transfers between the solitons and the emitted radiation reveals the origin of inelasticity in flat-top soliton collisions. We perform a variational calculation that accounts for the amount of radiation produced by the collision and hence provides further insight into the physics underlying the loss of elasticity of collisions.

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I. INTRODUCTION

The most interesting and appealing feature of solitons is that their integrity is preserved while propagating and after scattering with each other [1–4]. Such a unique feature suggests there may be a plethora of applications for solitons [5–8]. Within this context, bright solitons of the fundamental nonlinear Schrödinger equation (NLSE) have been investigated extensively [1–3]. Specifically, the interaction between bright solitons [9] and their scattering by potentials [10–14] have been well studied. It is established that collisions between bright solitons of the fundamental NLSE are elastic, and it is understood that this is a consequence of the integrability of the fundamental NLSE, which implies the existence of an infinite number of conserved quantities (see, for instance, Ref. [1]).

Soliton collisions have been studied extensively across various models [15–19]. The interactions of cosine-Hermite-Gaussian solitons have been analyzed using the two-dimensional NLSE [20]. Collisions between anisotropic dipolar solitons at different velocities and angles, examined through the three-dimensional mean-field Gross-Pitaevskii equation, show quasielastic behavior at high velocities with no deformation, and inelastic behavior with increasing deformation at lower velocities [21]. Inelastic soliton collisions

have also been explored in the nonlinear Klein-Gordon model [22]. Additionally, the $(3 + 1)$ -dimensional Sharma-Tasso-Olver-like model, applicable to dispersive wave phenomena in optics, reveals inelastic fusion or fission phenomena where some kink waves disappear or split into multiple waves [23].

Recently, an increased interest was directed towards another type of solitons, known as flat-top solitons [24–34]. This was triggered by the experimental realization of so-called quantum droplets, which are one form of flat-top solitons in mixtures of Bose-Einstein condensates [24,35–37] and dipolar condensates [38–40]. Flat-top solitons are supported by the NLSE with dual power-law nonlinearity. In mixtures of Bose-Einstein condensates, the competing nonlinearities are quartic and cubic. The former originates from quantum fluctuations, known as Lee-Huang-Yang (LHY) fluctuations [41,42], while the latter originates from interatomic interactions [43]. In nonlinear optics, dual nonlinearities are cubic and quintic, where the former, known as Kerr nonlinearity, corresponds to the nonlinear response of the medium, while the latter corresponds to higher-order corrections [6].

Many features of flat-top solitons have been studied, including their dynamics [44–48], collective modes [49], their scattering by potentials [50], and binding into molecules [51]. It is known that the NLSE with dual nonlinearities is not integrable [1]. Collisions of flat-top solitons are also known to be inelastic [52] and lead to the fragmentation of solitons [44]. Given the importance of understanding the behavior of flat-top solitons from fundamental and application points of view, in this paper we aim to investigate mutual collisions of flat-top solitons, with an emphasis on the elasticity and inelasticity of the collisions. In particular, we will answer the following questions: (i) Can collisions of flat-top solitons be elastic? (ii) What is the physics underlying inelasticity in

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the collisions of flat-top solitons? To answer these questions, we start by characterizing the outcome of collisions between flat-top solitons in terms of the solitons' parameters, namely initial relative phase, initial relative speed, initial separation, and soliton width, where the latter is determined by the strength of the nonlinearities. This results in a classification of scattering regimes into (i) nearly elastic scattering associated with width exchange, (ii) inelastic scattering with radiation production, and (iii) a single case of completely elastic scattering with no width exchange and no emitted radiation. Investigating the single case of completely elastic scattering, we find periodic occurrences of such a case in terms of any of the initial parameters of flat-top solitons, including their relative phase, separation, and speed. The second main finding is revealing the origin of inelasticity through a detailed analysis of the energy transfers between the different forms of energy. Specifically, we consider a completely inelastic case in which the two solitons coalesce after collision, and then we show that the permanent loss in the initial center-of-mass kinetic energy is mostly carried by dispersive radiation waves. We discuss this mechanism with respect to the nonintegrability of the NLSE with dual nonlinearity. This picture is confirmed through a variational calculation, where we treat the amplitude of the produced radiation as a variational parameter. Conservation of norm, energy, and momentum is then used to predict the norm of radiation that agrees favorably with the numerical results.

The rest of the paper is organized as follows. In Sec. II, we present the theoretical model and setup for the numerical simulations. In Sec. III, we present and discuss our results for characterizing the outcome of flat-top soliton scattering in terms of the solitons' parameters. In Sec. IV, we show results for periodic occurrences of elastic scattering in terms of the solitons' parameters. In Sec. V, we discuss energy transfers, shedding more light on the origin of inelasticity. In Sec. VI, we perform a variational calculation to account for the amount of radiation production. We end in Sec. VII with a summary of our main findings, conclusions, and an outlook for future work.

II. THEORETICAL MODEL AND NUMERICAL SETUP

A. Theoretical model

The dimensionless nonlinear Schrödinger equation with dual power nonlinearities is expressed as

$$i\frac{\partial}{\partial t}\psi(x,t) + g_1\frac{\partial^2}{\partial x^2}\psi(x,t) + g_2|\psi(x,t)|^p\psi(x,t) + g_3|\psi(x,t)|^q\psi(x,t) = 0, \quad (1)$$

where $\psi(x,t)$ represents a complex field corresponding, for example, to the amplitude of the electric (or magnetic) field in the case of optical solitons, while it corresponds to the condensate wave function in the case of Bose-Einstein condensates. The term g_1 is the dispersion coefficient, while g_2 and g_3 denote the strengths of the nonlinearity terms, and p and q are arbitrary real constants that are not necessarily integers. This is one of the nonlinear evolution equations that admits exact solutions corresponding to flat-top solitons [53]. Experimentally relevant situations, such as quantum droplets

of Bose-Einstein condensates, or optical solitons in fibers require $q = p/2$. In this case, Eq. (1) admits the following exact solution that corresponds to a one-parameter family comprising four fundamentally different solutions including flat-top solitons:

$$\psi(x,t) = \left\{ \frac{(p+2)u_0}{g_2 + g_2\sqrt{1+\gamma}\cosh[p\sqrt{\frac{u_0}{g_1}}(x-x_0-v_0t)]} \right\}^{1/p} \times e^{i[u_0t + \frac{v_0}{4g_1}[2(x-x_0)-v_0t+\phi_0]]}. \quad (2)$$

In this equation, u_0 , x_0 , ϕ_0 , and v_0 are parameters that denote the soliton's initial amplitude, peak position, phase, and velocity, respectively. The dimensionless parameter γ , expressed as $\gamma = g_3/g_{30}$, where $g_{30} = (p+1)g_2^2/[(p+2)^2u_0]$, is the single parameter that dictates the form of the soliton solution. Depending on γ , the soliton may manifest as a kink soliton ($\gamma = -1$), a flat-top soliton ($-1 < \gamma < 0$), a bright soliton ($\gamma = 0$), or a thin-top soliton ($\gamma > 0$), as discussed in [50].

For optical solitons in fibers, the nonlinearities are cubic and quintic, which correspond to $p = 2$, and for this case, Eq. (1) takes the form

$$i\frac{\partial}{\partial t}\psi(x,t) + g_1\frac{\partial^2}{\partial x^2}\psi(x,t) + g_2|\psi(x,t)|^2\psi(x,t) + g_3|\psi(x,t)|^4\psi(x,t) = 0, \quad (3)$$

and its solution, (2), simplifies to

$$\psi(x,t) = \sqrt{\frac{2u_0}{g_2\sqrt{1+\gamma}}} \times \frac{1}{\sqrt{\frac{1-\sqrt{1+\gamma}}{2\sqrt{1+\gamma}} + \cosh^2[p\sqrt{\frac{u_0}{g_1}}(x-x_0-v_0t)]}} \times e^{i[u_0t + v_0[2(x-x_0)-v_0t]/(4g_1)]}. \quad (4)$$

Figure 1 illustrates the soliton profiles corresponding to the four members of this γ family of solutions.

The width of the flat-top soliton, W , is defined by the half-maximum criterion, namely $|\psi(W/2, 0)| = (1/2)|\psi(x_0, 0)|$, which gives

$$W = 2\sqrt{\frac{g_1}{u_0}} \ln \left[4 + \frac{3}{\sqrt{1+\gamma}} + \frac{1}{2} \sqrt{-4 + \left(8 + \frac{6}{\sqrt{1+\gamma}} \right)^2} \right]. \quad (5)$$

In the limit $\gamma \rightarrow -1$, where γ approaches -1 from the right, the soliton width diverges, and the left and right edges of the flat-top soliton approach $-\infty$ and $+\infty$, respectively. The soliton, in this limit, will take the form of a finite constant value, as shown by the thin horizontal line in Fig. 1. However, since x_0 is an arbitrary constant, the solution can be shifted by $x_0 = W/2$ such that the left edge of the flat-top soliton is always at the origin. With this shift, the solution, in the limit $\gamma \rightarrow -1$, takes the shape of a kink localized at $x = 0$, while the right edge is shifted to $+\infty$, as shown by the inset in Fig. 1.

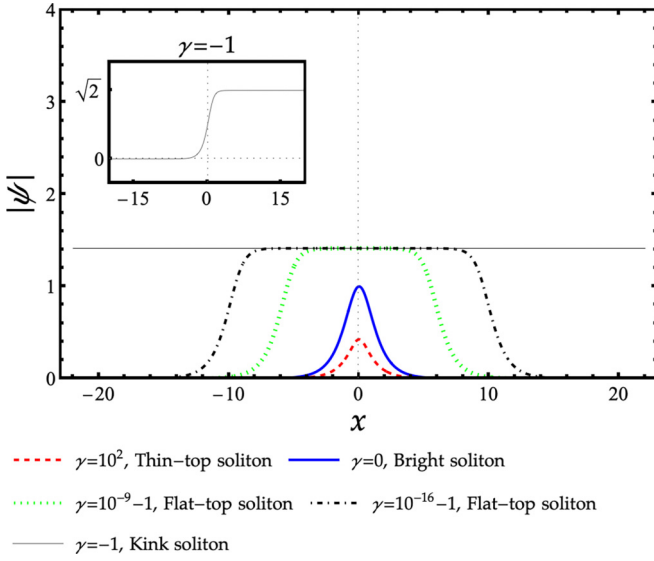


FIG. 1. Solution profile of Eq. (4) for different values of γ . Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $v_0 = 0$.

The energy functional corresponding to Eq. (3) is

$$E = \int_{-\infty}^{\infty} \left[g_1 \left| \frac{\partial \psi}{\partial x} \right|^2 - \frac{1}{2} g_2 |\psi|^4 - \frac{1}{3} g_3 |\psi|^6 \right] dx. \quad (6)$$

The first term in the energy functional gives the kinetic energy, which, for a quantum state, can be split into two terms, namely the *center-of-mass kinetic energy* and the *kinetic energy pressure*. The former corresponds to the translational motion of the flat-top soliton, which is analogous to the classical translational kinetic energy of rigid objects, while the second term originates from the curvature in the profile and has no classical analog. The two terms can be extracted from the total kinetic energy term, $\text{KE}_{\text{tot}} = g_1 \int_{-\infty}^{\infty} \left| \frac{\partial \psi}{\partial x} \right|^2 dx$, as follows. Consider the polar form of the wave function: $\psi(x, t) = A(x, t)e^{i\phi(x, t)}$, with real amplitude and phase. The total kinetic energy term can then be written as

$$\text{KE}_{\text{tot}} = g_1 \int_{-\infty}^{\infty} \left(\frac{\partial A}{\partial x} \right)^2 dx + g_1 \int_{-\infty}^{\infty} A^2 \left(\frac{\partial \phi}{\partial x} \right)^2 dx, \quad (7)$$

which defines the kinetic energy pressure

$$\text{KE}_{\text{pressure}} = g_1 \int_{-\infty}^{\infty} \left(\frac{\partial A}{\partial x} \right)^2 dx \quad (8)$$

where

$$F_2(\gamma) = -\frac{3}{512c^{5/2}(c+1)^{5/2}} \left\{ \frac{16c(c+1)}{\gamma+1} \left[\frac{\gamma}{\tan^{-1}\left(\frac{\sqrt{\gamma+1}-1}{\sqrt{\gamma}}\right)^2} \right]^{3/2} \left[\sqrt{c}\sqrt{c+1} - (2c+1) \tanh^{-1}\left(\sqrt{\frac{c}{c+1}}\right) \right] \right. \\ \left. + \gamma \left[\frac{\gamma}{(\gamma+1) \tan^{-1}\left(\frac{\sqrt{\gamma+1}-1}{\sqrt{\gamma}}\right)^2} \right]^{3/2} \left(3\sqrt{c}\sqrt{c+1}(2c+1) - (8c^2+8c+3) \tanh^{-1}\left(\sqrt{\frac{c}{c+1}}\right) \right) \right\} \quad (15)$$

and the center-of-mass kinetic energy

$$\text{KE}_{\text{com}} = g_1 \int_{-\infty}^{\infty} A^2 \left(\frac{\partial \phi}{\partial x} \right)^2 dx. \quad (9)$$

It should be noted here that this distinction between the two forms of energy may not be well defined for the general case when the amplitude, $A(x, t)$, and phase, $\phi(x, t)$, have a general time dependence, since in that case, each of the two types of energy may have contributions from both definitions (8) and (9). However, for solutions that depend on time only linearly through their translational motion, obtained by a Galilean transformation of a stationary solution, such as the one given by (4), the two types of kinetic energy are well defined. In this case, using the solution (4), the kinetic energy pressure takes the form

$$\text{KE}_{\text{pressure}} = \frac{1}{48g_1} g_2^2 N^3 F_1(\gamma), \quad (10)$$

where

$$F_1(\gamma) = \frac{3}{16} (-\gamma)^{-3/2} \left[\frac{\gamma}{\tan^{-1}\left(\frac{\sqrt{1+\gamma}-1}{\sqrt{\gamma}}\right)} \right]^{3/2} \\ \times \left[\sqrt{-\gamma} - 2(1+\gamma) \tanh^{-1} \left(\sqrt{\frac{2+\gamma-2\sqrt{1+\gamma}}{\gamma}} \right) \right] \quad (11)$$

and the center-of-mass kinetic energy takes the form

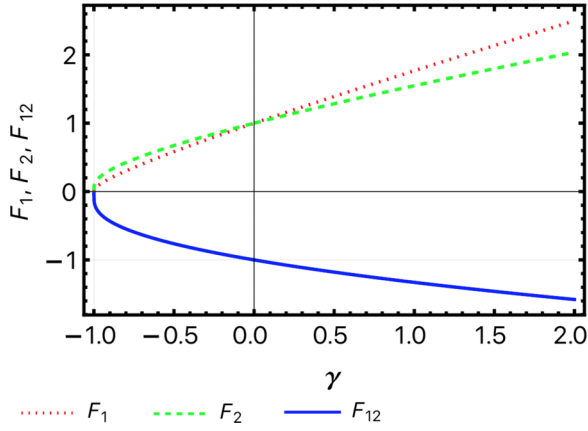
$$\text{KE}_{\text{com}} = \frac{1}{4g_1} N v_0^2, \quad (12)$$

where N is the total soliton intensity, or norm, defined by

$$N = \int_{-\infty}^{\infty} |\psi|^2 dx \\ = 8 \sqrt{\frac{g_1 u_0}{g_2^2 \gamma}} \tan^{-1} \left(\frac{\sqrt{1+\gamma}-1}{\sqrt{\gamma}} \right). \quad (13)$$

The total interaction energy is also calculated as

$$\text{IE} = \int_{-\infty}^{\infty} \left[-\frac{1}{2} g_2 |\psi|^4 - \frac{1}{3} g_3 |\psi|^6 \right] dx \\ = -\frac{1}{24g_1} g_2^2 N^3 F_2(\gamma), \quad (14)$$

FIG. 2. The function $F_{12} = F_1 - 2F_2$ given by Eq. (18).

and $c = \frac{1-\sqrt{\gamma+1}}{2\sqrt{\gamma+1}}$. The total energy is thus

$$E = \frac{1}{4g_1} N v_0^2 + \frac{1}{48g_1} g_2^2 N^3 F_1(\gamma) - \frac{1}{24g_1} g_2^2 N^3 F_2(\gamma). \quad (16)$$

While the analysis we perform below on the flat-top soliton dynamics requires the determination of each energy component separately, it is instructive to show that the last expression simplifies to

$$E = \frac{1}{4g_1} N v_0^2 + \frac{1}{48g_1} g_2^2 N^3 F_{12}(\gamma), \quad (17)$$

where

$$F_{12}(\gamma) = F_1 - 2F_2 = -\frac{3}{4 \tanh^{-1}\left(\frac{\sqrt{\gamma+1}-1}{\sqrt{-\gamma}}\right)^2} \times \left[1 + \frac{\sqrt{-\gamma}}{2 \tanh^{-1}\left(\frac{\sqrt{\gamma+1}-1}{\sqrt{-\gamma}}\right)}\right]. \quad (18)$$

The quantity $F_{12}(\gamma)$ is real for our relevant range of γ , namely $\gamma > -1$. Plotting this quantity, as shown in Fig. 2, shows that it equals -1 for the bright soliton ($\gamma = 0$), and 0 for the kink soliton ($\gamma = -1$). In the bright soliton limit, $\gamma \rightarrow 0$, both functions $F_1(\gamma)$ and $F_2(\gamma)$ equal 1, and the energy takes the correct form of bright soliton energy, namely $E_{BS} = \frac{1}{4g_1} N v_0^2 - \frac{1}{48g_1} g_2^2 N^3$. In the limit $\gamma \rightarrow -1$, which corresponds to the kink soliton, N diverges as $-\ln(1+\gamma)$ while the term $F_1 - 2F_2$ vanishes as $1/[\ln(1+\gamma)]^2$. Equation (17) shows that the energy, which is proportional to $N^3(F_1 - 2F_2)$, diverges as $-\ln(1+\gamma)$. Divergence in energy of the kink soliton is due to the infinite nonzero flat part of the kink soliton, which results in a diverging norm. In Fig. 3, we plot the width, W , the norm, N , and the energy, E , of a single moving soliton, as given by expressions (5), (13), and (17), respectively.

B. Setup for numerical simulations

We consider two equal flat-top solitons, namely a stationary soliton located at $x = -x_0$ and a second soliton at $x = x_0 > 0$ moving towards the first soliton with initial center-of-mass velocity $v_0 < 0$. The profile corresponding to this initial

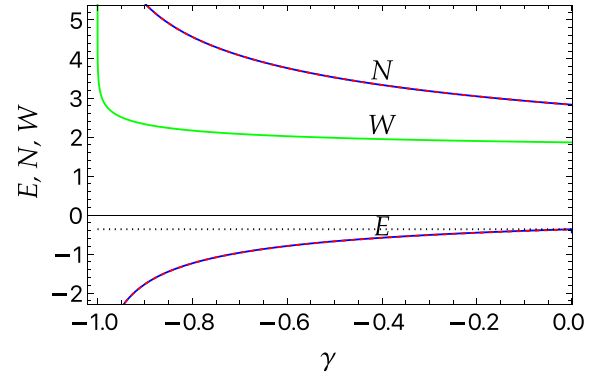


FIG. 3. Solid curves correspond to the analytical expressions of the total energy, E , and norm, N , of two solitons, and the width, W , of a single soliton, as is essentially given by Eqs. (17), (13), and (5), respectively. It should be noted that expression (17) gives the energy of a single soliton. The corresponding curve in this figure is for two solitons; initially one is stationary and the other is moving with a speed v_0 . Therefore, we have multiplied the interaction energy term in (17) by 2. Dashed red curves for E and N , which coincide with the analytical expressions, are obtained from the numerical solution. Horizontal dotted line marks the bright soliton energy limit. Parameters used are $u_0 = 1$, $g_1 = 1/2$, $g_2 = 1$.

setting is

$$\psi(x, 0) = \psi_l(x, 0) + \psi_r(x, 0), \quad (19)$$

where the initial profile of the stationary soliton is

$$\psi_l(x, 0) = \sqrt{\frac{2u_0}{g_2\sqrt{1+\gamma}}} \frac{1}{\sqrt{\frac{1-\sqrt{1+\gamma}}{2\sqrt{1+\gamma}} + \cosh^2\left[\sqrt{\frac{u_0}{g_1}}(x+x_0)\right]}}, \quad (20)$$

and the initial profile of the moving soliton is

$$\psi_r(x, 0) = \sqrt{\frac{2u_0}{g_2\sqrt{1+\gamma}}} \frac{1}{\sqrt{\frac{1-\sqrt{1+\gamma}}{2\sqrt{1+\gamma}} + \cosh^2\left[\sqrt{\frac{u_0}{g_1}}(x-x_0)\right]}} \times e^{i[v_0(x-x_0)/(2g_1)]}, \quad (21)$$

where we have set the initial phase to $\phi_0 = 0$, which will be the case for most of the numerical experiments. In Sec. IV, however, we relax this setting and allow this phase to change in order to investigate its effect on the periodic occurrence of elastic scattering. This initial profile is evolved using Eq. (3). We use the iterative power series finite-difference method [55,56], which proved to be highly accurate in simulating the scattering of very wide flat-top solitons by potential wells [54]. The method is superior to other finite-difference methods in accuracy and speed. We have shown in Ref. [54] that accuracy can reach machine precision, and speed is at least three times faster than the split-step finite-difference method.

It is instructive to find how the soliton's main features, namely energy, norm, and width, depend on the parameter γ . These three quantities are plotted in Fig. 3 for a single soliton moving with speed v_0 , using the exact analytical expressions (5), (13), and (17). As a confirmation of the accuracy of

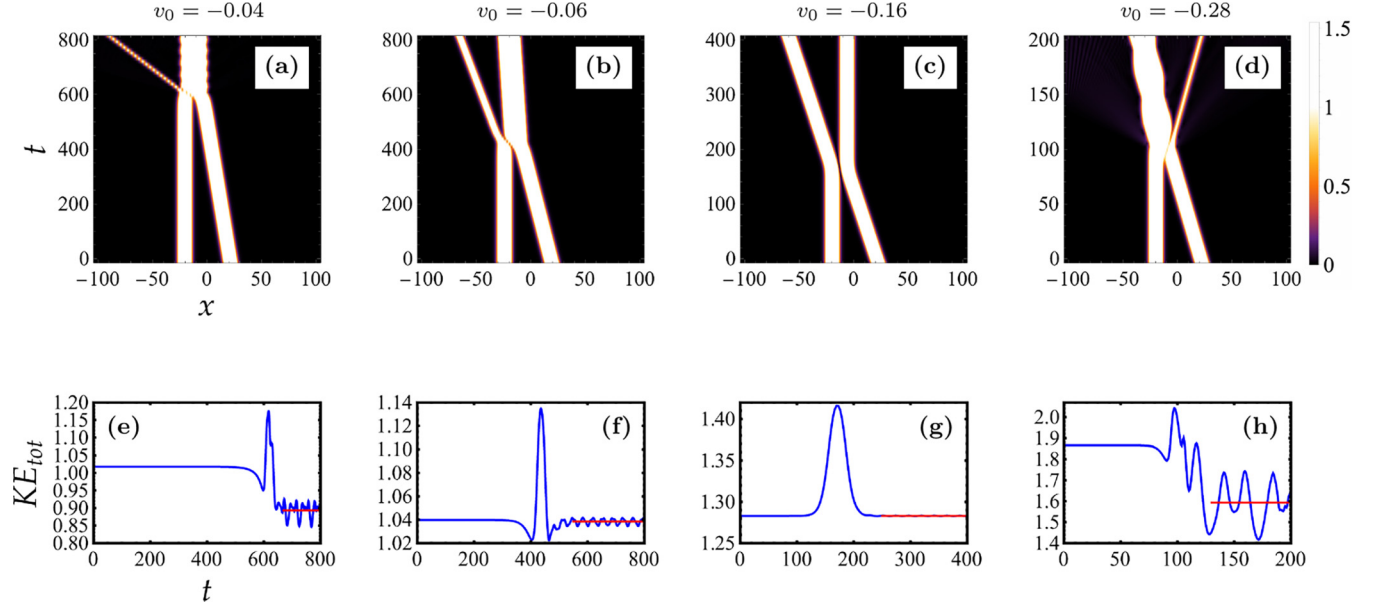


FIG. 4. Interaction of two flat-top solitons with widths defined by γ_9 (upper row) and the corresponding total kinetic energy as a function of time (lower row). The red line indicates the mean value. In each case, one soliton remains stationary while the other is moving towards it with a speed v_0 . The velocity and change in kinetic energy, respectively, are (a), (e) $v_0 = -0.04$, $|\Delta KE| = 1.243 \times 10^{-1}$; (b), (f) $v_0 = -0.06$, $|\Delta KE| = 1.374 \times 10^{-2}$; (c), (g) $v_0 = -0.16$, $|\Delta KE| = 2.074 \times 10^{-7}$; and (d), (h) $v_0 = -0.28$, $|\Delta KE| = 2.729 \times 10^{-1}$. Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$.

the numerical code, the energy and norm are also calculated numerically using the integrations in (8), (9), (13), and (14), where a perfect agreement with the analytical expressions is obtained.

A final remark about the values of γ that we consider in this work is in order. Our numerical investigations are restricted to flat-top solitons with $-1 < \gamma < 0$. For wide flat-top solitons, γ has to be very close to -1 . It is thus convenient to consider values of γ given by the function $\gamma_i = 10^{-i} - 1$, where $i = 0, 1, 2, \dots, 16$. In this manner, the width of the flat-top soliton, given by Eq. (5), increases almost linearly with i for $i > 2$. Here, γ_0 corresponds to the bright soliton and γ_{16} corresponds to the widest flat-top soliton that we consider here, with width $W \approx 14.3$ for $g_1 = 1/2$, $u_0 = 1$.

III. COMPUTATIONAL RESULTS AND ANALYSIS

Elastic scattering of classical rigid objects is characterized by the conservation of total kinetic energy. Numerical inspection showed that scattering of flat-top solitons is complicated by the various possible outcomes. Depending on the solitons' initial relative speed, phase, and width, the scattering may be, as in the classical case, completely elastic, completely inelastic (coalescence), or partially inelastic. These possible outcomes are complicated by the possibility of permanent average *width exchange*, excitation of internal modes such as width oscillations, and emitted radiation. We found that, while the collision may be nearly elastic, considerable width exchange may take place accompanied by very small width oscillations. It should be noted, however, that only for the completely elastic case is there no width exchange, no width oscillations, no radiation, and the kinetic energy is exactly conserved. Emitted radiation is always present in flat-top scat-

tering, except for the completely elastic scattering case, and it turns out to provide the key for understanding the source of inelasticity in flat-top scattering.

In this section, we examine the dynamics of interactions between two flat-top solitons with an emphasis on kinetic energy and width changes before and after scattering. In Sec. III A, we select a flat-top soliton with a width corresponding to γ_9 for the numerical study, outlining the parameter space for elastic and inelastic collisions, in addition to the width exchange between the two flat-top solitons postcollision. We also demonstrate that the two flat-top solitons merge into a single entity at very low and very high collision velocities, where the coalescence in both instances is accompanied by the ejection of small-amplitude propagating radiation waves. In Sec. III B, we broaden the study to encompass other flat-top solitons with varying widths corresponding to the range $\gamma_0 - \gamma_{16}$.

A. Collision of two flat-top solitons with widths corresponding to γ_9

For this part of our study, a flat-top soliton with a width corresponding to γ_9 is used to highlight multiple outcomes. Total kinetic energy is calculated before and after collision using Eq. (7). The difference in kinetic energy is then defined as $\Delta KE = KE(t = t_f) - KE(t = 0)$. Figure 4 demonstrates the degree of elasticity observed in the collision between two flat-top solitons. In (a), a stationary flat-top soliton with velocity $v_0 = 0$ is approached by another flat-top soliton with $v_0 = -0.04$. The collision is inelastic, as evidenced by the difference in kinetic energy before the collision and the average kinetic energy afterward, represented by the red line, accompanied by a significant width exchange. In (b), the

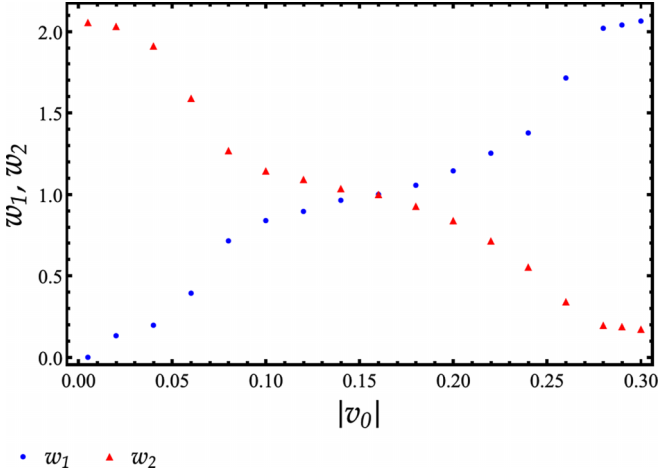


FIG. 5. Variation of flat-top solitons width ratios $w_1 = W_1(t = t_f)/W_1(t = 0)$ and $w_2 = W_2(t = t_f)/W_2(t = 0)$ vs γ . Here, w_1 and w_2 denote the width ratio of the first and second solitons, respectively. Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$.

flat-top soliton's velocity is increased to $v_0 = -0.06$, resulting in an oscillation of the kinetic energy around a value close to the initial energy, with $\Delta KE = 0.001\,378$. Given the near-zero value of ΔKE , this collision is nearly elastic, yet a noticeable width exchange between the solitons still occurs. Figure 4(c) presents an elastic collision with a velocity of $v_0 = -0.16$, where $\Delta KE \approx 0$ and no width exchange takes place. Lastly, Fig. 4(d) showcases an inelastic collision with a velocity of $v_0 = -0.28$, accompanied by a width exchange, with a recorded $\Delta KE = -0.281\,656$. This leads us to conclude that there are three possible outcomes for flat-top soliton collisions: (i) completely elastic collisions with no radiation and no width exchange, (ii) nearly elastic collision with almost no radiation but with width exchange, and (iii) inelastic collision with both emitted radiation and width exchange. The rather surprising appearance of a single case of completely elastic scattering, imbedded within a continuum of inelastic collisions, will be explained in the next section. It should be noted that, here and throughout, the spatial domain is taken large enough such that the backscattered radiation from the edges of the spatial grid does not reach the soliton during the evolution time.

To elucidate quantitatively the phenomenon of width exchange between two flat-top solitons, as observed in Fig. 4, we introduce the width ratios $w_1 = W_1(t = t_f)/W_1(t = 0)$ and $w_2 = W_2(t = t_f)/W_2(t = 0)$ to quantify the changes in their widths. Here, w_1 is defined as the ratio of the postinteraction width to the preinteraction width for the left flat-top soliton, while w_2 follows the same formula for the right flat-top soliton. These width ratios are plotted versus soliton speed in Fig. 5. From the figure, it is observed that at low velocities, the left flat-top soliton emerges from the collision with a reduced width compared to its original width, while the right flat-top soliton experiences the opposite effect. Upon increasing the velocity, a critical point is reached around $|v_0| = 0.16$, at which no width exchange is observed. Past this point, the left flat-top soliton begins to emerge postcollision with an increased width. To comprehend this behavior, we list the

values of ΔKE as a function of velocity in Table I. Notably, at $|v_0| = 0.16$, ΔKE attains its minimum value, registering at a nearly negligible -2.074×10^{-7} . Thus, indeed at this distinguished speed of $|v_0| = 0.16$, the collision is completely elastic and no width exchange takes place.

For an overview of the scattering outcomes, we identify in Fig. 6 the velocity ranges corresponding to elastic and inelastic domains, as well as regions exhibiting radiational effects, using the data presented in Table I. For low velocities $|v_0| < 0.08$, the collision is inelastic. As the collision velocity increases, the interaction transitions to being nearly elastic for velocities in the range $0.08 \leq |v_0| \leq 0.22$, marked by a reduction in ΔKE by two orders of magnitude. Nonetheless, width exchange is still ongoing in this regime, as shown in the table. For velocities above $|v_0| = 0.22$, we once again observe inelastic collisions. Notably, at relatively high collision velocities, $|v_0| > 0.30$, the inelastic collision not only results in a single flat-top soliton, but it also gives rise to small-amplitude density waves indicative of radiation. It is noteworthy that at $|v_0| \approx 0.16$, the collision becomes fully elastic, marked by an absence of width exchange, width oscillations, and radiation. The numerical value for ΔKE is reduced, at this point, by anywhere from two to five orders of magnitude compared to the other data in Table I. Strictly speaking, radiation is almost always produced in flat-top soliton scattering, but it is negligibly small in the region of nearly elastic scattering and is absent only in a single case of completely elastic scattering, as shown in Table I. As will be discussed in Sec. V, the completely elastic scattering case is part of periodic occurrences at specific values of the soliton parameters.

The emergence of a single flat-top soliton at high velocity, along with the symmetry in the data in Table I, i.e., ΔKE , is large at both low and high velocities due to the ejection of small density waves. This raises the question of whether two flat-top solitons might similarly merge at very low velocities. Figure 7 depicts this phenomenon, showing two flat-top solitons combining into a single entity at both very low ($|v_0| = 0.005$) and high ($|v_0| = 0.30$) collision velocities.

It is interesting to observe that the coalesced soliton acquires a positive momentum in spite of the fact that the total initial momentum is negative, which raises a question about the conservation of total momentum. To understand this unusual behavior, we first verify that total energy, norm, and momentum are indeed conserved, up to the accuracy of the numerical method, as shown in Fig. 8. Since the total momentum

$$J = \frac{i}{2} \int_{-\infty}^{\infty} \left(\frac{\partial \psi^*(x, t)}{\partial x} \psi(x, t) - \frac{\partial \psi(x, t)}{\partial x} \psi^*(x, t) \right) dx \quad (22)$$

is conserved, the total momentum before collision is negative, and the momentum of the coalesced solitons after collision is positive, some negative momentum should be carried by the scattered radiation for the total momentum to be conserved. We have calculated the momentum carried by the radiation part and the momentum carried by the solitons separately, as shown in Fig. 9. The figure shows that the momentum of the radiation scattered to the left is larger than the momentum of radiation scattered to the right, such that the total momentum carried by radiation is indeed negative, and the

TABLE I. Change in kinetic energy, ΔKE , radiation production, E_{rad} , and width exchange expressed by the width ratios w_1 and w_2 of different flat-top soliton collisions. Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$.

$ v_0 $	$ \Delta KE $	E_{rad}	w_1, w_2
0.005	1.494×10^{-1}	2.956×10^{-1}	0.0000 ^a , 2.0536
0.02	1.294×10^{-1}	9.981×10^{-2}	0.1313, 2.0322
0.04	1.243×10^{-1}	2.033×10^{-4}	0.1964, 1.9107
0.06	1.374×10^{-2}	1.149×10^{-7}	0.3929, 1.5893
0.08	2.567×10^{-3}	8.654×10^{-9}	0.7143, 1.2679
0.10	7.425×10^{-7}	4.428×10^{-9}	0.8393, 1.1429
0.12	4.397×10^{-7}	3.924×10^{-9}	0.8929, 1.0893
0.14	4.107×10^{-6}	1.009×10^{-9}	0.9643, 1.0357
0.16	2.074×10^{-7}	5.726×10^{-10}	1.0000, 1.0000
0.18	2.218×10^{-5}	1.920×10^{-8}	1.0536, 0.9286
0.20	1.443×10^{-4}	3.686×10^{-8}	1.1429, 0.8393
0.22	7.947×10^{-4}	7.642×10^{-7}	1.2500, 0.7143
0.24	8.219×10^{-3}	4.899×10^{-5}	1.3750, 0.5536
0.26	5.893×10^{-2}	1.402×10^{-3}	1.7143, 0.3393
0.28	2.729×10^{-1}	2.567×10^{-2}	2.0398, 0.1964
0.30	3.422×10^{-1}	1.874×10^{-1}	2.0621, 0.1712

^aTwo flat-top solitons combine after the collision to form a single flat-top soliton.

total momentum of both the coalesced soliton and radiation is consequently conserved.

B. Collision behaviors of flat-top solitons across widths spanning γ_0 to γ_{16}

It is instructive to investigate the elastic collisions between flat-top solitons of varying sizes. To achieve this, we conducted numerical calculations to outline the regions corresponding to elastic, inelastic, and radiative behaviors for flat-top solitons with widths corresponding to the range $\gamma_0 - \gamma_{16}$. We selected three representative velocity values to characterize the elastic, inelastic, and radiation regions: $|v_0| = 0.1$, $|v_0| = 0.26$, and $|v_0| = 0.31$, as illustrated in Fig. 6. In Fig. 10, we utilize these velocity values to plot ΔKE versus γ . For bright solitons (γ_0), our findings confirm that the kinetic energy remains constant before and after the collision, implying purely elastic behavior for all velocities. It is worth

noting that bright solitons are well-known for their exclusively elastic collision properties, as discussed above. In contrast, for $\gamma \neq 0$ we identify three distinct regions based on collision behavior. At $|v_0| = 0.1$, although there is width exchange, all collisions are nearly elastic, as $\Delta KE \approx 0$; at $|v_0| = 0.26$, they are inelastic, given that $\Delta KE \neq 0$. The degree of inelasticity varies with the width of the flat-top solitons. At $|v_0| = 0.31$, we observe an inelastic collision that is also accompanied by radiation. The findings indicate a notable divergence in the collisional properties of flat-top solitons compared to the traditional behaviors observed in the scattering dynamics of two bright solitons.

In Fig. 11, we examine the exchange of widths that occurs when two flat-top solitons collide with widths corresponding to the range $\gamma_0 - \gamma_{16}$. Referring to Fig. 4, a significant width exchange is evident at both low and high velocities. We investigated this phenomenon at three specific velocities: $|v_0| = 0.06$, $|v_0| = 0.16$, and $|v_0| = 0.26$. According to Fig. 4, there appears to be no width exchange around $|v_0| = 0.16$, prompting us to investigate whether this holds true for flat-top solitons of various widths. In Fig. 11(a), we observe that increasing the flat-top soliton width leads to an exchange in widths following the collision. Notably, the rate of width exchange stabilizes for flat-top solitons with widths corresponding to $|\gamma| > |\gamma_6|$. At this velocity, there is an absence of a noticeable width exchange for flat-top solitons with a width corresponding to γ_1 . Figure 11(b) presents a fascinating finding: there is minimal width exchange for all flat-top solitons of different widths, as $w_{1,2}$ is nearly equal to 1. This outcome confirms that at this velocity ($v_0 = -0.16$), the collision behavior of flat-top solitons resembles that of bright solitons, characterized by the absence of width exchange and a situation in which $\Delta KE = 0$. For $|v_0| = 0.26$, as shown in Fig. 11(c), we observe results akin to those in Fig. 11(a), with a width exchange also taking place. However, there is a distinct difference: while the left flat-top soliton had a smaller width after the collision at $|v_0| = 0.08$, it now exhibits a larger width after the collision.

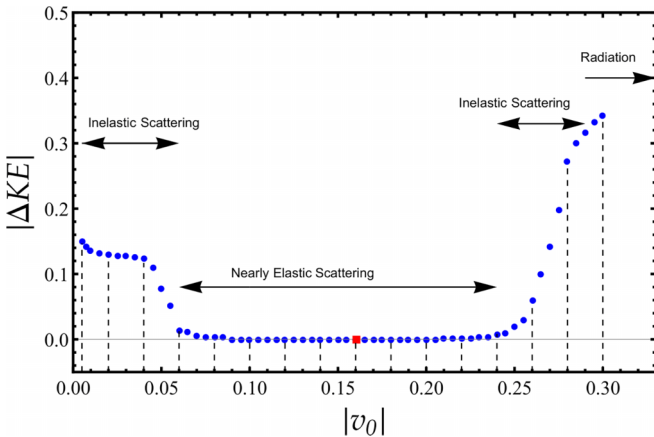


FIG. 6. Inelastic, nearly elastic, and radiation regimes for the flat-top soliton collisions with width defined by γ_9 . A completely elastic collision, indicated by the red square, occurs at $|v_0| = 0.16$.

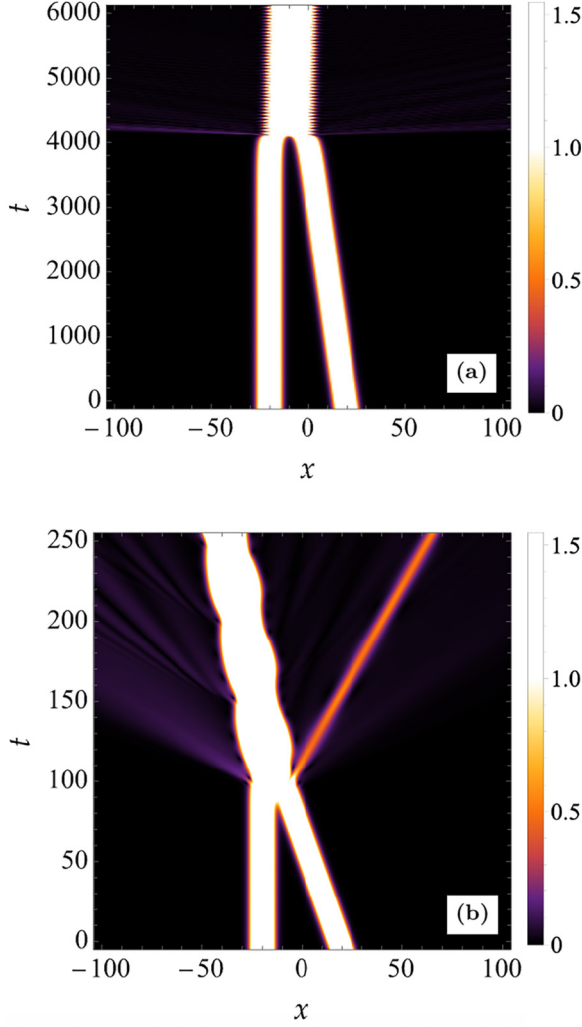


FIG. 7. Two flat-top solitons at very low and very high velocities. A stationary flat-top soliton is approached by another flat-top soliton moving with (a) $v_0 = -0.005$ and (b) $v_0 = -0.30$. Spatial domain taken is $L = 800$.

IV. PERIODIC OCCURRENCE OF ELASTIC SCATTERING

We found in the previous section that a completely elastic collision with no width exchange occurs only at a specific value of the initial soliton speed. Attempting to find an explanation for this behavior, we observe in Fig. 4(c) that flat-top solitons do not overlap in this case of completely elastic scattering. Invoking the well-established fact that solitons of π -phase difference repel each other [9], we conjecture that completely elastic collision without width exchange occurs when the two flat-top solitons have a π -phase difference just before their collision. In Fig. 12, this is confirmed by plotting the phase for a completely elastic case and one inelastic case. The present section is devoted to further proofs and consequences of this conjecture.

The first test of our conjecture is to repeat the completely elastic scattering case of Fig. 4(c) [or equivalently Fig. 12(a)], namely with $|v_0| = 0.16$, but with a constant phase, ϕ_0 , added to the moving soliton. The initial profile, in this case, takes the

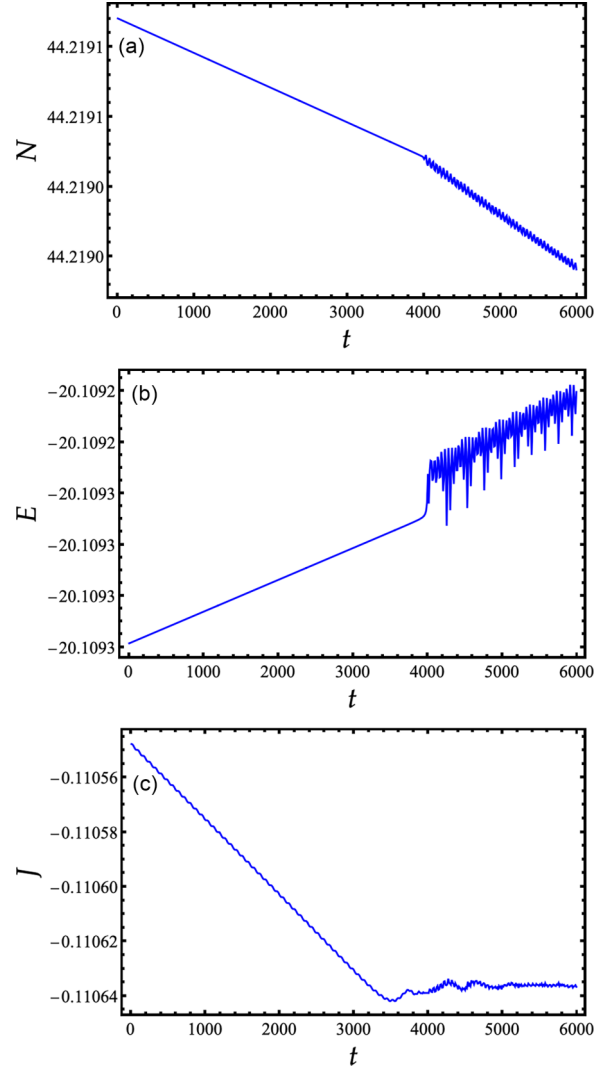


FIG. 8. Time evolution of norm (a), total energy (b), and momentum (c) of the flat-top soliton shown in Fig. 7(a), showing their conservation, within the accuracy of the numerical method.

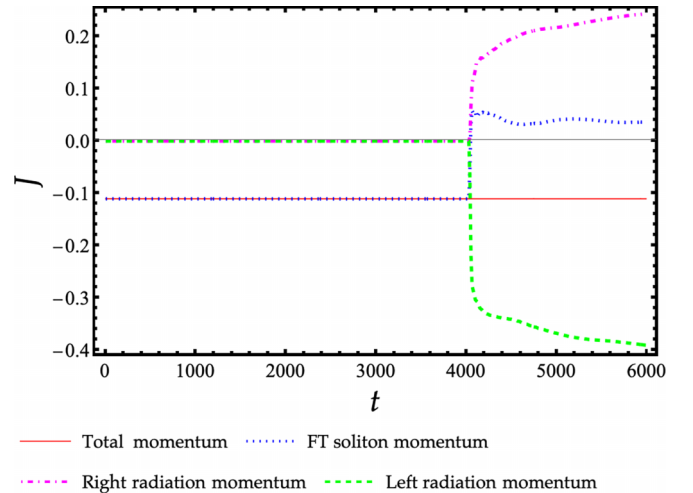


FIG. 9. Time evolution of the momentum of the coalesced flat-top soliton and the associated radiation for the scattering shown in Fig. 7(a).

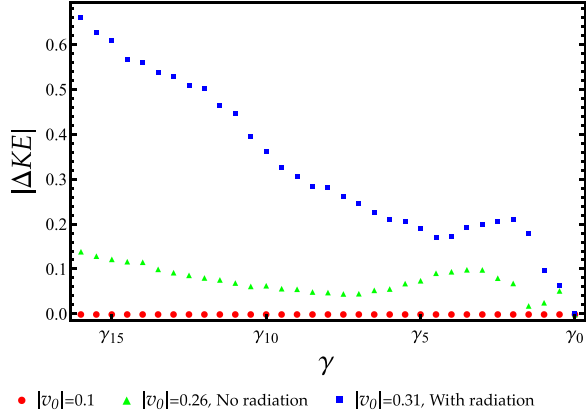


FIG. 10. Variation of the change in kinetic energy, $|\Delta KE|$, and γ for three flat-top soliton collisions, each characterized by the velocity of the moving flat-top soliton. Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$.

form

$$\psi(x, 0) = \psi_l(x, 0) + \psi_r(x, 0) e^{i\phi_0}, \quad (23)$$

where $\psi_l(x, 0)$ is the initial profile of the left (stationary) soliton, as given by Eq. (4) with $t = 0$ and $v_0 = 0$, and $\psi_r(x, 0)$ is the initial profile of the right (moving) soliton, as given by Eq. (4) with $t = 0$.

If our conjecture is correct, we should see the periodic appearance of completely elastic scattering in terms of ϕ_0 at values of relative phase $\phi_0 = \pi + 2m\pi$, where m is an integer. Indeed, this is what we see in Fig. 13, where a completely elastic collision is attained for $\phi_0 = \pi$ and $\phi_0 = 3\pi$. For values of $\phi_0 \neq \pi + 2m\pi$, an overlap between the two flat-top solitons occurs, and whenever there is an overlap, there is a kinetic energy loss. It is observed that the maximum energy loss occurs at $\phi_0 = 0, 2\pi, 4\pi$, which correspond to in-phase solitons. It is only when the two solitons are completely out-of-phase, $\phi_0 = \pi + 2m\pi$, that the repulsion between the two flat-top solitons is strong enough such that they do not overlap and thus do not lose any of their initial kinetic energy. This connection between overlap and loss in kinetic energy will be helpful to understand the physics underlying the inelasticity, discussed in more detail in the next section.

Additional evidence manifesting the dependence of collision elasticity on the relative phase is its dependence on the initial separation between the two flat-top solitons. Since phase is a dynamic quantity that evolves with time, it is expected that the phase at which the two flat-top solitons meet will be a function of the distance that the initial soliton had to cross from the initial time until the time of the collision. We fix the left soliton and change the initial position of the right soliton and then calculate the kinetic energy change in terms of the initial separation. Plotting in Fig. 14 the change in kinetic energy in terms of the initial separation, we see the periodic appearance of the completely elastic scattering, as anticipated. The positions at which completely elastic scattering occurs are $x_0 = 14.7, 24.6, 34.4$. The difference between any two consecutive values of these positions, say the first two, is about $\Delta x_0 \approx 10.0$. Our conjecture that elastic collision occurs

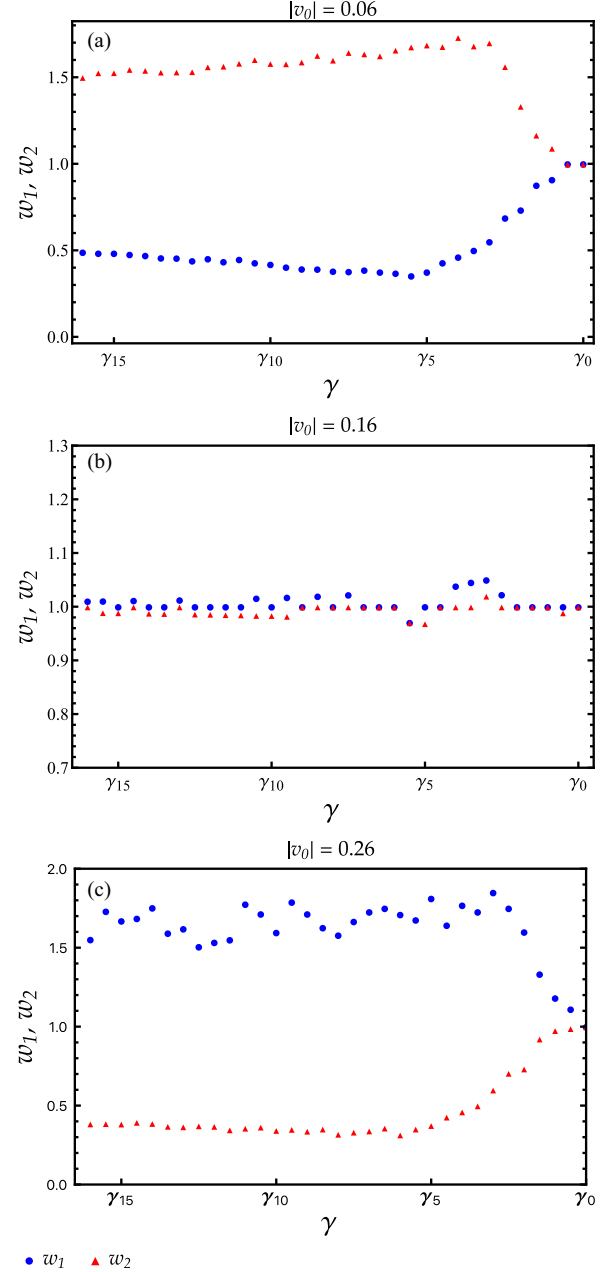


FIG. 11. Variation of width ratios of flat-top solitons with γ from γ_0 to γ_{16} for three flat-top soliton collisions, each characterized by the velocity of the moving flat-top soliton. Here, w_1 and w_2 denote the width ratio of the first and second solitons, respectively. (a) $|v_0| = 0.06$, (b) $|v_0| = 0.16$, and (c) $|v_0| = 0.26$. Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$.

when the relative phase between the two solitons equals π plus a multiple of 2π can now be verified by explaining two features of this curve. First, the distance between the initial positions of any two successive elastic collisions should correspond to a phase change that is a multiple of 2π . Secondly, the initial separation between the two solitons for any of the elastic scattering cases should correspond to a π -phase difference when the two solitons collide, plus a multiple of 2π . The phase of the incident soliton (on the right side), given

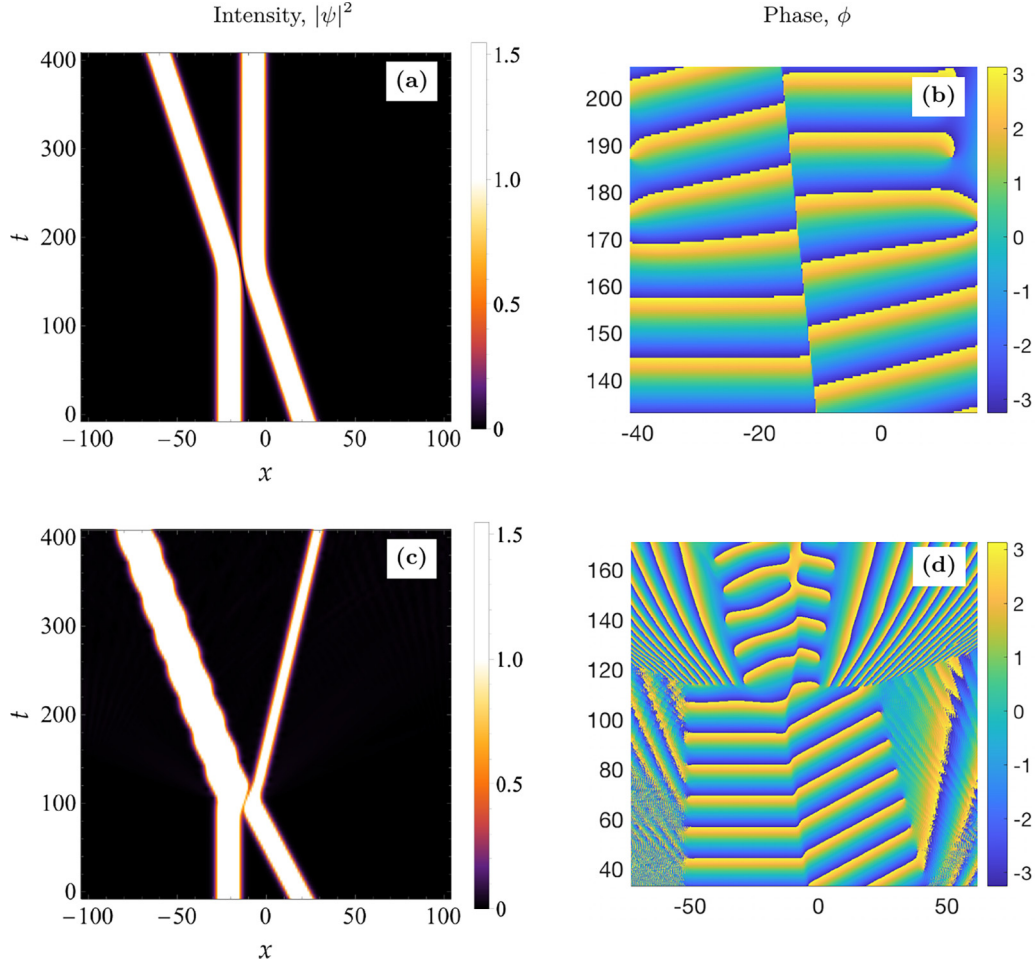


FIG. 12. Density and phase plots of collisions between a stationary flat-top soliton and a moving flat-top soliton, respectively. (a), (b) Elastic collision with $v = 0.16$, and (c), (d) inelastic collision $v = 0.26$. Values of other parameters are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$.

by Eq. (4), is

$$\phi_r(x, t) = u_0 t + v_0 [2(x - x_0) - v_0 t] / (4g_1), \quad (24)$$

and the phase of the stationary soliton (on the left side) is

$$\phi_l(x, t) = u_0 t. \quad (25)$$

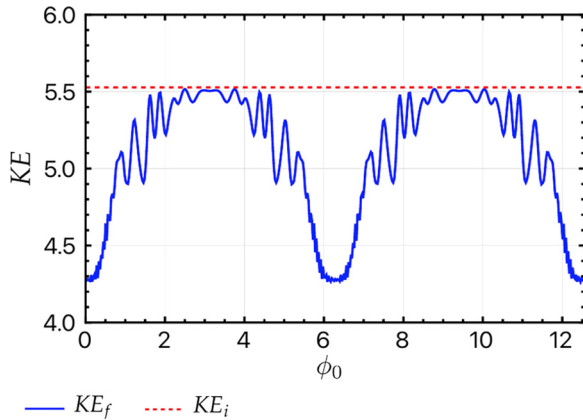


FIG. 13. Kinetic energy after collision of two flat-top solitons (solid blue) and their kinetic energy before collision (dashed red), in terms of the initial relative phase, ϕ_0 . Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $v_0 = -0.64$, $x_0 = 20.0$.

It should be mentioned that these expressions describe almost exactly the evolution of phase for both solitons since the overlap between them is negligible, except just before the collision time. Therefore, these expressions are expected to give an accurate estimate of the relative phase between the two solitons just before they collide. To explain the first feature, namely the periodic appearance of elastic collision, the change in the relative phase, $\Delta(\delta\phi)$, due to a change in the position of the right soliton, Δx_0 , must be a multiple of 2π . Here, $\delta\phi$ is defined as the relative phase

$$\delta\phi = \phi_r(x, t) - \phi_l(x, t) = \frac{v_0 [2(x - x_0) - v_0 t]}{4g_1}. \quad (26)$$

The two solitons meet at $x = -x_0$ at a time $t = -2x_0/v_0$. Their relative phase when they meet is thus

$$\delta\phi(x_0, v_0) = \frac{x_0 v_0}{2g_1}. \quad (27)$$

When the initial position is changed from x_{01} to x_{02} , where x_{01} and x_{02} correspond to two elastic scattering cases, the phase difference changes as

$$\Delta(\delta\phi) = \delta\phi(x_{02}, v_0) - \delta\phi(x_{01}, v_0) = \frac{\Delta x_0 v_0}{2g_1}, \quad (28)$$

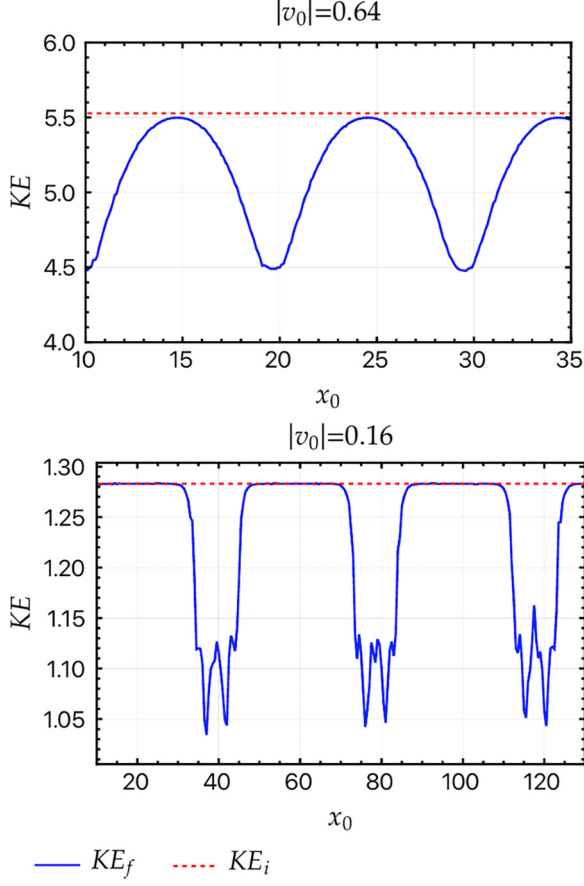


FIG. 14. Kinetic energy after collision of two flat-top solitons (solid blue) and their kinetic energy before collision (dashed red), in terms of the initial separation between flat-top solitons, for two values of the initial relative speed. Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $\phi_0 = 0$.

where $\Delta x_0 = x_{02} - x_{01}$. For the first period of the curve in Fig. 14 for $|v_0| = 0.64$, we read off $\Delta x_0 = 24.6 - 14.7$, which corresponds to a phase difference, according to (28), that equals $\Delta(\delta\phi) = 1.008411 \times 2\pi$. For the second period with $\Delta x_0 = 34.4 - 24.6$, the phase difference is $\Delta(\delta\phi) = 0.99822 \times 2\pi$. The second feature to be confirmed is that values of x_0 at minimum kinetic energy loss (peaks of the curves) should correspond to a relative phase, $\delta\phi$, that equals π . For the present case, Eq. (27) gives $\delta\phi = 2.99466 \times \pi$, $5.01147 \times \pi$, $7.00791 \times \pi$, corresponding to peak positions at $x_0 = 14.7$, 24.6 , 34.4 , respectively.

For further confirmation, we repeat this calculation for a soliton initial speed of $|v_0| = 0.16$. In Fig. 14 for $|v_0| = 0.16$, the loss in kinetic energy for this case is closest to the initial kinetic energy at $x_0 \approx 20, 59.5, 100$, which correspond to relative phases $\delta\phi = 1.01859 \times \pi$, $3.03031 \times \pi$, $5.093 \times \pi$, respectively. Clearly, the out-of-phase condition for elastic scattering is also verified here as well.

Initial soliton speed is another parameter that affects the relative phase of solitons, as seen by Eq. (27). Therefore, as an additional proof of the role of the phase, we fix the initial soliton position and study the effect of initial soliton speed

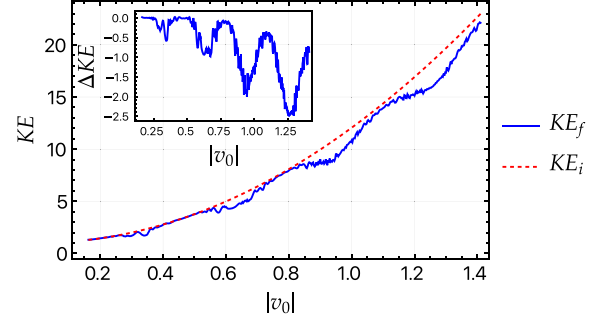


FIG. 15. Kinetic energy after collision of two flat-top solitons (solid blue) and their kinetic energy before collision (dashed red), in terms of their initial relative speed, which equals the speed of the right soliton. Inset shows the kinetic energy change. Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $x_0 = 20.0$, $\phi_0 = 0$.

on elasticity. In Fig. 15, we plot kinetic energy versus initial soliton speed, v_0 . Indeed, the periodic appearance of elastic scattering occurs at speeds $|v_0| = 0.470, 0.785, 1.100$. Equation (28) gives the phase difference in terms of the change in the initial speed as follows:

$$\Delta(\delta\phi) = \frac{x_0 \Delta v_0}{2g_1}. \quad (29)$$

Relative phases corresponding to the above initial soliton speeds are $2.99211 \times \pi$, $4.99747 \times \pi$, $7.00282 \times \pi$, which correspond to out-of-phase solitons and hence provide additional evidence that elastic scattering occurs when solitons are out-of-phase.

V. ENERGY TRANSFER ANALYSIS: ORIGIN OF INELASTICITY

Loss in the kinetic energy signifies inelasticity in soliton scattering, as shown in Fig. 16, which corresponds to a completely inelastic scattering where the two solitons continue to be coalesced after scattering. The reduction in the total kinetic energy is balanced by an equal increase in the interaction energy, such that the total energy is constant. However, this energy transfer does not explain why the two solitons continue to be coalesced and do not retain their state of two separate solitons as bright solitons do after coalescence. Scattering

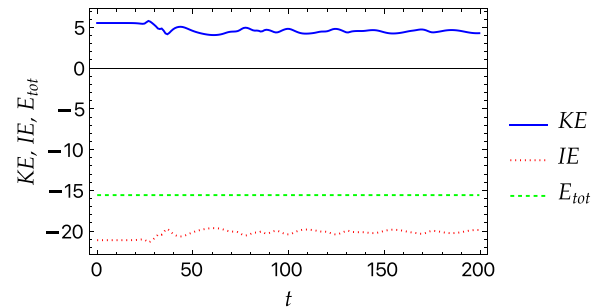


FIG. 16. Kinetic energy, KE, interaction energy, IE, and total energy, $E_{\text{tot}} = KE + IE$, including both solitons and radiation. Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $v_0 = -0.64$.

TABLE II. Energy components of solitons and radiation before and after scattering.

Energy components	Soliton		Radiation	
	Initial	Final	Initial	Final
KE_{com}	4.52794	2.10299	0	1.58479
KE_{pressure}	0.999485	0.464666	0	0.126536
KE_{tot}	5.52793	2.56765	0	1.71133
IE	-21.1096	-19.6569	0	-0.20418
N	44.2191	41.2722	0	2.9469
$v_{\text{com}} = \sqrt{2KE_{\text{com}}/N}$	0.639998	0.319231	0	1.0371

of bright solitons is completely elastic, although solitons do coalesce momentarily, and part of the kinetic energy transfers to interaction energy. Bright solitons somehow have an “internal memory” such that they retain their status before collision of two separate solitons with exactly the same kinetic energy. The question we try to answer here is as follows: why does this not happen to flat-top solitons, and why do flat-top solitons not retain their state of two separate solitons as before scattering? As we have seen in the previous section, flat-top soliton scatterings are always inelastic, except for cases when the two solitons are completely out-of-phase at the moment of scattering, where the force of repulsion is strong enough to prevent the solitons from overlapping. Inelasticity always occurs only when the profiles of the two solitons overlap. So, we rephrase the above questions as follows: when two flat-top solitons scatter, why is the reduction in the kinetic energy, which is supposed to transfer momentarily to interaction energy, not retained? Had solitons retained their original kinetic energy, they would have the momentum to separate after coalescence. But with a kind of “permanent” loss in the kinetic energy, the interaction energy of the coalesced solitons overcomes the kinetic energy such that the two solitons are “locked” into this state of coalescence. The follow-up question is, what causes this permanent loss in the kinetic energy, and why in bright soliton scattering does it seem like there is no such permanent loss in the kinetic energy? Our inspection showed that the key answer to these questions is in radiation production. Emitted radiation carries with it some kinetic energy and momentum, which are permanently lost from the solitons since radiation is a dispersive type of wave and it propagates faster than solitons. Bright solitons, on the other hand, do not emit radiation during their scattering since they are exact solutions to a completely integrable system, namely the fundamental NLSE, i.e., Eqs. (3) and (4) with $g_3 = \gamma = 0$. Integrable systems are characterized by an infinite number of conserved quantities. The NLSE with dual power-law nonlinearities is not integrable and hence radiation is always produced during the scattering of its exact solutions, which leads to an irreversible loss in kinetic energy and hence inelastic scattering of flat-top solitons.

To verify this explanation, we perform a detailed analysis on the energy transfers of the different energy components. We take the specific case of $\gamma = \gamma_9 = 10^{-9} - 1$ corresponding to a left-moving soliton with velocity $v_0 = -0.64$ scattered by a stationary soliton, leading to a coalesced single soliton accompanied by radiation. The energy components of solitons and radiation are calculated

before and after scattering separately, as listed in Table II. Before scattering, energies are calculated at the initial time. After scattering, the energies are recorded after an evolution time sufficient to reach a steady state. Center-of-mass kinetic energy, $KE_{\text{com}} = 4.52794$, is carried initially by the left-moving soliton; the other soliton is initially stationary. Total kinetic energy pressure, $KE_{\text{pressure}} = 0.999485$, is the sum of two equal kinetic energy pressure values of the two solitons. Total kinetic energy is the sum of these two kinetic energy components: $KE_{\text{tot}} = KE_{\text{com}} + KE_{\text{pressure}} = 5.52743$. After collision, KE_{com} reduces to 2.10299, KE_{pressure} reduces to 0.464666, and thus KE_{tot} reduces to 2.56765. Interaction energy, on the other hand, increases from -21.1096 to -19.6569. Total kinetic energy decreases by 2.95977 while interaction energy increases only by 1.45270. Thus, out of the total reduction in kinetic energy, 2.95977, only 1.45270 transfers to interaction energy while the rest, $2.95977 - 1.45270 = 1.50707$, transfers to radiation. Examining the Radiation column in the table, we see that a total kinetic energy of 1.71133 is gained by radiation accompanied by -0.20418 interaction energy. The total energy carried by radiation is thus $1.71133 - 0.20418 = 1.50715$, which equals to the energy reduction by solitons’ kinetic energy. Another important feature to point out is that most of the radiation energy is center-of-mass kinetic energy. This shows that about 50% of the reduction in the kinetic energy of the solitons transfers mainly to a center-of-mass kinetic energy of the radiation. The norm of solitons reduces from 44.2191 for the two incident solitons to 41.2722 for the coalesced solitons. The reduction in norm equals the radiation norm, 2.9469. The center-of-mass speed of the incident soliton is 0.64, the center-of-mass speed of the coalesced solitons is 0.319231, and the effective center-of-mass speed for radiation is 1.0371, which is almost three times faster than the speed of the coalesced solitons. This analysis shows that a reduction in the solitons’ kinetic energy is transferred to the center-of-mass kinetic energy of radiation, which is never retained by the solitons due to the dispersive nature of radiation.

It is instructive to see how the above-described picture changes in terms of the width of the scattering solitons. To find out, we repeat the detailed analysis of energy transfer for a range of γ values. We consider the range $\gamma \in [-0.95, 10^{-13} - 1]$. For this range, the coalesced soliton is located almost at the same position at the final time of evolution. This enables us to calculate the energy components of the soliton and radiation separately using fixed integration limits. In Fig. 17, we show that the total kinetic energy

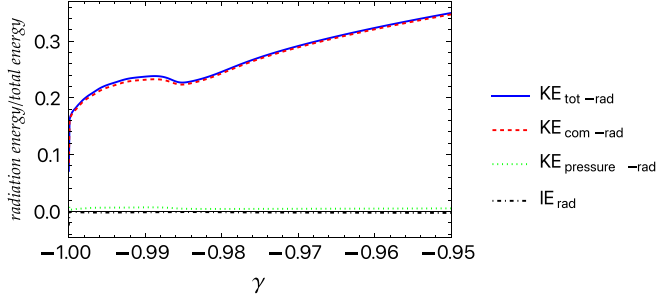


FIG. 17. Energy components of emitted radiation. $KE_{\text{tot-rad}}$: total radiation kinetic energy, $KE_{\text{com-rad}}$: center-of-mass radiation kinetic energy, $KE_{\text{pressure-rad}}$: radiation kinetic energy pressure, IE_{rad} : radiation interaction energy. Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $v_0 = -0.64$.

of radiation is mainly of the center-of-mass type. The radiation and interaction energies are very small compared to the center-of-mass or total energy. In Fig. 18, we plot the difference between the final energy components of solitons and radiation relative to the total energy. The first subfigure shows that the kinetic energy pressure of the initial solitons is mainly transferred to interaction energy. The second subfigure shows that the reduction in the center-of-mass kinetic energy of the initial solitons transfers mainly into the form

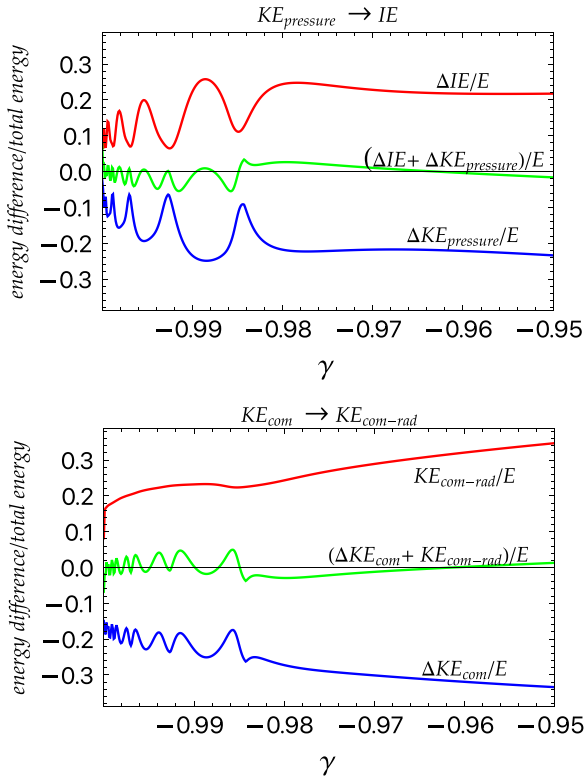


FIG. 18. Change in the kinetic and interaction energies of the flat-top solitons and radiation as a result of soliton collision. Left: mainly transfer from solitons' kinetic energy pressure to solitons' interaction energy. Right: mainly transfer from center-of-mass kinetic energy of solitons to center-of-mass kinetic energy of radiation. Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1$, $v_0 = -0.64$.

of radiation center-of-mass kinetic energy. The latter energy transfer is the main cause of inelasticity in flat-top soliton collisions.

VI. VARIATIONAL CALCULATION

In this section, we perform a variational calculation that explains the numerical results. In particular, it provides an analytical account of the amount of radiation production, and hence it reveals the physics underlying the degree of analyticity of collisions between flat-top solitons.

The main goal of the variational calculation in this section is to provide an analytical account for the amount of radiation in terms of the parameters of the system. Specifically, we want to confirm that the amount of radiation increases as the width of the flat-top soliton increases, i.e., when γ approaches -1 . This should explain the above numerical results, which showed that completely elastic collisions for $\gamma = 0$ (bright solitons) start to increasingly lose their elasticity as γ approaches -1 .

The variational calculation, in this section, is divided into two steps. At first, we choose an ansatz function and perform a variational calculation for a single flat-top soliton to calculate the equilibrium ground-state energy and equilibrium width of the soliton. The purpose of this step is to ensure that the selected ansatz function leads to the correct static properties as compared with those obtained from the exact solution. In the second step, we use the same ansatz function for the soliton in the presence of an unknown amount of radiation background. The amplitude of the background radiation is treated as a variational parameter, which is then determined by minimizing the energy with respect to the amount of radiation.

A. Single flat-top soliton

1. Static case

Choosing the appropriate ansatz function representing the profile of the flat-top soliton is a delicate step since the trial function needs to account for the “flatness” of the top of the soliton through an additional variational parameter. One needs also to compromise between the accuracy of the variational results and the difficulty of calculating the relevant integrations. In the literature, super-Gaussian is typically used [57,58],

$$\psi_{\text{sup.Gauss.}}(x, t) = A e^{-\left(\frac{x-x_0}{w}\right)^s} e^{i(v(x-x_0)+\beta(x-x_0)^2+\phi)}, \quad (30)$$

where A , w , v , x_0 , β , ϕ are variational parameters corresponding to amplitude, width, velocity, position, chirp, and phase, respectively. The variational parameter, s , is specific for flat-top solitons and it controls its degree of flatness. If one is interested in only the static properties, the variational parameters will be independent of time; if the dynamics of the soliton is to be obtained through the equations of motion, the variational parameters will be time-dependent.

One may consider other forms of the ansatz function, such as the following one constructed from tanh functions:

$$\psi_{\text{tanh}}(x, t) = \frac{1}{2\sqrt{c+1}} \{\tanh[d(x+w)] - \tanh[d(x-w)]\}, \quad (31)$$

where $d = \sqrt{u_0/g_1}$ and w is the variational width. Here, u_0 is determined by the norm, N , as given by Eq. (13). Another ansatz can be written in terms of the piecewise function

$$\psi_{\text{piecewise}}(x, t) = \begin{cases} \frac{1}{\sqrt{1+c}}, & |x - x_0| \leq w, \\ 0, & |x - x_0| > w. \end{cases} \quad (32)$$

While the above three ansatz functions result in accurate estimates of the equilibrium energy and width, employing the exact solution with a variational width and amplitude provides another ansatz that leads to the exact width and energy. This ansatz takes the form of the exact solution, (4), but with width, w , and amplitude, A , added as variational parameters,

$$\psi(x, t) = A f \frac{1}{\sqrt{c + \cosh^2\left[\frac{d}{w}(x - x_0 - v_0 t)\right]}} \times e^{i\{u_0 t + v_0[2(x - x_0) - v_0 t]/(4g_1)\}}, \quad (33)$$

where $f = \sqrt{\frac{2u_0}{g_2\sqrt{1+\gamma}}}$ and

$$u_0 = g_2^2 N^2 \gamma / \left(64g_1 \tan^{-1} \left[\frac{\sqrt{1+\gamma} - 1}{\sqrt{\gamma}} \right]^2 \right). \quad (34)$$

For $A = w = 1$, this function becomes the exact solution (4), which is normalized to N . Requesting the ansatz function to be normalized to N for general A and w , we get

$$A = \frac{1}{\sqrt{w}}. \quad (35)$$

The energy functional, (6), is then calculated using this ansatz to give

$$E = I_x \frac{g_1}{w^2} - I_4 \frac{g_2}{2w} - I_6 \frac{g_3}{3w^2}, \quad (36)$$

where

$$I_x = \frac{1}{d} \int_{-\infty}^{\infty} \left(\frac{f d \cosh(x) \sinh(x)}{(c + \cosh^2(x))^{3/2}} \right)^2 dx, \quad (37)$$

$$I_4 = \frac{1}{d} \int_{-\infty}^{\infty} \frac{f^4}{(c + \cosh^2(x))^2} dx, \quad (38)$$

$$I_6 = \frac{1}{d} \int_{-\infty}^{\infty} \frac{f^6}{(c + \cosh^2(x))^3} dx. \quad (39)$$

Minimizing the energy with respect to w , the condition $dE/dw = 0$ gives

$$w_{\text{var}} = \frac{4(g_1 I_x - g_3 I_6/3)}{g_2 I_4}. \quad (40)$$

The width of the soliton at half-maximum is obtained from the condition $\psi(W) = \psi(0)/2$, and this will be given by

$$\begin{aligned} W &= w_{\text{var}} \frac{\cosh^{-1}(\sqrt{4+3c})}{d} \\ &= 2\sqrt{\frac{g_1}{u_0}} \ln \left[4 + \frac{3}{\sqrt{1+\gamma}} \right. \\ &\quad \left. + \frac{1}{2} \sqrt{-4 + \left(8 + \frac{6}{\sqrt{1+\gamma}} \right)^2} \right], \end{aligned} \quad (41)$$

which equals the width obtained from the exact solution, (5). This confirms the accuracy of the ansatz function (33) as a variational function to the flat-top soliton.

2. Dynamic case

For the dynamic case, we use the ansatz

$$\begin{aligned} \psi(x, t) &= \frac{A(t) f}{\sqrt{c + \cosh^2\left[\frac{d}{w(t)}[x - x_0(t) - v_0(t)t]\right]}} \\ &\times e^{i\{u_0 t + v_0(t)[2(x - x_0(t)) - v_0(t)t] + \beta(t)[x - x_0(t)]^2/(4g_1)\}}. \end{aligned} \quad (42)$$

We have added a chirp variational parameter, $\beta(t)$, in the phase to account for the width variations with time. The Lagrangian

$$L = \int_{-\infty}^{\infty} \frac{i}{2} \left(\frac{\partial \psi}{\partial t} \psi^* - \frac{\partial \psi^*}{\partial t} \psi \right) dx - E \quad (43)$$

is then calculated, from which the Euler-Lagrange equations give the equations of motion for the variational parameters:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_0} - \frac{dL}{dx_0} = 0 \rightarrow \frac{d}{dt} v_0(t) = 0, \quad (44)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{v}_0} - \frac{dL}{dv_0} = 0 \rightarrow \frac{d}{dt} x_0(t) = v_0(t), \quad (45)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{w}} - \frac{dL}{dw} = 0 \rightarrow \dot{\beta}(t) = f_3(w, \beta, \gamma), \quad (46)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}} - \frac{dL}{d\beta} = 0 \rightarrow \dot{w}(t) = f_4(w, \beta, \gamma), \quad (47)$$

where $f_3(w, \beta, \gamma)$ and $f_4(w, \beta, \gamma)$ are rather lengthy expressions and hence not shown here. The equilibrium value of w is obtained from the last equation for $w'(t) = 0$, which is identical to the result of the static case, (40). Clearly, the first two equations of motion correspond to uniform center-of-mass motion. The last two equations of motion give the dynamics of width and amplitude ($A = 1/\sqrt{w}$). We have verified, by solving the equations of motion numerically, that with an initial width not equal to the equilibrium width, the soliton acquires breathing oscillations, while with an initial width that equals the equilibrium width, the soliton's width remains constant during the time evolution.

B. Radiation production

Having established confidence in the variational ansatz function, we use it in this section to calculate the amount of radiation produced when two flat-top solitons scatter. The setup is the same as above, namely two equal solitons; one is stationary at $x = -x_0$ and another starting motion from $x = x_0$ towards the first soliton with initial speed v_0 . We consider the specific case when the two solitons coalesce and move with a speed v_c . An amount of radiation is produced after scattering, which carries some norm, energy, and momentum. The main goal here is to calculate the amount (norm) of radiation in terms of γ that will confirm and explain the numerical results.

The calculation is outlined as follows. We calculate the energy, norm, and momentum of the two solitons before col-

lision. Then we calculate the same for the coalesced solitons after collision together with radiation. Imposing the conservation of energy, norm, and momentum leads to the optimum amount of radiation produced at a specific value of γ . Good agreement with numerical results is then obtained.

Before collision, we use the ansatz function (33) to calculate the energy, which turns out to be

$$E_b = \text{KE}_{\text{com}} + 2\text{KE}_{\text{pressure}} + \text{IE}_1 + \text{IE}_2, \quad (48)$$

where

$$\text{KE}_{\text{com}} = g_1 N v_0^2, \quad (49)$$

$$\begin{aligned} \text{KE}_{\text{pressure}} &= \frac{g_1 A^2 d f^2}{4[c(1+c)]^{3/2} w} \\ &\times (\sqrt{c(1+c)}(1+2c) - \sinh^{-1}(\sqrt{c})), \end{aligned} \quad (50)$$

$$\begin{aligned} \text{IE}_1 &= \frac{g_2 A^4 f^4 w}{2[c(1+c)]^{3/2} d} \\ &\times \left[\sqrt{c(1+c)} \sinh^{-1} \sqrt{c} - 2c \tanh^{-1} \left(\sqrt{\frac{c}{1+c}} \right) \right], \end{aligned} \quad (51)$$

$$\begin{aligned} \text{IE}_2 &= -\frac{g_3 A^6 f^6 w}{12[c(1+c)]^{5/2} d} [-3\sqrt{c(1+c)}(1+2c) \\ &+ [3+8c(1+c)] \sinh^{-1}(\sqrt{c})]. \end{aligned} \quad (52)$$

The $\text{KE}_{\text{pressure}}$ term in Eq. (48) is multiplied by 2 to account for two solitons, and the terms IE_1 and IE_2 are the interaction energies of the cubic and quintic nonlinearities, respectively.

After collision, the two solitons coalesce and an amount of radiation is produced. The total norm of the coalesced solitons is less than the sum of the two norms of the solitons before scattering, namely

$$N_a = N_b - \delta, \quad (53)$$

where $N_b = 2N$, with N being the norm of any of the two equal solitons before scattering, and δ is the total norm of radiation produced after scattering.

The variational function after scattering is written as a superposition of a soliton part and a radiation part,

$$\psi(x, t) = \psi_{\text{soliton}} + \psi_{\text{rad}}. \quad (54)$$

The soliton part, ψ_{soliton} , takes the same form as that of a single flat-top soliton, (33), but with a norm equal to $N_c = 2N - \delta$ and consequently a different value of γ denoted as γ_c ,

$$\begin{aligned} \psi_{\text{soliton}}(x, t) &= \sqrt{\frac{2u_c}{g_2 \sqrt{1+\gamma_c}}} \\ &\times \frac{1}{\sqrt{\frac{1-\sqrt{1+\gamma_c}}{2\sqrt{1+\gamma_c}} + \cosh^2 \left[\sqrt{\frac{u_0}{g_1}} (x - x_0 - v_c t) \right]}} \\ &\times e^{i[u_c t + v_c (2x - v_c t)/(4g_1)]}, \end{aligned} \quad (55)$$

where

$$u_c = g_2^2 N_c^2 \gamma / \left(64g_1 \tan^{-1} \left[\frac{\sqrt{1+\gamma_c} - 1}{\sqrt{\gamma_c}} \right]^2 \right) \quad (56)$$

and

$$\gamma_c = \frac{4 \tan^2 [N_c \sqrt{g_3/g_1}/(2\sqrt{3})]}{(\tan^2 [N_c \sqrt{g_3/g_1}/(2\sqrt{3})] - 1)^2}, \quad (57)$$

which are obtained from Eq. (34) after replacing γ by γ_c and N by N_c and then solving for γ_c . The center-of-mass speed of the coalesced soliton, v_c , is determined from the conservation of momentum as follows:

$$N v_0 = (2N - \delta) v_c, \quad (58)$$

where the left-hand side is the momentum of the two solitons before collision—one soliton is stationary—and the right-hand side is the momentum of the coalesced solitons. We have ignored the momentum carried by the radiation part since it is negligible compared to that of the solitons part.

The radiation part, ψ_{rad} , is a linear dispersive wave

$$\psi_{\text{rad}} = A_r e^{i(kx - \omega t)} \quad (59)$$

with wave number k and frequency ω , and amplitude $A_r = \sqrt{\frac{\delta}{L}}$. The parameter L is the length of the spatial domain over which the radiation part is to be integrated and gives a norm that equals δ , namely

$$\delta = \int_{-L/2}^{L/2} |\psi_{\text{rad}}|^2 dx. \quad (60)$$

The dispersion relation for the wave number and frequency of the radiation part is determined from the condition that ψ_{rad} is a solution to the NLSE, (3). Inserting ψ_{rad} from Eq. (59) in the NLSE, (3), this condition gives

$$\omega = g_2 A_r^2 + g_3 A_r^4 - g_1 k^2. \quad (61)$$

Comparing our ansatz for radiation, ψ_{rad} , with the exact constant-wave (CW) solution

$$\psi_{\text{CW}}(x, t) = \pm \sqrt{\frac{\pm g_2 - \sqrt{g_2^2 + 4g_3 u_0}}{g_3}} e^{i(u_0 t + \frac{v_0}{4g_1} (2x - v_0 t))}, \quad (62)$$

we obtain $\omega = u_0 - v_0^2/(4g_1)$, which upon using in (61) leads to

$$k = \frac{1}{g_1} \sqrt{A_r^2 g_2 + A_r^4 g_3 - u_0 + \frac{1}{4g_1} v_0^2}. \quad (63)$$

The energy of the coalesced soliton in the presence of radiation should be calculated using the ansatz function (54). Since the contribution from the overlap between the soliton part and the radiation part is much smaller than that of the soliton or radiation, we ignore all terms corresponding to the overlap and thus the total energy after scattering will be approximated by that of the coalesced soliton plus that of the radiation. Finally, equating the energy of the two flat-top solitons before collision, E_b , with the energy of the coalesced soliton and radiation, and solving for δ , we obtain the amount of radiation produced. In Fig. 19, we plot the radiation norm, δ , versus γ and compare the results with the numerical solution, where good agreement is obtained. In particular, the specific case of γ_0 , which was examined above numerically, is represented in the inset of Fig. 19 by the point plotted with $\times$. The numerical

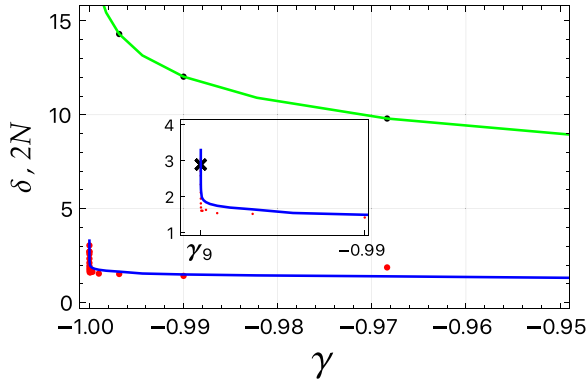


FIG. 19. Lower curve: Radiation norm, δ . Upper curve: total norm, $2N$. Values of γ range from -0.95 to $\gamma_{10} = 10^{-10} - 1$. Points correspond to the numerical calculation, and the curves are the results of the variational calculation. The inset shows the specific case discussed in the text, with the $\times$ corresponding to the value of δ found from the numerical simulations. Parameters used are $g_1 = 1/2$, $g_2 = 1$, $u_0 = 1/2$, $v_0 = -0.64$.

solution gives a value of $\delta = 2.99$, while the variational calculation estimate is $\delta = 3.1$, which is in good agreement with the numerical value. The calculation shows also that indeed, when γ approaches -1 , i.e., the soliton gets deeper into the flat-top regime, the amount of radiation increases, which increases the inelasticity of flat-top solitons, as we found in the numerical simulations.

VII. CONCLUSIONS AND OUTLOOK

There are three main findings in this paper: (i) While collisions of flat-top solitons are in general inelastic, a regime in the solitons' parameter space exists such that nearly elastic collisions occur. Although this regime is characterized by minimal width oscillations and radiation production, a remarkable exchange in the widths of the solitons takes place. Such a case is shown in Figs. 4(b) and 4(f). This behavior may find an application in optical data processing as data may be stored in the width of the solitons. (ii) There is a periodic occurrence of completely elastic collision between flat-top solitons at specific values of the initial solitons' parameters, including their relative phase, relative speed, and separation, as shown in Figs. 13–15. This periodic occurrence takes place at values of the parameters such that the two solitons are out-of-phase, i.e., their relative phase is π , just before their collision. In this particular case, no width exchange takes place, no width oscillations occur, and no radiation is produced. (iii) The origin of inelasticity lies in the production of radiation during the scattering process; part of the solitons' initial center-of-mass kinetic energy will be irreversibly carried away by the dispersive radiation. This is verified by Figs. 16–18, where the different components of the solitons and radiation energies are plotted separately. We show that most of the radiation energy is in the form of translational kinetic energy, and a much smaller part of it is in the form of kinetic energy pressure and interaction energy.

It is observed that coalesced solitons acquire breathinglike oscillations. These oscillations can be understood within the

context of Ref. [51]. In this work, the potential of interaction between two flat-top solitons is calculated and shown to be molecular-type in terms of the separation between solitons, i.e., there is a nonzero “bond length” where the energy is minimum. Any initial state of two solitons with separation slightly deviated from the equilibrium bond length undergoes bond oscillations. Due to the considerable overlap between the solitons, these bond oscillations resemble breathing oscillations of a single soliton. The frequency of the oscillation can be calculated analytically by linearizing in small-amplitude oscillations around the minimum of the potential. The frequency turns out to be a function of γ and the norm of the soliton, δ . In the present work, the oscillations observed in our simulations after the two colliding solitons coalesce can be viewed as those of the two-soliton molecule. From the variational calculation in Sec. VI, the norm of the coalesced soliton is expressed as the total norm of the two initial solitons, $2N$, minus the norm of radiation produced, δ . In addition, γ is also different from that of the initial solitons. The frequency of the coalesced soliton is thus a function of the norm of radiation, δ and γ .

The main questions posed in the Introduction of this paper are thus answered. Collisions of flat-top solitons can be elastic as long as the relative phase between the two colliding solitons equals π . In such a case, the solitons do not overlap at the closest approach as a result of the repulsive force. Consequently, no production of radiation occurs and hence total kinetic energy will be conserved indicating an elastic scattering. The relative phase at collision can be monitored by the initial parameters of the solitons, namely their initial relative phase, initial relative speed, and initial separation. When any of these initial parameters change by an amount leading to a change in the relative phase that is a multiple of 2π , the same scattering outcome will be obtained, which explains the periodic appearance of elastic scattering.

Inelasticity is directly caused by the production of radiation, and the production of radiation is caused by the nonintegrability of the NLSE equation with dual power-law nonlinearity, which is confirmed here.

On a fundamental level, we believe the present work lays down the essentials of two theoretical investigations. First, we investigate the link between nonintegrability and inelasticity through the calculation of conserved quantities. One needs to find out which physical quantity is not conserved and is responsible for inelasticity. Our preliminary investigations show that this quantity must be other than norm, energy, or momentum, as these three quantities turn out to be conserved for the NLSE with dual nonlinearities. The second investigation is a variational calculation that accounts for the intensity of radiation and loss in the solitons' kinetic energy. In view of the above findings, specifically Figs. 17 and 18, the trial function may be set up with a soliton part and a radiation part. The variational parameters may be the amplitudes and speeds of the soliton and radiation. These two theoretical works are left for future investigations.

Finally, the results we found here may be useful for applications of flat-top solitons such as data carriers, where elastic collisions are favored due to minimization of errors [59–61]. Specifically, optical data transfer and data processing favor elastic scattering in order to minimize errors. Data may

be coded in the amplitude or width of the soliton. Regimes of elastic scattering identified in this paper would be useful in this case. In addition, the interesting regimes of nearly elastic scattering but with considerable exchange in width (the “nearly elastic scattering” regime in Fig. 6) would be particularly important. Data may be coded in the ratio between

the widths of the solitons. Scattering will change this ratio. The change depends on several parameters, including initial separation, relative velocity, relative phase, and gamma. By manipulating these parameters, a certain protocol may be designed to obtain the function of operations such as logic gates.

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